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Stochastically Resilient Design of Mixed H_2-Dissipative Observers for Discrete-Time Nonlinear Systems

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Abstract: A linear matrix inequality based mixed H_2-Dissipative type state observer design approach is presented for smooth discrete time nonlinear systems with finite energy disturbances. This observer is designed to maintain H_2 type estimation error performance together with either H_∞ or a passivity type disturbance reduction performance in case of randomly varying perturbations in its gain. A linear matrix inequality is used at each time instant to find the time-varying gain of the observer. Simulation studies are included to explore the performance in comparison to the extended Kalman filter.

1. INTRODUCTION

An observer that diverges or significantly deteriorates in performance by a small perturbation in the observer gain is referred to as a "fragile" or "non-resilient" observer. The resilience problem, after the publication of (Keel and Bhattacharyya, 1997), has gained attention, e.g. (Dorato, 1998; Famularo, *et al.*, 1998; Jadbabaie, *et al.*, 1998; Keel, *et al.*, 1998) to name a few. One reason for the importance of resilience is that, in applications, the observer gains are calculated offline using available software, hence there is a need to address the consequences, in practice, of the computation error. Also, sometimes during implementation, it is necessary to resort to manual tuning to improve performance. In some other cases, the gains may slowly drift. Due to these reasons, the observer must be able to tolerate some perturbations in the observer coefficients.

On the other hand, the research in nonlinear observer design has resulted in many new state observation techniques for various classes of systems: feedback linearization (Bestle and Zeitz, 1983; Isidori, 1985; Krener and Respondek, 1985), variable structure techniques (Walcott and Zak, 1987; Walcott, *et al*., 1987; Yaz and Azemi, 1993a; Yaz and Azemi, 1993b; Azemi and Yaz, 2000) extended linearization (Baumann and Rugh, 1986), high gain observers (Bornard and Hammouri, 1991), Lyapunov-based observer design (Thau, 1973; Kou, *et al*., 1980; Vidyasagar, 1980; Yaz and Azemi, 1993c), observers as limiting cases stochastic estimators (Baras, *et al*., 1988), set-valued estimation (James, and Petersen, 1998), State Dependent Riccati Equation (SDRE) estimator design of observers (Jaganath, *et al*., 2005), etc, among others.

In this paper, we introduce a resilient design of mixed H_2-dissipative type state observer for discrete-time nonlinear systems with smooth nonlinearities and finite energy (ℓ_2) type disturbances. Linear matrix inequalities (LMIs) (Boyd, et al., 1994) are used as the main mathematical tool. In this work, in contrast to the previous work that uses a single LMI and a constant gain, the time-varying gain of the observer is found by solving a difference LMI at each point in time. This result is a natural extension of the constant gain LMI-based resilient linear observers in (Yaz, et al., 2005) and (Yaz, et al., 2006), an extension to resilient nonlinear observer case of mixed performance criteria results (Yaz, *et al.,* 1992; Haddad, *et al.,* 1994; Xiang and Zhou, 2001) and can also be viewed as a stochastically resilient dissipative version of the Extended Kalman filter (EKF) because of its form, the time-varying nature of its gain, and locally H_2 performance criterion it satisfies. Therefore, some simulation examples are also included to compare the performance of the new nonlinear observer to the EKF.

2. MAIN RESULTS

Consider the state space representation of the following nonlinear system with a nonlinear measurement equation:

$$\begin{aligned} x_{k+1} &= f(x_k) + B_k w_k \\ y_k &= h(x_k) + D_k w_k \end{aligned} \tag{1}$$

where $x_k \in R^n$ is the state, $y_k \in R^p$ is the measured output, w_k is an ℓ_2 type disturbance and the nonlinearities f and h are smooth.

Expanding the nonlinearities around the current state estimate $\hat{x}_k$ into Taylor series, we have

$$f(x_k) \cong f(\hat{x}_k) + A_k e_k,\ \ h(x_k) \cong h(\hat{x}_k) + C_k e_k$$

where $e_k = x_k - \hat{x}_k$ denotes the estimation error. Then equation (1) can be approximately rewritten as

$$x_{k+1} \cong f(\hat{x}_k) + A_k e_k + B_k w_k$$
$$y_k \cong h(\hat{x}_k) + C_k e_k + D_k w_k \quad (2)$$

Let $\hat{x}_k$ obey the following nonlinear Luenberger observer (or EKF form) equation :

$$\hat{x}_{k+1} = f(\hat{x}_k) + (K_k + \Delta_k)(y_k - h(\hat{x}_k)) \quad (3)$$

where Δ_k represents the time-varying error made in implementing the observer gain. As explained before, this can be due either to computational / modeling errors or random changes during operation. In this work, a general stochastic description of the error in the filter gain is given as follows:

$$\Delta_k = \sum_{i=1}^{M} \gamma_k^i K^i$$

where γ_k^i are mutually uncorrelated, scalar, standard (zero mean and unit variance) white noise sequences and K^i are known perturbation matrices. The zero mean property chosen for the multiplicative noise means perturbations can take on positive or negative values in an equally likely manner. The general time varying property of the gain perturbations γ_k^i as random sequences rather than random constants is useful in allowing different amounts of perturbations that may occur at different times during operation. If only an a priori computational error in the gain is to be considered, then γ^i can be modelled as random constants and not as random sequences.

Substituting from equations (2) and (3), we find that the error dynamics locally obey

$$e_{k+1} \cong (A_k - (K_k + \Delta_k)C_k)e_k + (B_k - (K_k + \Delta_k)D_k)w_k \quad (4)$$

Let z_k denote the performance output where

$$z_k = C_z e_k + D_z w_k \quad (5)$$

and consider the conditional expectation

$$E\{V_{k+1} - V_k + e_k^T Q e_k + \in \|z_k\|^2 + \delta \|w_k\|^2 - \beta z_k^T w_k \mid e_k, e_{k-1}, \cdots, e_0\} \le 0 \quad (6)$$

for an energy function $V_k = e_k^T P_k e_k$ where $P_k > 0$. $Q > 0$ is a weight matrix which determines the relative weighting of the H_2 vs. the other criteria. In this formulation, it is also possible to minimize H_2 norm (by maximizing the minimum eigenvalue of Q) while satisfying the other criteria at the same time. Note that upon summation over k, taking expectation and using the expectation property $E\{E\{x/y\}\} = E\{x\}$, (6) yields the performance criterion

$$\sum_{k=0}^{N-1} E\{e_k^T Q e_k\} \le V_N + \sum_{k=0}^{N-1} E\{e_k^T Q e_k\}$$
$$\le E\{e_0^T P_0 e_0\} - \sum_{k=0}^{N} E\{\in \|z_k\|^2 + \delta \|w_k\|^2 - \beta z_k^T w_k\} \quad (7)$$

that allows several mixed criteria formulations possible in a unified eigenvalue problem (Boyd, et al., 1994) framework. The suggested performance criteria with different design parameters $\in$, β, and δ are given in Table 1.

∈	β	δ	Performance criteria
1	0	<0	Suboptimal H_2 -H_∞ design
0	1	0	H_2 –Passivity
0	1	>0	H_2 -Input Strict passivity
>0	1	0	H_2 –Output strict passivity
>0	1	>0	H_2 –Very strict passivity

Table 1. Various dissipative performance criteria in a common framework.

The following is the main result of this paper:

Theorem 1. Given model (1) and the nonlinear observer (3) for the performance output (5) and objective (7). For $P_0 > 0$, let the following LMI

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} \ge 0 \quad (8)$$

hold for $P_k > 0$, Y_k, $k \ge 0$, and

$$q_{11} = P_k - \in C_z^T C_z - Q - C_k^T \sum(P) C_k$$
$$q_{12} = -\in C_z^T D_z - C_k^T \sum(P) D_k + 0.5\beta C_z^T$$
$$q_{13} = A_k^T P_{k+1} - C_k^T Y_{k+1}^T$$
$$q_{21} = q_{12}^T$$
$$q_{22} = -\delta I - \in D_z^T D_z - D_k^T \sum(P) D_k + \frac{\beta}{2}(D_z + D_z^T)$$
$$q_{31} = q_{13}^T$$
$$q_{23} = B_k^T P_{k+1} - D_k^T Y_{k+1}$$
$$q_{32} = q_{23}^T$$
$$q_{33} = P_{k+1}$$

where $\sum(P) = \sum_{i=1}^{M} K^{i^T} P_{k+1} K^i$. Then, the performance criteria (7) is satisfied and the gain of the resilient observer which satisfies the performance objective (7) is found from $K_k = P_{k+1}^{-1} Y_{k+1}$.

Proof:

We substitute for the terms in (6) to obtain the following inequality

$$E\{e_{k+1}^T P_{k+1} e_{k+1} - e_k^T P_k e_k + e_k^T Q e_k + \in z_k^T z_k + \delta w_k^T w_k - \beta z_k^T w_k \mid e_k, e_{k-1}, \cdots, e_0\} \le 0 \quad (9)$$

or

$$E\left\{ [e_k^T \; w_k^T] \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} e_k \\ w_k \end{bmatrix} \right\} \le 0 \quad (10)$$

for

$$S_{11} = (A_k - (K_k + \Delta_k) C_k)^T P_{k+1} (A_k - (K_k + \Delta_k) C_k) + \in C_z^T C_z - P_k + Q$$
$$S_{12} = (A_k - (K_k + \Delta_k) C_k)^T P_{k+1} (B_k - (K_k + \Delta_k) D_k) + \in C_z^T D_z - 0.5 \beta C_z^T$$
$$S_{21} = S_{12}^T$$
$$S_{22} = (B_k - (K_k + \Delta_k) D)^T P_{k+1} (B_k - (K_k + \Delta_k) D_k) + \delta I + \in D_z^T D_z + \frac{\beta}{2}(D_z + D_z^T)$$

Taking the expectation and considering the matrix in the middle in (10) and rearranging yields

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} - \begin{bmatrix} R_1^T \\ R_2^T \end{bmatrix} P_{k+1} [R_1 , \; R_2] \ge 0 \quad (11)$$

$$R_1 = A_k - K_k C_k$$
$$R_2 = B_k - K_k D_k$$

Using the Schur complement (Boyd, et al., 1994) in (11), we obtain the LMI (8), where $Y_{k+1} = P_{k+1} K_k$ and the necessary estimator gain is found from $K_k = P_{k+1}^{-1} Y_{k+1}$. Therefore, for the performance index (7) to hold for the linearized error equation in (4) and the disturbance input $w \in \ell_2$, (8) must be feasible for $P_k > 0$, Y_k for all $k \ge 0$.

Remark 1. Note that an LMI needs to be solved in each step instead of a Riccati equation for EKF to find the necessary gain. The performance of this observer is compared to that of the EKF in the next section.

Remark 2. The resilience comparison of the new observer with that of the EKF is also provided in simulation studies. Since this observer is made more resilient by design, it is expected to perform better in this sense than EKF.

Remark 3. Note that *Q*, $\in$, β, and δ are design parameters which can be used to obtain better performance. By varying *Q*, it is also possible to attach more or less importance to H_2- vs. H_∞ or various passivity criteria.

Remark 4. Since a time-varying LMI with different parameter values is used at different times, a change in the feasibility conditions can be expected in time. Note however that the same problem may also be faced in EKF implementation which may lead to premature convergence or divergence.

3. SIMULATION RESULTS

<u>Example 1.</u> Since both the new approach and the EKF are based on linearization approximation, it is fair to compare this approach to EKF. Because EKF convergence was considered before in (Zhai, et al., 2003) for various types of scalar discrete nonlinear systems, we compare our results with the ones in (Zhai, et al., 2003), which uses an EKF in one-step form for the scalar model

$$x_{k+1} = f(x_k)$$
$$y_k = x_k + w_k$$

Also, the same categorization of systems introduced in (Zhai, et al., 2003) is used here. All simulation results are averages of 30 runs of the same experiment.

a) Type I system ($\sup_k |A_k| < 1$ (Zhai, et al., 2003))

Consider the following system model

$$f(x_k) = 0.5 \sin(x_k)$$

where w_k is e^{-k}. Figure 1. shows the simulation results of both the sample squared error and P_k in (8) with respect to the iteration time k.

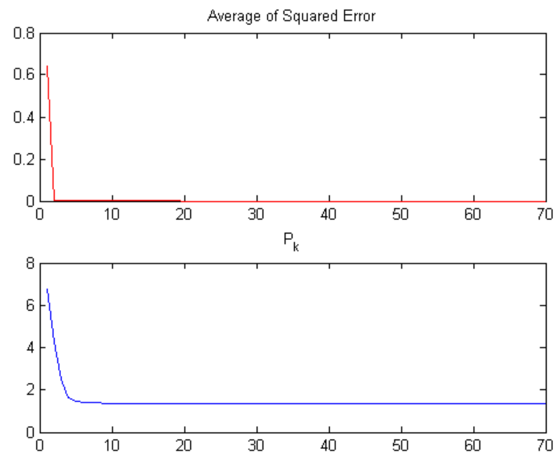

Fig. 1. The change of sample squared error & P_k vs. iteration time k.

Figure 2. is on the comparison of sample squared error between the results of the new LMI approach and the traditional EKF approach. As is shown in Fig. 2, the new LMI approach has smaller squared error value both during the transient stage and the steady state than that of EKF approach.

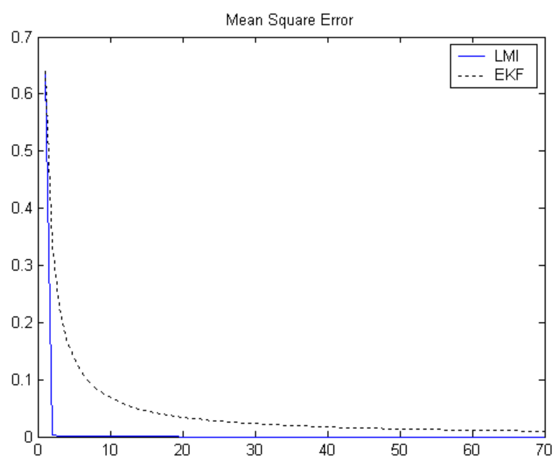

Fig. 2. Type I system, squared error comparison between the LMI method and EKF vs. iteration time

b) Type II system ($\sup_k |A_k| > 1$ (Zhai, et al., 2003))

The Skew Tent map with a trajectory that exhibits chaotic behaviour is considered. Skew Tent map is defined by

$$f(x_k) = \begin{cases} x_k / a, & 0 \le x_k \le a \\ (1-x_k)/(1-a), & a < x_k \le 1 \end{cases}$$

The value 0.6 was chosen for a for which the trajectory exhibits chaotic behaviour. Output equation y_k has w_k = 0.9^k. Note that at a single point, the nonlinearity is not differentiable; however this did not create any problems in the simulation studies. Figure 3. shows the simulation results for the sample squared error and P_k with respect to the iteration time k. Figure 4. shows the comparison of the sample squared error between the results of the new LMI approach and the EKF approach. We can see that the LMI approach has smaller sample squared error than that of EKF.

c) Type III system ($\sup_k |A_k| = 1$ (Zhai, et al., 2003))

Consider the following nonlinearity:

$$f(x_k) = \sin(x_k)$$

where w_k is e^{-k}. Figure 5. shows the simulation results of both the sample squared error and P_k vs. the iteration time k. As k increases, sample squared error decreases quickly.

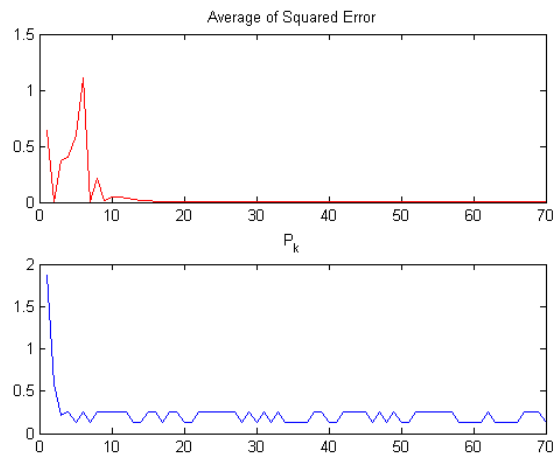

Fig. 3. The change of sample squared error & P_k vs. iteration time k.

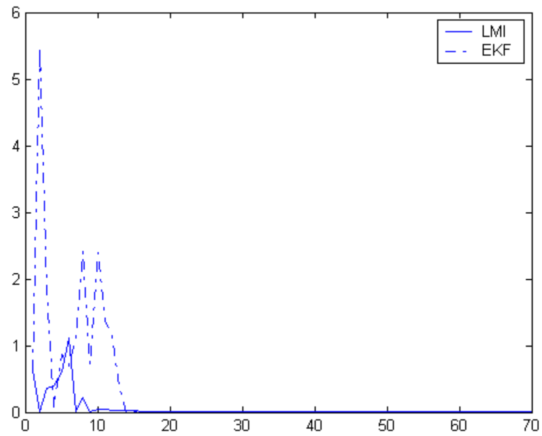

Fig. 4. Type II system, comparison of sample squared error between the new LMI method and EKF vs. iteration time

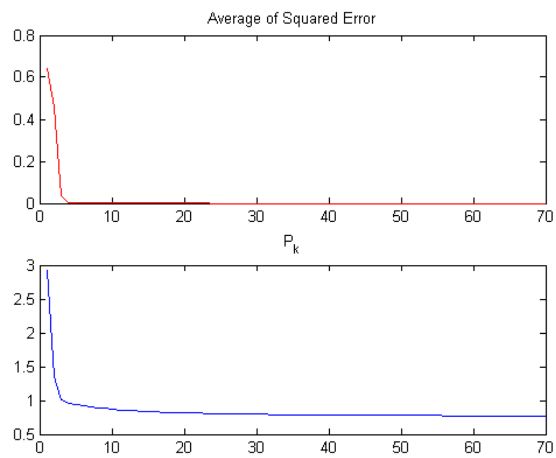

Fig. 5. The change of the sample squared error & P_k vs. iteration time k.

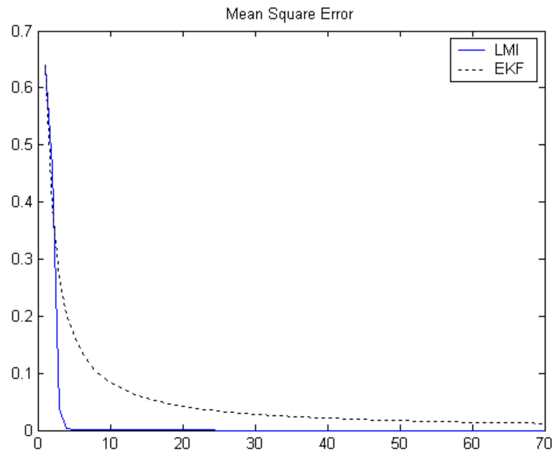

Fig. 6. Type III system, comparison of the sample squared error between LMI method and EKF vs. iteration time k.

As is shown in Fig. 6., the new LMI approach shows smaller sample squared error value than that of EKF approach.

Table 2. shows the sum (over k) of the sample squared errors for both the new LMI approach and the traditional EKF approach for those three types of systems. As shown in Table 2., the LMI approach has smaller amount of sample squared error than that of the EKF approach.

	LMI	EKF
Type I	0.6434	3.2523
Type II	3.5547	19.1467
Type III	1.1403	3.8067

Table 2. The Sum of the sample squared error for the new LMI and EKF approaches

The values of design parameters used in the simulations are given in Table 3.

	C_z	D_z	β	$\in$	δ	Q
Type I	1	1	0	0.1	1	0.5
Type II	1	1	0	0.09	0.923	0.001
Type III	1	1	0	0.1	1	0.1

Table 3. Design Parameter Values

Example 2. This example contains a simulation of the error responses of observers for stochastic error Δ_k made in implementing the observer gain to test the resilience of our design. Type II system (Skew Tent map) is chosen for this simulation. The known perturbation matrices (K^i) is 0.25 for this example with $w_k = 0.8^k$. This system is simulated by running the model and observers 30 times and taking ensemble averages. In each simulation, a sequence of standard white noise is generated and added to both the new observer and the EKF gains for comparison. The actual state (x_k) and estimated state ($\hat{x}_k$) are depicted in Fig.7. Figure 8. is the comparison of sample mean square error between the results of the LMI method and the traditional EKF method.

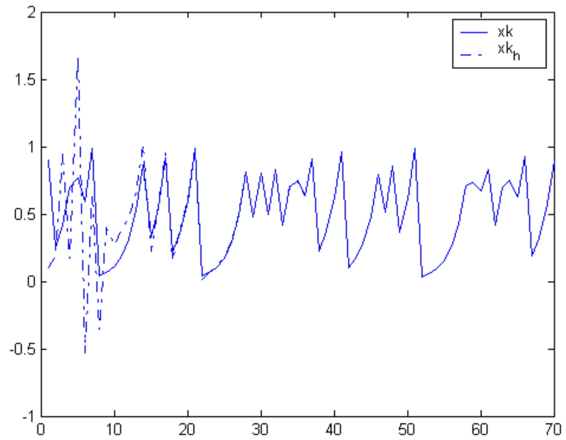

Fig. 7. Type II system, actual state and estimated state by LMI method.

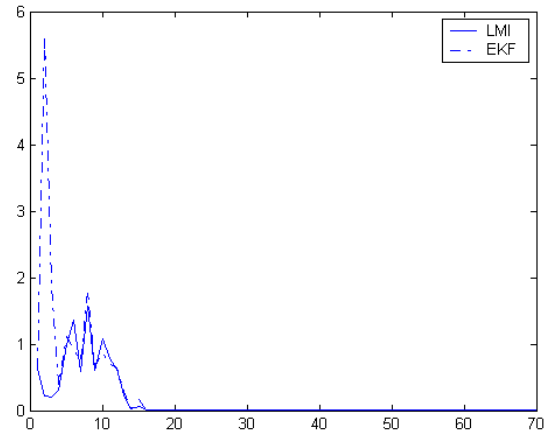

Fig. 8. Comparison of the MSE between LMI method and EKF vs. iteration time k.

In Fig. 9, the sum of $\sum_{k=0}^{N-1} z_k^2$ where z_k is performance output in (5) vs. iteration time k of both the new LMI method and EKF method are depicted. As shown in Fig.s 8 and 9, the new LMI method has smaller mean square error and performance output energy than that of EKF method.

4. CONCLUSION

A new stochastically resilient state observer which satisfies mixed H_2 - dissipativity criteria has been presented for smooth discrete-time nonlinear systems. The technique uses the solution of an LMI at each stage to compute the time-varying observer gain. The observer was tested with various types of first order nonlinear dynamic systems. Initial comparison studies between this new LMI based technique and the existing EKF based state estimation are presented. Simulation examples show superior performance of the new LMI based technique.

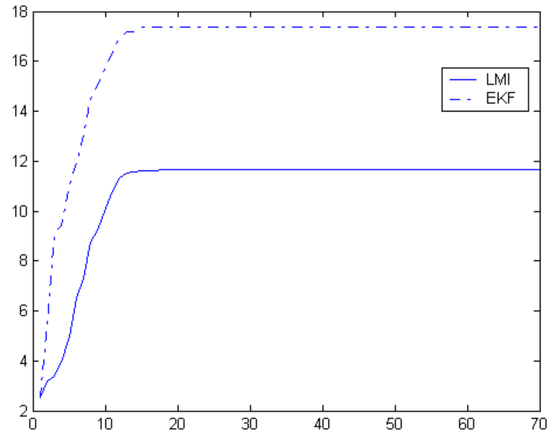

Fig. 9. Comparison of the $\sum_{k=0}^{N-1} z_k^{\ 2}$ between LMI method and EKF method vs. iteration time k.

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