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Stability and $\mathcal{L}_2$-Norm Bound Conditions for Takagi-Sugeno Descriptor Systems

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Abstract: In this paper, the stability of Takagi-Sugeno (TS) descriptor systems is studied. In most of previous works concerning TS descriptor systems, the authors claimed that the study of polytopic matrix pencil $(\sum_i \lambda_i(t)E_i, \sum_i \lambda_i(t)A_i)$ reduces to the study of an augmented polytopic matrix pencil $(E^*, \sum_i \lambda_i(t)A_i^*)$ with a common matrix E^*. The approach they have used is based on a state augmentation. In this paper, it is proved that this transformation introduces impulsive terms, because time derivate of the state variables are added in the state vector. The major contribution of this paper is to avoid this state augmentation. A new sufficient stability condition is established. Stability with guaranteed decay rate and $\mathcal{L}_2$-norm bound are also studied. All results are given in the linear matrix inequality formalism.

Keywords: Singular systems; stability analysis.

1. INTRODUCTION

In the last two decades, the Takagi-Sugeno systems (TS), proposed by Takagi and Sugeno (1985), has received a considerable amount of attention, due to its ability to describe nonlinear systems. In Wang et al. (1996), stability analysis is addressed and controller design is derived in the LMI formalism. Relaxed sufficient conditions for fuzzy controllers and observers are proposed in Tanaka et al. (1998), taking benefit from properties of the activating functions. A multiple Lyapunov function is defined in Tanaka et al. (2003) to study the stability of TS systems. The multiple Lyapunov function approach is appealing in order to relax the conservativeness of stability and stabilization problems.

As pointed in Dai (1989b), the descriptor formalism is very attractive for system modeling , consequently much attention has been paid to descriptor systems. Controllability and observability of descriptor systems were studied in Cobb (1984), and H_2 and H_∞-controllers are designed in Ikeda et al. (2000) and Uezato and Ikeda (1999) respectively.

The TS model has been generalized to descriptor systems in Taniguchi et al. (1999) and Taniguchi et al. (2000). The stability and the design of state-feedback controllers for TS descriptors systems (TSDS) are characterized via LMI in Taniguchi et al. (1999). The particular problem of nonlinear model following is treated in Taniguchi et al. (2000). The study of TSDS is envisaged with interval methods in Wang et al. (2001). In Marx and Ragot (2006), stability, controller and observer design are envisaged under pole clustering constraint, and LMI-based solutions are given.

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All the cited works use a particular state augmentation to reduce the study of the general case of polytopic matrix pencils $(\sum_i \lambda_i(t)E_i, \sum_i \lambda_i(t)A_i)$ to the special case of an augmented polytopic matrix pencil $(E^*, \sum_i \lambda_i(t)A_i^*)$ with a common E^* matrix. This state augmentation will be shown to be highly detrimental since it may artificially introduce impulsive terms in the state response, and causes poor controllability and observability.

In this paper we consider the class of TS descriptor systems, and no rank assumption is made on the E_i matrices: the E_i matrices can be of different ranks, and rank deficient. Stability and $\mathcal{L}_2$-norm bound are investigated. All the results are given in LMI formalism, since this approach is numerically efficient (Gahinet et al. (1995)). The paper is organized as follows. In section 2, the TSDS is presented and the motivations of this paper are discussed. The section 3 is dedicated to the establishment of the stability conditions. Emphasis is put on the relaxation of the LMI condition, and an α-stability condition is given in order to ensure a prescribed decay rate. In section 4 the $\mathcal{L}_2$-norm bound condition is stated. Before concluding, a numerical example is given.

Notation 1. For any matrix M, M^T is the transpose of M, $M < 0$ (resp. $M > 0$) stands for M is negative (resp. positive) definite and $\mathbb{S}(M)$ is defined by $\mathbb{S}(M) = M + M^T$.

2. PROBLEM FORMULATION

In this paper, we consider the TSDS, which is an extension of the Takagi-Sugeno models, defined in Tanaka et al. (1998), to the descriptor case. The TS system is described by fuzzy *if-then* rules, which represent linear input-output descriptor relations. The class of TSDS can be defined by

$$\begin{aligned} &Rule\ i\ :\ If\ z_1(t)\ is\ M_{i1}\ and\ \ldots\ \ and\ z_l(t)\ is\ M_{il} \\ &Then \begin{cases} E_i\dot{x}(t) = A_ix(t) + B_iu(t) \\ \ \ y(t) = C_ix(t) \end{cases} \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^p$ is the input vector, and $y(t) \in \mathbf{R}^m$ are the measurements. The premise variable, $z(t) \in \mathbf{R}^l$ is defined by $z(t) = [z_1(t)\ \ldots\ z_l(t)]^T$. The matrices E_i, A_i, B_i, and C_i, for $i = 1, \ldots, N$, are known constant real matrices of appropriate dimensions. The matrices E_i may be rank deficient, their rank are denoted $rankE_i = r_i \leq n$. M_{ij} is a fuzzy set and N is the number of $if - then$ rules. The overall model, defined by Taniguchi et al. (2000), is

$$\sum_{i=1}^{N} h_i(z(t))E_i\dot{x}(t) = \sum_{i=1}^{N} h_i(z(t))\left(A_ix(t) + B_iu(t)\right) \tag{2}$$

$$y(t) = \sum_{i=1}^{N} h_i(z(t))C_ix(t) \tag{3}$$

where the normalized activating functions $h_i(z(t))$ are required to be $\mathcal{C}^1$ functions and to verify the following constraints

$$\sum_{i=1}^{N} h_i(z(t)) = 1 \ \text{ and } \ h_i(z(t)) \geq 0,\ i = 1, \ldots, N, \forall t \tag{4}$$

Most of the works addressing the class of T-S fuzzy descriptor systems (e.g. Taniguchi et al. (1999), Taniguchi et al. (2000), Wang et al. (2001)), use the following state augmentation from (E_i, A_i, B_i, C_i) to $(E^*, A_i^*, B_i^*, C_i^*)$, rewriting the system (2-3) into the pretended equivalent system :

$$E^*\dot{x}^*(t) = \sum_{i=1}^{N} h_i(z(t))\left(A_i^*x^*(t) + B_i^*u(t)\right) \tag{5}$$

$$y(t) = \sum_{i=1}^{N} h_i(z(t))C_i^*x^*(t) \tag{6}$$

where x^*, E^*, A_i^*, B_i^*, C_i^* are defined by :

$$x^*(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}\ E^* = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}\ A_i^* = \begin{bmatrix} 0 & I_n \\ A_i & -E_i \end{bmatrix}\ B_i^* = \begin{bmatrix} 0 \\ B_i \end{bmatrix}$$
$$C_i^* = [C_i\ 0] \tag{7}$$

Unfortunately, this augmentation causes major drawbacks. Even for a non impulsive trajectory of the system $x(t)$, the time derivative of $x(t)$, introduced in the augmented state vector $x^*(t)$, may be impulsive (discontinue). In fact, unless the matrices E_i are full rank (in other words, the pencil-matrices (E_i, A_i) are not differential-algebraic, but usual dynamic systems), the pencil-matrices (E^*, A_i^*) are necessary impulsive. Moreover the systems (E^*, A_i^*, C_i^*) are not impulse observable, and the condition for impulse controllability of (E^*, A_i^*, B_i^*) is more restrictive than the one concerning the original systems (E_i, A_i, B_i).

Proposition 1. The following statement are equivalent, for $i = 1, \ldots, N$:

(i) $rankE_i = n$
(ii) the system (E^*, A_i^*) is impulse free
(iii) the system (E^*, A_i^*, C_i^*) is impulse observable

The following statements are equivalent, for $i = 1, \ldots, N$:

(iv) the system (E^*, A_i^*, B_i^*) is impulse controllable
(v) $rank\,[E_i\ B_i] = n$

Proof: $(i) \Leftrightarrow (ii)$. Let recall that (E^*, A_i^*) $i = 1, \ldots, N$ is impulse free if and only if (see Dai (1989a))

$$\begin{aligned} 2n &= rank \begin{bmatrix} E^* & A_i^* \\ 0 & E^* \end{bmatrix} - rankE^* \\ &= rank \begin{bmatrix} I_n & 0 & 0 & I_n \\ 0 & 0 & A_i & -E_i \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - n \end{aligned} \tag{8}$$

which is obviously equivalent to $rankE_i = n$.
$(i) \Leftrightarrow (iii)$. Let us recall that (E^*, A_i^*, C_i^*) $i = 1, \ldots, N$ is impulse observable if and only if (see Dai (1989b))

$$2n + rankE^* = rank \begin{bmatrix} A_i^{*T} & E^{*T} & C_i^{*T} \\ E^{*T} & 0 & 0 \end{bmatrix} = 2n + rankE_i^T \tag{9}$$

which is obviously equivalent to $rankE_i = n$.
$(iv) \Leftrightarrow (v)$. Let us recall that (E^*, A_i^*, B_i^*) $i = 1, \ldots, N$ is impulse controllable if and only if (see Dai (1989b))

$$2n + rankE^* = rank \begin{bmatrix} A_i^* & E^* & B_i^* \\ E^* & 0 & 0 \end{bmatrix} = 2n + rank\,[E_i\ B_i] \tag{10}$$

□

Impulsive terms in the time response of a descriptor system may be highly detrimental for its operation. The impulse controllability (resp. the impulse observability) is the ability to cancel (resp. reconstruct) these undesirable impulsive terms. As a result of the above proposition, if rank $E_i \neq n$ the design methods based on the state augmentation introduce impulsive terms which cannot be observed. Moreover the impulse controllability of the original systems (E_i, A_i, B_i) do not imply the impulse controllability of the systems (E^*, A_i^*, B_i^*), since the condition (v) is more restrictive than

$$rank \begin{bmatrix} A_i & E_i & B_i \\ E_i & 0 & 0 \end{bmatrix} = n + rankE_i \tag{11}$$

The previous considerations highlight that most of the results concerning TS fuzzy descriptor systems are not efficient for algebraic-differential systems but mainly dedicated to descriptor systems (i.e. $E_i \neq I_n$, with $rankE_i = n$), which is very restrictive. In such a case all the different systems must be of the same order, despite one of the main interest in TS fuzzy singular systems is the ability to model systems with different orders behavior.

The aim of the paper is to propose a method which can be applied to T-S singular systems, even if the matrices E_i are not of full rank. The assumption made throughout this paper is the following.

Assumption 1: There exist real positive scalars ν_i verifying

$$\left|\dot{h}_i(z(t))\right| \leq \nu_i,\ for\ i = 1, \ldots, N. \tag{12}$$

The determination of the ν_i is discussed in Tanaka et al. (2003) and Jadbabaie (1999).

In the remaining of the paper some technical results are often used, it is recalled in the following lemma.

Lemma 1. For any X and Y of appropriate dimension, and for any real positive definite matrix M the following inequality holds

$$X^TY + Y^TX \leq X^TMX + Y^TM^{-1}Y \quad (13)$$

The free positive definite matrix M is used to limit the conservatism if the matrices X and Y are numerically far (*e.g.* M may be set to the ratio of the condition number of Y to the one of X).

Proof: The inequality (13) is equivalent to

$$(X^TM^{1/2} - Y^TM^{-1/2})(M^{1/2}X - M^{-1/2}Y) \geq 0 \quad (14)$$

which is obvious for any positive definite matrix M. □

Lemma 2. The condition (15) is satisfied if there exist symmetric matrices T_{ij} satisfying (16) and (17).

$$\sum_{i=1}^{N}\sum_{j=1}^{N} h_i(z(t))h_j(z(t))x^T(t)G_{ij}x(t) < 0 \quad (15)$$

$$\frac{1}{2}(G_{ij} + G_{ji}) < T_{ij},\ for\ 1 \leq i \leq j \leq N \quad (16)$$

$$\begin{bmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{12} & T_{22} & \dots & T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1N} & T_{2N} & \dots & T_{NN} \end{bmatrix} \leq 0 \quad (17)$$

Proof: See Tanaka and Wang (2001) (chap. 12). □

A classical method to satisfy condition (15) is to impose $G_{ij} < 0$. The relaxation introduced by lemma 2 is obvious since the solution $T_{ij} = 0$ leads to impose $G_{ij} < 0$.

Notation 2. In the remaining of the paper, the activating functions $h_i(z(t))$ are shortened to $h_i(t)$, and $h_{ij}(t)$ denotes $h_{ij}(t) = h_i(t)h_j(t)$.

3. STABILITY ANALYSIS OF TSDS

In this section, a stability condition is proposed, in terms of LMI. The major benefit of the proposed method is that different E_i matrices are handled simultaneously, then the previously discussed state augmentation can be avoided, and consequently no impulsive term is artificially introduced. We first establish a basic result obtained by differentiating a quadratic Lyapunov function, then a less restrictive LMI condition will be stated, based on properties of the activating functions.

3.1 LMI stability condition

Consider the input-free T-S descriptor system defined by

$$\sum_{i=1}^{N} h_i(t)E_i\dot{x}(t) = \sum_{i=1}^{N} h_i(t)A_ix(t) \quad (18)$$

The following theorem gives a sufficient strict LMI condition for the stability of a TS fuzzy singular system.

Theorem 1. The TSDS (18) is quadratically stable if there exist positive definite matrices $P \in \mathbb{R}^{n\times n}$, $X_{ij} \in \mathbb{R}^{n\times n}$ and matrices $Q_j \in \mathbb{R}^{(n-r)\times n}$ verifying the LMI (19) for $1 \leq i \leq j \leq N$and (20) for $1 \leq i \leq j \leq N$.

$$\begin{bmatrix} \frac{\Phi_{i,j}+\Phi_{j,i}}{2} & E_1^TPE_1 & E_2^TPE_1 & \dots & E_N^TPE_N \\ E_1^TPE_1 & -2X_{11} & 0 & \dots & 0 \\ E_1^TPE_2 & 0 & -2X_{12} & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ E_N^TPE_N & 0 & 0 & 0 & -2X_{NN} \end{bmatrix} < 0 \quad (19)$$

$$E_i^TPE_j + E_j^TPE_i \geq 0 \quad (20)$$

where $\Phi_{i,j}$ is defined by (21), and where the full column rank matrix $E_0 \in \mathbb{R}^{n\times(n-r)}$ is a base of the intersection of the null spaces of E_i^T defined by (22)

$$\Phi_{i,j} = \sum_{u=1}^{N}\sum_{v=1}^{N}\frac{(\nu_u+\nu_v)^2}{2}X_{uv} + \mathbb{S}(A_i^T(PE_j+E_0Q_j)) \quad (21)$$

$$[E_1\ E_2\ \dots\ E_N]^T E_0 = 0 \quad (22)$$

Remark: If the intersection of the null spaces of E_i is empty, then $E_0 = 0$.

Proof: Consider the following Lyapunov function of quadratic form

$$V(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N} h_i(t)h_j(t)x^T(t)E_i^TPE_jx(t) \quad (23)$$

Obviously, $V(0,t) = 0$ and (20) implies that $V(x,t)$ is nonnegative for $x \neq 0$. The time derivate of $V(x,t)$, along the trajectory of the system (18), is given by the following relation.

$$\dot{V}(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\Big(\dot{h}_{ij}(t)x^T(t)E_i^TPE_jx(t) + h_{ij}(t)\left(\dot{x}^T(t)E_i^TPE_jx(t) + x^T(t)E_i^TPE_j\dot{x}(t)\right)\Big)$$

Since, for $i = 1,\dots,N$, we have $E_i^TE_0 = 0$, the previous equation is equivalent to

$$\dot{V}(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\dot{h}_{ij}(t)x^T(t)E_i^TPE_jx(t) + \sum_{i=1}^{N}\sum_{j=1}^{N}h_{ij}(t)\Big(\dot{x}^T(t)E_i^T(PE_j+E_0Q_j)x(t) + x^T(t)(PE_i+E_0Q_i)^TE_j\dot{x}(t)\Big) \quad (24)$$

According to the definition of the system (18), and permuting the subscripts i and j in the last term, the previous equality becomes

$$\dot{V}(x,t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\dot{h}_{ij}(t)x^T(t)E_i^TPE_jx(t) + \sum_{i=1}^{N}\sum_{j=1}^{N}h_{ij}(t)x^T(t)\Big(A_i^T(PE_j+E_0Q_j) + (PE_j+E_0Q_j)^TA_i\Big)x(t) \quad (25)$$

With the use of lemma 1, the following inequality holds for positive definite matrices X_{ij}

$$\dot{h}_{ij}(t)x^T(t)E_i^TPE_jx(t) \leq \frac{\left(\dot{h}_{ij}(t)\right)^2}{2}x^T(t)X_{ij}x(t) + \frac{1}{2}x^T(t)E_j^TPE_iX_{ij}^{-1}E_i^TPE_jx(t) \quad (26)$$

Substituting (26) in (25), we obtain the following inequality

$$\begin{aligned}\dot{V}(x,t) \leq & \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\left(\dot{h}_{ij}(t)\right)^2}{2}x^T(t)X_{ij}x(t) \\ & + \sum_{i=1}^{N}\sum_{j=1}^{N}h_{ij}(t)x^T(t)\mathbb{S}\left(A_i^T(PE_j+E_0Q_j)\right)x(t) \\ & + \frac{1}{2}x^T(t)E_j^TPE_iX_{ij}^{-1}E_i^TPE_jx(t)\end{aligned} \quad (27)$$

Taking the assumption 1 into account, and since $h_i(t) \leq 1$, it is possible to bound $\dot{h}_{ij}(t)$, by

$$|\dot{h}_{ij}(t)| = |\dot{h}_i(t)h_j(t) + h_i(t)\dot{h}_j(t)| \leq \nu_i + \nu_j \quad (28)$$

Substituting (28) in (27) and factorizing by $h_{ij}(t)$, we have

$$\begin{aligned}\dot{V}(x,t) \leq & \sum_{i=1}^{N}\sum_{j=1}^{N}h_{ij}(t)x^T(t)\Bigg[\sum_{u=1}^{N}\sum_{v=1}^{N}\frac{(\nu_u+\nu_v)^2}{2}X_{uv} \\ & + \frac{1}{2}E_v^TPE_uX_{uv}^{-1}E_u^TPE_v + \mathbb{S}\left(A_i^T(PE_j+E_0Q_j)\right)\Bigg]x(t)\end{aligned} \quad (29)$$

and with N^2 Schur complements the LMI (19) is obtained. Note that this LMI implies that the X_{ij} are positive definite. Due to the positiveness of $h_{ij}(t)$, if the LMI (19) are satisfied, then $\dot{V}(x,t) < 0$, and consequently the quadratic stability of the system (18) is proved. □

3.2 Relaxed stability condition

In this section a relaxed stability condition for TSDS is proposed. In order to avoid the conservatism introduced by the extensive use of lemma 1 in the previous results, an interesting property of the activating functions $h_i(t)$ is exploited. Since $\sum_{i=1}^{N} h_i(t) = 1$, we have

$$\sum_{i=1}^{N}\dot{h}_i(t) = 0 \Leftrightarrow \dot{h}_N(t) = -\sum_{i=1}^{N-1}\dot{h}_i(t) \quad (30)$$

Taking benefit of (30), the relaxed stability condition is established.

Theorem 2. The TSDS (18) is quadratically stable if there exist positive definite matrices $P \in \mathbb{R}^{n\times n}$, symmetric matrices $T_{ij} \in \mathbb{R}^{n\times n}$ and matrices $Q_j \in \mathbb{R}^{(n-r)\times n}$ verifying the LMI (31-32) for $1 \leq i \leq j \leq N$, (32), (34) and (34) for $1 \leq i \leq j \leq N-1$.

$$\frac{1}{2}\left(\Psi_{ij}+\Psi_{ji}\right) < T_{ij} \quad (31)$$

$$E_i^TPE_j + E_j^TPE_i \geq 0 \quad (32)$$

$$\begin{bmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{12} & T_{22} & \dots & T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1N} & T_{2N} & \dots & T_{NN} \end{bmatrix} \leq 0 \quad (33)$$

$$(E_i-E_N)^TP(E_j-E_N)+(E_j-E_N)^TP(E_i-E_N) \geq 0 \quad (34)$$

where the full column rank matrix $E_0 \in \mathbb{R}^{n\times(n-r)}$ is a base of the intersection of the null spaces of E_i^T defined by (22), and where Ψ_{ij} is defined by

$$\begin{aligned}\Psi_{ij} = & \sum_{u=1}^{N-1}\sum_{v=1}^{N-1}\nu_u\nu_v(E_u-E_N)^TP(E_v-E_N) \\ & + \frac{1}{2}\mathbb{S}(E_i^TPE_j) + \mathbb{S}\left(A_i^T(PE_j+E_0Q_j)\right)\end{aligned} \quad (35)$$

Proof: The Lyapunov function $V(x,t)$ is defined by (23), obviously (32) implies $V(x,t) \geq 0$. The function $V(x,t)$ is derivated along the trajectory of the system (18). Let us denote the first term of the right-hand side of (25) by

$$f(t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\dot{h}_{ij}(t)x^T(t)E_i^TPE_jx(t) \quad (36)$$

With (30), $f(x,t)$ can be developed in

$$\begin{aligned}f(t) = & x^T(t)\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\dot{h}_i(t)h_j(t)E_i^TPE_j + \sum_{i=1}^{N}\sum_{j=1}^{N}h_i(t)\dot{h}_j(t)E_i^TPE_j\right)x(t) \quad (37) \\ = & x^T(t)\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N}\dot{h}_i(t)h_j(t)(E_i-E_N)^TPE_j + \sum_{i=1}^{N}\sum_{j=1}^{N-1}h_i(t)\dot{h}_j(t)E_i^TP(E_j-E_N)\right)x(t) \quad (38)\end{aligned}$$

Permuting the subscripts i and j in the second term, and noticing that $f(x,t)$ is a scalar function, we have

$$\begin{aligned}f(t) = & 2x^T(t)\left(\sum_{i=1}^{N-1}\dot{h}_i(t)(E_i-E_N)^T\right)P\left(\sum_{j=1}^{N}h_j(t)E_jx(t)\right) \\ \leq & x^T(t)\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\dot{h}_i(t)\dot{h}_j(t)(E_i-E_N)^TP(E_j-E_N) + \sum_{i=1}^{N}\sum_{j=1}^{N}h_{ij}(t)E_i^TPE_j\right)x(t)\end{aligned} \quad (39)$$

with (4), (12), (34) and lemma 1, we have

$$f(t) \leq x^T(t)\left(\sum_{u=1}^{N-1}\sum_{v=1}^{N-1}\nu_u\nu_v(E_u-E_N)^TP(E_v-E_N) + \sum_{i=1}^{N}\sum_{j=1}^{N}h_{ij}(t)E_i^TPE_j\right)x(t) \quad (40)$$

Substituting (40) in (25), and factorizing by $h_{ij}(t)$ we have (41), where Ψ_{ij} is defined by (35).

$$\dot{V}(x,t) \leq \sum_{i=1}^{N}\sum_{j=1}^{N}h_i(t)h_j(t)x^T(t)\Psi_{ij}x(t) \quad (41)$$

With lemma 2, it is clear that (31-33) are sufficient conditions for (41). Consequently, (31-34) implies $\dot{V}(x,t) < 0$, and thus the stability of the TSDS (18) is proved. □

The stability condition given by theorem 2 is less restrictive than in theorem 1 since the number of positive definite or semi positive definite terms on the left hand side of (31) is $(N-1)^2$, whereas in (29) it is N^2. Moreover, if the different matrices E_i are defined by different operating point, then different matrices E_i may have several entries in common, and thus $(E_N - E_i)$ may be sparse matrices or at least with entries close to zero. Then the positive semi definite terms $(E_N - E_i)^T P(E_N - E_i)$ appearing in the negative definite part of the LMI are sparse and are not restrictive. A numerical example will illustrate the benefit of the relaxed stability condition.

3.3 Relaxed α-stability condition

The vocable α-stability denotes the property of stability with a prescribed decay rate α of the state variable $x(t)$. In fact, in most control applications, stability is not a satisfying objective since it is important to also consider the time of response to reach performance objectives. This criteria is related to the decay rate of $x(t)$, *i.e.* to the largest α, such that $\lim_{t\to\infty} e^{\alpha t}||x(t)|| = 0$. A sufficient condition to ensure a decay rate of $x(t)$ larger or equal to α is that there exists a Lyapunov function verifying $\dot{V}(x,t) + 2\alpha V(x,t) \leq 0$, see Boyd et al. (1994).

Theorem 3. The TSDS (18) is quadratically α-stable if there exist a positive definite matrix $P \in \mathbb{R}^{n\times n}$, symmetric matrices $T_{ij} \in \mathbb{R}^{n\times n}$ and matrices $Q_j \in \mathbb{R}^{(n-r)\times n}$ verifying the LMI (31-34), where the full column rank matrix $E_0 \in \mathbb{R}^{n\times(n-r)}$ is a base of the intersection of the null spaces of E_i^T defined by (22), and where Ψ_{ij} is defined by

$$\Psi_{ij} = \sum_{u=1}^{N-1}\sum_{v=1}^{N-1} \nu_u \nu_v (E_u - E_N)^T P(E_v - E_N) + \left(\frac{1+2\alpha}{2}\right) \mathbb{S}(E_i^T P E_j) + \mathbb{S}\left(A_i^T (PE_j + E_0 Q_j)\right)$$

<u>*Proof:*</u> The proof is derived of the proof of theorem 2, by verifying $\dot{V}(x,t) + 2\alpha V(x,t) \leq 0$, with $V(x,t)$ defined by (23). □

4. $\mathcal{L}_2$-NORM BOUND CONDITION

The $\mathcal{L}_2$-gain of a nonlinear system is a useful performance criterion to quantify disturbance attenuation, control or filtering performance (for linear time invariant system it coincides with the H_∞-norm). It is defined by the maximum of the ratio of the L_2-norm of the output signal and the L_2-norm of the input signal, where the L_2-norm of a signal $w(t)$ is defined by $||w(t)||_2^2 = \int_0^\infty w^T(t)w(t)dt$. The $\mathcal{L}_2$-gain of a system of input $u(t)$ and output $y(t)$ is said to be bounded by a positive real γ if the following condition is true

$$\int_0^\infty (y^T(t)y(t) - \gamma^2 u^T(t)u(t))dt < 0 \tag{42}$$

Considering the TSDS defined by

$$\sum_{i=1}^{N} h_i(t) E_i \dot{x}(t) = \sum_{i=1}^{N} h_i(t)(A_i x(t) + B_i u(t)) \tag{43}$$

$$y(t) = \sum_{i=1}^{N} h_i(t)(C_i x(t) + D_i u(t)) \tag{44}$$

the objective of this section is to give a condition enabling to test if the $\mathcal{L}_2$-gain of (43-44) is lower than a prescribed $\gamma \in \mathbb{R}^+$. Usually, $\mathcal{L}_2$-norm bound condition are used in disturbance rejection or filtering.

Theorem 4. The $\mathcal{L}_2$-gain of the TSDS (43-44) is bounded by γ if there exist positive definite matrices $P \in \mathbb{R}^{n\times n}$, symmetric matrices $T_{ij} \in \mathbb{R}^{(n+p)\times(n+p)}$ and matrices $Q_j \in \mathbb{R}^{(n-r)\times n}$ verifying the LMI (45), (46-47), for $1 \leq i \leq j \leq N$ and (48), for $1 \leq i \leq j \leq N-1$.

$$\begin{bmatrix} T_{11} & T_{12} & \dots & T_{1N} \\ T_{12} & T_{22} & \dots & T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1N} & T_{2N} & \dots & T_{NN} \end{bmatrix} \leq 0 \tag{45}$$

$$\begin{bmatrix} \frac{\Psi_{ij}+\Psi_{ji}+\mathbb{S}(C_i^T C_j)}{2} & \frac{C_i^T D_j + C_j^T D_i}{2} \\ \frac{D_i^T C_j + D_j^T C_i}{2} & \frac{\mathbb{S}(D_i^T D_j)}{2} - \gamma^2 I_p \end{bmatrix} < T_{ij} \tag{46}$$

$$\mathbb{S}(E_i^T P E_j) \geq 0 \tag{47}$$

$$\mathbb{S}\left((E_i - E_N)^T P(E_j - E_N)\right) \geq 0 \tag{48}$$

where the full column rank matrix $E_0 \in \mathbb{R}^{n\times(n-r)}$ is a base of the intersection of the null spaces of E_i^T defined by (22), and where Ψ_{ij} is defined by (35).

<u>*Proof:*</u> As mentioned in Boyd et al. (1994), the $\mathcal{L}_2$-gain of a system is bounded by γ if there exist a Lyapunov function $V(x,t)$ verifying the following constraint

$$\frac{dV(x,t)}{dt} + y^T(t)y(t) - \gamma^2 u^T(t)u(t) < 0 \tag{49}$$

Let consider the function $V(x,t)$ defined by (23). Obviously, from (44) we have

$$y^T(t)y(t) - \gamma^2 u^T(t)u(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} h_{ij}(t) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} C_i^T C_j & C_i^T D_j \\ D_i^T C_j & D_i^T D_j - \gamma^2 I_p \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{50}$$

With (41), we have

$$\dot{V}(x,t) + y^T(t)y(t) - \gamma^2 u^T(t)u(t) \leq \sum_{i=1}^{N}\sum_{j=1}^{N} h_{ij}(t) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} \Psi_{ij} + C_i^T C_j & C_i^T D_j \\ D_i^T C_j & D_i^T D_j - \gamma^2 I_p \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \tag{51}$$

With lemma 2 and similarly to the proof of theorem 2, (45-48) are sufficient conditions to ensure $\dot{V}(x,t) + y^T(t)y(t) - \gamma^2 u^T(t)u(t) < 0$, and then to ensure that the $\mathcal{L}_2$-norm of the system (43-44) is bounded by γ. □

5. NUMERICAL EXAMPLE

A simple numerical example is computed with the Matlab LMI toolbox to compare the results obtained with theorem 1 and 2. Consider the system (18) defined by

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; A_1 = \begin{pmatrix} -6 & -a \\ 16 & -9 \end{pmatrix} \; E_2 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \; A_2 = \begin{pmatrix} -5 & -2 \\ b & -7 \end{pmatrix}$$

Figure 1 displays the comparison of the feasible areas of theorem 1 and 2. For $\nu_1 = \nu_2 = 0.35$, the stability conditions of theorem 1 are satisfied for $a \geq 0$ and $14 \geq b \geq -10$, whereas the stability conditions of theorem 2 are satisfied for $25 \geq a \geq -25$ and $35 \geq b \geq -25$. For $\nu_1 = \nu_2 = 1$, the stability conditions of theorem 1 are not satisfied for any values of the pair (a,b) in

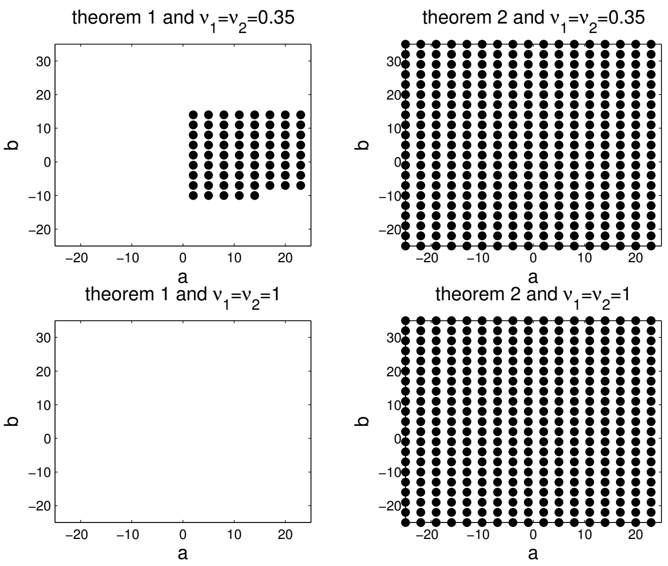

Fig. 1. Comparison of the feasible areas of theorem 1 (left column) and theorem 2 (right column), for $\nu_1 = \nu_2 = 0.35$ (first row) and $\nu_1 = \nu_2 = 1$ (second row).

$(\{-25, 25\}, \{-25, 35\})$, whereas the stability conditions of theorem 2 are satisfied for all values of (a, b) in these intervals.

The state augmentation made in Taniguchi et al. (1999, 2000) and Wang et al. (2001), would have lead to the system defined by $E^* = diag(1, 1, 0, 0)$ and

$$A_1^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & -a & -1 & 0 \\ 16 & -9 & 0 & 0 \end{pmatrix} \quad A_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -2 & -2 & -1 \\ b & -7 & 0 & 0 \end{pmatrix}$$

Obviously, (E^*, A_1^*) and (E^*, A_2^*) are not impulse free, thus the basic requirement of admissibility of the local systems is not verified. In other words, the analysis based on this state augmentation would lead to conclude that the system is not admissible, whereas we have shown it is. This simple example illustrates that the systems (E_1, E_2, A_1, A_2) and (E^*, A_1^*, A_2^*) are not equivalent, and shows the contribution of the proposed approach which allows to avoid the state augmentation.

6. CONCLUSION

In this paper a solution to characterize the stability of Takagi-Sugeno descriptor systems (TSDS) is presented. The major contribution of the proposed approach is that it was shown that the existing results in Takagi-Sugeno descriptor systems are based on a state augmentation which is very conservative, whereas the results obtained in this note do not need any state augmentation. Stability, α-stability and $\mathcal{L}_2$-norm bound sufficient conditions are derived from a quadratic Lyapunov function. In this note, only stability constraints or prescribed decay rate constraints where envisaged, but future works may explore the stability submitted to pole clustering constraints. This lead to the concept of $\mathcal{D}$ admissibility of descriptor systems Marx et al. (2003), which is also of interest. The controller and observer designs derived from the stability condition are also under study.

REFERENCES

S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in Systems and Control Theory.* SIAM, 1994.

D. Cobb. Controllability, observability, and duality in singular systems. *IEEE Trans. on Automatic Control*, 29:1076–1082, 1984.

L. Dai. Impulsive modes and causality in singular systems. *International Journal of Control*, 50 (4):1267–1281, 1989a.

L. Dai. *Singular Control Systems.* Springer, 1989b.

P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali. LMI *Control Toolbox.* The Mathworks Inc, 1995.

M. Ikeda, T.K. Lee, and E. Uezato. A strict LMI condition for H_2 control of descriptor systems. In *Proc. of the 39th IEEE Conf. on Decision and Control*, pages 601–604, 2000.

A. Jadbabaie. A reduction in conservatism in stability and $\mathcal{L}_2$ gain analysis of Takagi-Sugeno fuzzy systems via linear matrix inequalities. In *Proc of the 14th IFAC Triennal World Congress, Beijing, China*, pages 285–289, 1999.

B. Marx, D. Koenig, and D. Georges. Robust pole-clustering in LMI regions, a characterization via strict LMI. In *Proc. of the European Control Conf., paper No.189*, 2003.

B. Marx and J. Ragot. Controller and observer designs for a class of ts descriptor systems with pole placement constraint. In *Proc. of the IEEE Conf. on Decision and Control*, pages 6355–6360, 2006.

T Takagi and M Sugeno. Fuzzy identification of systems and its application to modeling and control. *IEEE Trans. on Systems, Man and Cybernetics*, 15:116–132, 1985.

K. Tanaka, T. Hori, and H.O. Wang. A multiple lyapunov function approach to stabilization of fuzzy control systems. *IEEE Trans. on Fuzzy Systems*, 11:582–589, 2003.

K Tanaka, T. Ikeda, and H.O. Wang. Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs. *IEEE Trans. on Fuzzy Systems*, 6: 250–265, 1998.

K. Tanaka and H.O. Wang. *Fuzzy Control Systems Design and Analysis, a Linear Matrix Inequality Approach.* Wiley-Interscience, 2001.

T. Taniguchi, K. Tanaka, and H.O. Wang. Fuzzy descriptor systems and nonlinear model following control. *IEEE Trans. on Fuzzy Systems*, 8:442–452, 2000.

T. Taniguchi, K. Tanaka, K.i Yamafuj, and H.O. Wang. Fuzzy descriptor systems: Stability analysis and design via LMIs. In *Proc. of the IEEE American Control Conf.*, pages 1827–1831, 1999.

E. Uezato and M. Ikeda. Strict LMI condition for stability, robust stabilization, and H_∞-control of descriptor systems. In *Proc. of the IEEE Conf. on Decision and Contol*, pages 4092–4097, 1999.

H.O. Wang, K. Tanaka, and M.F. Griffin. An approach to fuzzy control of nonlinear systems : Stability and design issues. *IEEE Trans. on Fuzzy Systems*, 4:14–23, 1996.

Y. Wang, Q.L. Zhang, and W.Q. Liu. Stability analysis and design for T-S fuzzy descriptor systems. In *Proc. of the IEEE Conf. on Decision and Control*, pages 3962–3967, 2001.