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Dynamic Linear System Synthesis with Account of Phase and Control Restrictions via Sigma-function Feedback

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Abstract: In the paper linear dynamic systems problem stabilization with account of state variables and controls restrictions synthesis methods are suggested. The suggested approach is based on the block synthesis method and the use of sigma-functions in the feedback chain, which are approximations of discontinuous functions. Besides the procedure decomposition into simple subproblems of lower dimension, the fact that state vector coordinates and a selection in the form of sigma functions are used as fictitious control (restricted in amplitude) provided an opportunity to solve the stabilization problem with account of state variables and controls restrictions.

1. INTRODUCTION

In the paper a sufficiently simple solution of the stabilization problem of the dynamic systems with account of phase variables and controls restrictions with the use of σ-functions in the feedback chain is suggested.

As known (Utkin, 1992), under the synthesis of systems with discontinuous controls functioning in the sliding mode, the limitations on the value of the discontinuous functions amplitudes are accounted for at the synthesis stage. Under the use of various approximations of the discontinuous controls by the continuous functions, including the σ- functions, the physical limitations on the controls amplitude at the synthesis stage are also accounted for. The use in the paper of the block representation (Drakunov *et al.*, 1990) allows to expose the system structure with the use of controllability index only. With this, the further synthesis of independently solvable stabilization subproblems in each of the basic blocks with the use of sigmoid feedback (considered as fictitious controls) allows the accounting of with account of phase variables and true controls restrictions (Utkin, 2002).

The paper is organized as follows. In section 2, the main idea of the block control principle is provided. In section 3, the algorithm to solve the stabilization problem with account of phase variables and controls restrictions based on linear local feedback is developed. In section 4, the block control principle is extrapolated onto nonlinear (sigmoid) local feedback. In section 5, the one-parameter procedure of tuning of the sigmoid function parameters in the solution of the stabilization problem with account of state vector variables and controls restrictions. In section 6, an illustrative example is considered.

2. BACKGROUND. THE BLOCK CONTROL

Let us consider the problem of stabilizing a control plant whose mathematical model is described by a linear stationary differential equations system in the form

$$\dot{x} = Ax + Bu \ , \tag{1}$$

where $x \in R^n$, $u \in R^p$ are the state and controls vectors, correspondingly, A and B are constant matrices of coordinated dimensions, the pair (A, B) is controllable.

A known block method result of the stabilization problem consists of transforming the original controllable system (1) into a block control form (Utkin, 2001)

$$\dot{x}_i = A_i x_i^* + B_i x_{i-1}, \ i = \overline{r,1}; \ \ \dot{x}_0 = A_0 x_0^* + B_0 u, \tag{2}$$

where $x_i^* = \mathrm{col}(x_r, ..., x_i)$, $\dim x_i = \mathrm{rank} B_i = p_i$, $\dim x_{i-1} \le$ $\le \dim x_i$, $\sum_{i=0}^{r} p_i = n$, p_i are indexes of controllability, r is a indication of controllability. Here and below $i = \overline{r,1}$ stands for $i = r, r-1, ..., 1$.

Thus, the initial system is decomposed into $r+1$ consequently coupled subsystems representing basic blocks (the dimensions of the state vector and fictitious controls coincide), and, accordingly, the control synthesis is selected as a sequential solution of the eigenvalue assignment problem in each of these blocks.

The stabilization system (2) problem synthesis procedure consists of a feedback matrix choice F_i $(i = \overline{r,0})$ with dimensions $p_i \times p_i$.

At the first step of the procedure, the r block fictitious control formation in the form of

$$x_{r-1} = B_r^+(-A_r x_r + F_r x_r) \tag{3}$$

leads to the motion equations with given eigenvalue $\dot{\bar{x}}_r = F_r \bar{x}_r$, $\bar{x}_r = x_r$. The equations, relative to the variables

$$\bar{x}_{r-1} = x_{r-1} - B_r^+(-A_r x_r + F_r \bar{x}_r) \ , \tag{4}$$

having the sense of fictitious controls mismatches x_{r-1} from the desired values (3) take the form

$$\dot{\bar{x}}_{r-1} = A^*_{r-1}\bar{x}^*_{r-1} + B_{r-1}\bar{x}_{r-2},$$

where $\bar{x}^*_{r-1} = \mathrm{col}(\bar{x}_r, \bar{x}_{r-1})$, matrix A^*_{r-1} depends on r and $(r-1)$ blocks and given matrix F_r.

Because, by design, the left pseudoinverse matrices B_i^+ for matrices B_i, $(i=\overline{r,0})$ exists, there exists such a sequence of transformations similar to (4)

$$\bar{x}_r = x_r,\ \bar{x}_{i-1} = x_{i-1} - B_i^+(-A_i^*\bar{x}_i^* + F_i\bar{x}_i),\ i=\overline{r,1}, \tag{5}$$

where $\bar{x}_i^* = \mathrm{col}(\bar{x}_r, \bar{x}_{r-1}, ..., \bar{x}_i)$ and, accordingly, controls of the form

$$u = B_0^+(-A_0^*\bar{x}_0^* + F_0\bar{x}_0) \tag{6}$$

such that closed system motion (2) is described by equations of the form

$$\dot{\bar{x}}_i = F_i\bar{x}_i + B_i\bar{x}_{i-1},\ i=\overline{r,1};\ \dot{\bar{x}}_0 = F_0\bar{x}_0. \tag{7}$$

The eigenvalues of the system (7) coincide to the eigenvalues of the matrices $F_i, i=\overline{r,0}$, which are chosen arbitrarily. The sequential transformations (5) and the choice of controls (6) determine the decomposition of the initial n-parameter problem into independently solved elementary subproblems, in each of which the problem of eigenvalue assignment in the blocks of dimension $p_i < p$ is solved.

3. THE BLOCK CONTROL PRICIPLE WITH ACCOUNT OF STATE VECTOR AND CONTROLS RESTRICTIONS

The fact that the procedure from section 2 decomposes the synthesis problem into independently solvable elementary subproblems of lower dimension also allows for a step-by-step solution of the stabilization problem with account of state variables and controls restrictions.

Under the assumption that the state vector components and the controls are bounded by known values in the from $\|x_i\| \le X_i = \mathrm{const}$, $\|u\| \le U = \mathrm{const}$, $i=\overline{r,0}$, we will now briefly state the procedure of system (2) stabilization problem synthesis.

Step 1. In the procedure's first step fictitious controls x_{r-1} of the r-th block (2) must satisfy the condition

$$\|x_{r-1}\| = \|B_r^+(-A_r x_r + F_r x_r)\| \le X_{r-1},$$

which can be evaluated by the inequality

$$\|B_r^+\|(\|A_r\| + \|F_r\|)X_r \le X_{r-1}, \tag{8}$$

where $\|(.)\|$ are rows matrix norms. Inequality (8) may be assured to hold true by a choice of feedback matrix F_r from a class of Hurwitz matrices.

If inequality (8) cannot be assured to hold true under any F_r then system (2) cannot be stabilized with account of restrictions, and the procedure ends. If inequality (8) holds true then we transition to the next step.

Step 2. As the second step, considering the imposed limitations $\|x_{r-2}\| \le X_{r-2}$, let us verify the truthfulness of inequality

$$\|B_{r-1}^+(-A^*_{r-1}x_{r-1} + F_{r-1}x_{r-1})\| \le X_{r-2},$$

where matrix A^*_{r-1} depends on the matrices of the r-th and $(r-1)$-th blocks and matrix F_r given in the first step.

Let us use the evaluation similar to the one in step one in the form

$$\|B_{r-1}^+\|(\|A^*_{r-1}\| + \|F_{r-1}\|)X_{r-1} \le X_{r-2}. \tag{9}$$

To assure inequality (9), there exist two possibilities: decreasing norm $\|F_{r-1}\|$ and norm $\|A^*_{r-1}(F_r)\|$ by correcting matrix F_r chosen in step one.

Final step. Continuing the thought, the final step must assure that equation $u = B_0^+(-A_0^*x_0^* + F_0x_0)$ holds true. Considering $\|u\| \le U = \mathrm{const}$, we get evaluation $\|B_0^+\|(\|A_0^*\| + \|F_0\|)X_0 \le U$ where matrix A_0^* depends on matrix A_i and chosen matrices F_i of all the preceding blocks.

Thus, the choice of matrices F_i, $i=\overline{r,0}$ can assure the stabilization of system (2) with account of restrictions.

Note: matrix F_i is assigned at the i-th step. If, with this choice, the system converges to zero, then matrices F_i, $i=\overline{r,i+1}$ remain unchanged. In case convergence is not assured at the *i*-th step, matrix F_i is corrected first, and if this does not suffice, matrix F_{i+1} is corrected, etc.

4. STABILIZATION PROBLEM SYNTHESIS BASED ON SIGMOIDAL FUNCTIONS

The idea of the present paper consists of replacing the linear feedback in procedures presented in sections 2 and 3 with sigmoid functions of the form

$$\sigma(x) = X\left[\frac{2}{1+\exp(-x/\tau)} - 1\right],\ X = \mathrm{const}.$$

The rich content of a sigmoid function consists of the given function being approximated by a linear function with $\tau \to \infty$, and being approximated by a signum function $\sigma(x) \to X\text{sign}(x)$ with $\tau \to 0$. At the same time, this function is convenient from the mathematical point of view in that it is continuously differentiated with $\tau \neq 0$.
Let us repeat the step-by-step problem solving procedure in the preceding section with the use of sigmoid feedback without state variables restrictions.

Procedure 1

Step 1. In the r-th equation of system (2) $\dot{x}_r = A_r x_r + B_r x_{r-1}$ let us chose fictitious controls in the form of a σ-function

$$x_{r-1} = B_r^+[-A_r \bar{x}_r - \sigma(\bar{x}_r)],\ \bar{x}_r = x_r, \tag{10}$$

wherein and further on $\sigma(\bar{x}_i) = \text{col}(\sigma(\bar{x}_{i1}), ..., \sigma(\bar{x}_{ip_i})) \in R^{p_i}$.
Then the equations of the first block after feedback closure will take the from

$$\dot{\bar{x}}_r = -\sigma(\bar{x}_r) + B_r \bar{x}_{r-1}. \tag{11}$$

Step 2. The transformations of this step consist of assuring equation (10). Let us write the differential equation relative to variable (10) $\dot{\bar{x}}_{r-1} = A_{r-1}^* x_{r-1}^* + B_r^+ \dot{\sigma}(\bar{x}_r) + B_{r-1} x_{r-2}$, where $x_{r-1}^* = \text{col}(x_r, x_{r-1})$, matrix A_{r-1}^* satisfies equation

$$A_{r-1}^* x_{r-1}^* = A_{r-1} x_{r-1}^* + B_r^+ A_r [A_r x_r^* + B_r x_{r-1}].$$

Let us form a fictitious controls in the from

$$x_{r-2} = B_{r-1}^+[-A^*{}_{r-1} x_{r-1}^* - B_r^+ \dot{\sigma}(\bar{x}_r) - \sigma(\bar{x}_{r-1})]$$

and introduce a new variables

$$\bar{x}_{r-2} = x_{r-2} - B_{r-1}^+[-A^*{}_{r-1} x_{r-1}^* - B_r^+ \dot{\sigma}(\bar{x}_r) - \sigma(\bar{x}_{r-1})]. \tag{12}$$

Then, after feedback closure the second block is described by a definitely stable system of the form

$$\dot{\bar{x}}_{r-1} = -\sigma(\bar{x}_{r-1}) + B_{r-1} \bar{x}_{r-2},$$

which assures the truthfulness of equation (10).
Step 3. The transformations of this step consist of assuring the truthfulness of equation (12). Let us rewrite the equation with respect to variable (12)

$$\dot{\bar{x}}_{r-2} = A_{r-2}^* x_{r-2}^* + B_{r-2} x_{r-3} + B_{r-1}^+[B_r^+ \ddot{\sigma}(\bar{x}_r) + \dot{\sigma}(\bar{x}_{r-1})],$$

where $x_{r-2}^* = \text{col}(x_r, x_{r-1}, x_{r-2})$,

$$A_{r-2}^* x_{r-2}^* = A_{r-2} x_{r-2}^* + B_{r-1}^+ \overline{A}_{r-1} \dot{x}_{r-1}^*.$$

Let us chose a fictitious controls in the form

$$x_{r-3} = B_{r-2}^+ \times$$
$$\times[-A^*{}_{r-2} x_{r-2}^* - B_{r-1}^+(B_r^+ \ddot{\sigma}(\bar{x}_r) + \dot{\sigma}(\bar{x}_{r-1})) - \sigma(\bar{x}_{r-2})]$$

and introduce a new variables

$$\bar{x}_{r-3} = x_{r-3} - B_{r-2}^+[-A_{r-2}^* x_{r-2}^* - B_{r-1}^+(B_r^+ \ddot{\sigma}(\bar{x}_r) + \dot{\sigma}(\bar{x}_{r-1})) - \sigma(\bar{x}_{r-2})],$$

which assures the stability of the third block

$$\dot{\bar{x}}_{r-2} = -\sigma(\bar{x}_{r-2}) + B_{r-2} \bar{x}_{r-3}.$$

The three presented steps of the procedure make obvious the following structure of system (2) state vector transformation:

$$\begin{aligned} &\bar{x}_r = x_r,\ \bar{x}_{r-1} = x_{r-1} + B_r^+(A_r x_r + \sigma(\bar{x}_r)), \\ &\bar{x}_{r-2} = x_{r-2} + B_{r-1}^+(A_{r-1}^* x_{r-1}^* + B_r^+ \dot{\sigma}(\bar{x}_r) + \sigma(\bar{x}_{r-1})), \ldots, \\ &\bar{x}_0 = x_0 + B_1^+[A_1^* x_1^* + B_2^+[B_3^+ ...[B_{r-2}^+[B_{r-1}^+[B_r^+ \sigma^{(r-1)}(\bar{x}_r) + \\ &+ \sigma^{(r-2)}(\bar{x}_{r-1})] + \sigma^{(r-3)}(\bar{x}_{r-2})] + ...] + \dot{\sigma}(\bar{x}_2)] + \sigma(\bar{x}_1)] + B_0 u. \end{aligned} \tag{13}$$

As can be seen, the substitution of variables (13) is nonsingular due to the lower triangular form of the transformation matrix. Let us note that the reversed transformation is nonlinear, but it is not necessary to realize it in the system (2) stabilization problem because the stabilization problem solution in the new coordinates automatically solves the stabilization problem in the original coordinates.
The differential equations rewritten with respect to new variables have the following structure

$$\begin{aligned} &\dot{\bar{x}}_{r-\mu} = A_{r-\mu} x_{r-\mu}^* + B_{r-\mu} x_{r-\mu-1} + B_{r-\mu-1}^+[B_{r-\mu-2}^+[...[B_{r-1}^+[B_r^+ \times \\ &\times \sigma^{(\mu)}(\bar{x}_r) + \sigma^{(\mu-1)}(\bar{x}_{r-1})] + \sigma^{(\mu-2)}(\bar{x}_{r-2})] + ...] + \\ &+ \ddot{\sigma}(\bar{x}_{\mu-1})] + B_{r-\mu-1}^+ \dot{\sigma}(\bar{x}_{r-\mu-1}),\ \mu = \overline{0, r-1}, \\ &\dot{\bar{x}}_0 = A_0 x_0^* + B_0 u + \\ &+ B_1^+[...[B_{r-1}^+[B_r^+ \sigma^{(r)}(\bar{x}_r) + \sigma^{(r-1)}(\bar{x}_{r-1})] + \\ &+ \sigma^{(r-2)}(\bar{x}_{r-2})] + ...] + B_1^+ \dot{\sigma}(\bar{x}_1). \end{aligned} \tag{14}$$

System (14) stabilization problem synthesis procedure consists of sequential synthesis of fictitious controls

$$\begin{aligned} &x_{r-\mu-1} = B_{r-\mu}^+[-A_{r-\mu} x_{r-\mu}^* - B_{r-\mu-1}^+[B_{r-\mu-2}^+[...[B_{r-1}^+[B_r^+ \times \\ &\times \sigma^{(\mu)}(\bar{x}_r) + \sigma^{(\mu-1)}(\bar{x}_{r-1})] + \sigma^{(\mu-2)}(\bar{x}_{r-2})] + ...] + \\ &+ \ddot{\sigma}(\bar{x}_{\mu-1})] + \dot{\sigma}(\bar{x}_{r-\mu-1})] - \sigma(\bar{x}_{r-\mu}) \end{aligned} \tag{15}$$

and of choosing true controls of the form

$$\begin{aligned} &u = B_0^+[-A_0 x_0^* - B_1^+[B_2^+ ...[B_{r-1}^+[B_r^+ \sigma^{(r)}(\bar{x}_r) + \\ &+ \sigma^{(r-1)}(\bar{x}_{r-1})] + \sigma^{(r-2)}(\bar{x}_{r-2})] + ...] + \dot{\sigma}(\bar{x}_1)] - \sigma(\bar{x}_0). \end{aligned} \tag{16}$$

After ($r+1$) steps of the above procedure, closed system (2) will be presented in the form

$$\dot{\bar{x}}_i = -\sigma(\bar{x}_i) + B_i\bar{x}_{i-1}\,,\ i = \overline{r,1}\,;\ \dot{\bar{x}}_0 = -\sigma(\bar{x}_0)\,. \qquad (17)$$

As can be seen, system (17) is stabilized sequentially, bottom up.

5. ONE PARAMETER PROCEDURE OF TUNING WITH ACCOUNT OF STATE VECTOR AND CONTROLS RESTRICTIONS

Let us consider the use of control system synthesis procedure based on σ-functions to solve the stabilization problem with state vector restrictions.
Let system (2) be bounded by restrictions of the form

$$\|x_i\| \le X_i = \text{const},\ i = \overline{r,0};\ \|u\| \le U = \text{const}\,.$$

Obviously, under these restrictions the analysis of the closed system stability is impossible within the framework of the linear theory. Considering that the σ-functions used in the preceding section are modulo restricted, let us attempt to use the possibility of accounting for the state vector restrictions at the synthesis stage. Let us present the step-by-step procedure with respect to a common system transformed into a block form of controllability of from (2).
The amplitude of each of the σ-functions chosen below does not exceed the set restrictions of the corresponding coordinates $X_i^* \le X_{i-1},\ i = \overline{r,1}$. The problem consists of choosing parameter τ so that the set restrictions on the coordinates of the state and control vector are assured to be true. Let us complete procedure 1, but with account of phase restrictions.

Procedure 2

Step 1. According to the first step of procedure 1, r-th subsystem appears as $\dot{\bar{x}}_r = -\sigma(\bar{x}_r) + B_r\bar{x}_{r-1}$. Under the assumption that the following steps assure the truthfulness of expression $\bar{x}_{r-1} = 0$, stabilization is assured with any parameter τ. Let us note that with $\tau \to 0$ a sliding mode will appear that will assure the convergence of variables to zero in a finite time and the invariability with respect to nonzero values $\|\bar{x}_{r-1}\| \le \bar{X}_{r-1}$. With this, naturally, the speed of feedback function growth tends to infinity. With account of restrictions $\|x_{r-1}\| \le X_{r-1}$, the truthfulness of inequality $\|\ B_r^+[-A_r\bar{x}_r - \sigma(\bar{x}_r)]\ \| \le X_{r-1}$ or

$$X_{r-1} \ge \left\|B^+{}_r\right\|(\|A_r\|X_r + X_r^*)$$

is necessary from which follows a restriction on the choice of the σ-function amplitude:

$$X_r^* \le \frac{X_{r-1}}{\|B_r^+\|} + \|A_r\|X_r\,.$$

If the stated condition is not achievable system stabilization is not possible.
Step 2. At the second step, considering the imposed restrictions $\|x_{r-2}\| \le X_{r-2}$ we verify the truthfulness of inequality $\left\|B_{r-1}^+[-A^*{}_{r-1}x_{r-1}^* - B_r^+\dot{\sigma}(\bar{x}_r) - \sigma(\bar{x}_{r-1})]\right\| \le X_{r-2}$. An evaluation similar to step one is conducted

$$X_{r-1}^* \le$$
$$\le \frac{X_{r-2}}{\left\|B_{r-1}^+\right\|} + \left\|A_{r-1}^*\right\|\left\|X_{r-1}\right\| + \left\|B_r^+\right\|\frac{X_r^*}{2\upsilon}(\|A_r\|X_r + \|B_r\|X_{r-1}),$$

where $\left\|A_{r-1}^*\right\|$ is the row norm of matrix A_{r-1}^* satisfying equation

$$A_{r-1}^*x_{r-1}^* = A_{r-1}x_{r-1}^* + B_r^+A_r[A_rx_r^* + B_rx_{r-1}]\,,$$

$\|X_{r-1}\| = \max\{X_{r-1},\ X_r\}$. To stabilize system (2), it is necessary for the following equation to hold true:

$$X_{r-1}^* \le \frac{X_{r-2}}{\left\|B_{r-1}^+\right\|} + \left\|A_{r-1}^*\right\|\left\|X_{\kappa-1}\right\|.$$

If this condition does not hold true, then by increasing parameter τ it is possible to assure the truthfulness of the indicated equation.
Continuing this course of action, at the i-th step we verify the truthfulness of equation

$$\begin{aligned}&|\ B_{i+1}^+\{-A_{i+1}^*x_{i+1}^* - B_{i+2}^+\big[B_{i+3}^+...\big[B_{r-2}^+\big[B_{r-1}^+\big[B_r^+\sigma^{(i+2)}(\bar{x}_r) + \\ &+\sigma^{(i+3)}(\bar{x}_{r-1})] - \sigma^{(i+1)}(\bar{x}_{r-2})] + ...] + \dot{\sigma}(x_{i+2})] - \sigma(\bar{x}_{i+1})\}\ | \le X_i\end{aligned}$$

and evaluate the amplitude of the σ-function: we select parameter τ, with which the said equation holds true.
At the following step of the procedure, true controls are formed, which solves system (2) stabilization problem with account of restrictions on state variables and controls.
In conclusion of the paper we will present an example of use of the suggested approach in the problem of mathematical pendulum control.

6. EXAMPLE

For illustration the procedure proposed above let us consider the mathematic pendulum location stabilization problem described by differential equation

$$\dot{x}_2 = x_1,\ \dot{x}_1 = x_0 + \sin x_2,\ \dot{x}_0 = u, \qquad (18)$$

where $x = \text{col}(x_1, x_2, x_0) \in R^3$, $u \in R^1$ are state vector and control, correspondently.
As stabilizing feedbacks let us choose a σ-vector function of the form

$$\sigma(x_i) = X_i^*\left[\frac{2}{1+\exp(-x_i/\tau_i)} - 1\right],\ X_i^* = \text{const},\ i = \overline{1,3}.$$

We will further need the estimates of first and second time derivatives of the sigma-functions

i) $\sigma(x) = X\left[\frac{2}{1+\exp(-x/\tau)} - 1\right] \Rightarrow (1+\exp(-x/\tau) = \frac{2X}{\sigma + X}$,

$$\sigma_x'(\mathrm{x}) = \frac{2X}{\tau}\frac{\exp(-x/\tau)}{[1-\exp(x/\tau)]^2} = \frac{2X}{\tau}\left(\frac{2X}{\sigma+X} - 1\right)\frac{(\sigma+X)^2}{4X^2} =$$

$$= \frac{1}{2X\tau}(X^2 - \sigma^2) \geq 0,\ \dot{\sigma}(x) = \sigma_x'(x)\dot{x}.$$

The maximum value of the particular derivative is reached at point $x = 0$ and is equal to $\max\sigma_x' = X/2\tau$.
Thus, the valid estimate is

$$|\dot{\sigma}| \leq \frac{X}{2\tau}|\dot{x}|. \qquad (19)$$

ii) $\ddot{\sigma}(\mathrm{x}) = \frac{1}{2X\tau}[-2\sigma\dot{\sigma}\dot{x} + (X^2 - \sigma^2)\ddot{x}] =$

$$= -\frac{\sigma(X^2-\sigma^2)}{2X^2\tau^2}\dot{x}^2 + \frac{X^2-\sigma^2}{2X\tau}\ddot{x}.$$

The multiplier before $\dot{x}^2$ reaches a maximum at point $\sigma = \pm x/\sqrt{3}$, and before $\ddot{x}$ at point $\sigma = 0$. Thus, the valid estimate is

$$|\ddot{\sigma}| \leq \frac{1}{3\sqrt{3}X\tau^2}|\dot{x}^2| + \frac{X}{2\tau}|\ddot{x}|. \qquad (20)$$

Let us apply the step-by-step procedure 2 to system (18).
Step 1. Let us designate $x_2 = \bar{x}_2$ and let us choose local feedback in the first equation (18) as $x_1 = -\sigma(x_2)$. To assure this equation be true, it is necessary solve the stabilization problem with respect to a new variable $\bar{x}_1 = x_1 + \sigma(x_2)$ whose behavior is described by a differential equation

$$\dot{\bar{x}}_1 = x_0 + \sin x_2 + \dot{\sigma}(x_2).$$

Then, the closed first equation (18) will take the form $\dot{\bar{x}}_2 = -\sigma(\bar{x}_2) + \bar{x}_1$.
Step 2. Let us choose fictitious controls x_0 in the form $x_0 = -\sigma(x_1) - \dot{\sigma}(x_2) - \sin x_2$ and introduce a new variable $\bar{x}_0 = x_0 + \sigma(\bar{x}_1) + \dot{\sigma}(\bar{x}_2) + \sin x_2$, described by equation

$$\dot{\bar{x}}_0 = u + \dot{\sigma}(\bar{x}_1) + \ddot{\sigma}(x_2) + \dot{x}_2\cos x_2.$$

Then the second equation takes form $\dot{\bar{x}}_1 = -\varphi_1(\bar{x}_1) + \bar{x}_0$.
Step 3. Let us chose a true control in the form

$$u = -\sigma(\bar{x}_0) - \dot{\sigma}(\bar{x}_1) - \ddot{\sigma}(\bar{x}_2) - \dot{x}_2\cos x_2.$$

Then the third equation (18) takes form $\dot{\bar{x}}_0 = -\varphi_0(\bar{x}_0)$. As a result, the closed system id described by the system of equations of the following form

$$\dot{\bar{x}}_1 = -\sigma(\bar{x}_1) + \bar{x}_0,\ \dot{\bar{x}}_2 = -\sigma(\bar{x}_2) + \bar{x}_1,\ \dot{\bar{x}}_0 = -\sigma(\bar{x}_0). \qquad (21)$$

System (21) is stabilized sequentially, bottom up.
Now, let us consider the mathematic pendulum stabilization problem under the restrictions imposed on the coordinates of the state vector and control:

$$|x_i| \leq X_i = \text{const},\quad |u| \leq U = \text{const},\ \mathrm{i} = \overline{1,3}.$$

Let us apply the step-by-step procedure 2 to system (18) with aforementioned restrictions.
Step 1. Let us let $x_2 = \bar{x}_2$ and chose $x_1 = -\sigma(\bar{x}_2)$. Due to restriction $|x_1| \leq X_1$, follows the choice of a coefficient in $\sigma(\bar{x}_2)$-function $X_2^* \leq X_1$. The new variable $\bar{x}_1 = x_1 + \sigma(\bar{x}_2)$ is described by equation

$$\dot{\bar{x}}_1 = x_0 + \sin x_2 + \dot{\sigma}(\bar{x}_2), \qquad (22)$$

and the first equation (18) takes form $\dot{\bar{x}}_2 = -\sigma(\bar{x}_2) + \bar{x}_1$.

Step 2. Considering coordinate x_0 in equation (22) as a fictitious control, we will let it be equal

$$x_0 = -\sigma(\bar{x}_1) - \dot{\sigma}(\bar{x}_2) - \sin x_2.$$

The choice of the amplitude $\sigma(\bar{\mathrm{x}}_1)$ is determined by relation $1 < X_1^* \leq X_0$. The first iteration in choosing parameter τ is executed under condition $X_1^* > |\mathrm{sinx}_2 + \dot{\sigma}(\bar{x}_2)|$. Considering (19)

$$\dot{\sigma}(\bar{\mathrm{x}}_2) = \sigma(\bar{x}_2 = 0)|\dot{\bar{x}}_2| < \frac{X_2^*}{2\tau}X_1,$$

we get the upper estimate $|\mathrm{sinx}_2 + \dot{\sigma}(\bar{x}_2)| \leq 1 + X_2^* X_1/2\tau$ and under condition $X_1^* > 1$ we find the upper estimate of parameter τ_1:

$$X_1^* > 1 + \frac{X_2^* X_1}{2\tau_1} \Rightarrow \tau_1 > \frac{X_2^* X_1}{2(X_1^* - 1)}.$$

Let us note that condition $X_0 > 1$ is the necessary condition for the controllability of system (18).
Let us introduce a new variable

$$\bar{x}_0 = x_0 + \sigma(\bar{x}_1) + \dot{\sigma}(\bar{x}_2) + \sin x_2$$

described by equation

$$\dot{\bar{x}}_0 = \mathrm{u} + \dot{x}_2 \cos x_2 + \dot{\sigma}(\bar{x}_1) + \ddot{\sigma}(\bar{x}_2)\,. \qquad (23)$$

Then the second equation of system (18) takes form

$$\dot{\bar{x}}_1 = -\sigma(\bar{x}_1) + \bar{x}_0\,.$$

Step 3. In equation (23) let us choose control in the form

$$u = -\sigma(\bar{x}_0) - \dot{x}_2 \cos x_2 + \dot{\sigma}(\bar{x}_1) - \ddot{\sigma}(\bar{x}_2)\,.$$

Then the third equation of system (18) takes form:

$$\dot{\bar{x}}_0 = -\varphi_0(\bar{x}_0)\,.$$

Now follows the second iteration in choosing parameter τ accounting for condition

$$U^* > \left|\dot{x}_2 \cos x_2 + \dot{\sigma}(\dot{x}_1) + \ddot{\sigma}(\bar{x}_2)\right|.$$

Taking estimates (19) and (20) into account, we have

$$\left|\dot{x}_2 \cos x_2 + \dot{\sigma}(x_2) + \ddot{\sigma}(\bar{x}_1)\right| < X_1 + \frac{X_1^*}{2\tau}\left|\dot{\bar{x}}_1\right| + \frac{1}{3\sqrt{3}X_2^*\tau^2}\left|\dot{\bar{x}}_2^{\,2}\right| +$$

$$+ \frac{X_2^*}{2\tau}\left|\ddot{\bar{x}}_2\right| = X_1 + \frac{X_1^*}{2\tau}\left(X_0 + 1 + \frac{X_2^*}{2\tau}\,|\,\dot{\bar{x}}_2\,|\right) + \frac{1}{3\sqrt{3}X_2^*\tau^2}X_1^2 +$$

$$+ \frac{X_2^*}{2\tau}\left(X_0 + 1\right) = X_1 + \frac{(X_0+1)(X_1^* + X_2^*)}{2\tau} +$$

$$+ \frac{3\sqrt{3}X_2^{*2}X_1^*X_1 + 4}{12\sqrt{3}X_2^*\tau^2} < U.$$

Introducing designation

$$X_1 = A,\ \frac{(X_0+1)(X_1^* + X_2^*)}{2} = B,\ \frac{3\sqrt{3}X_2^{*2}X_1^*X_1 + 4}{12\sqrt{3}X_2^*} = C,$$

we solve the quadratic equation $\tau^2(2A - U) + 2B\tau + 2C < 0$ and choose τ_2 from two possible values – the positive and the minimal one. As result, the value of parameter τ should be chosen equal to maximum from values of each step of the procedure $\tau^* = \max\{\tau_i\},\ i = 1,2$.

Note that on the last step of the procedure the control action may be chosen as a discontinuous function $u = -M\mathrm{sign}\bar{x}_0$, $0 < M \le U$ and, under sufficiently large amplitude in system (23), the sliding mode will appear, and the stabilization problem will be solved in finite time.

7. CONCLUSIONS

In the paper a new approach to the synthesis of control systems with account of state vector and controls restrictions based on the block control principle and nonlinear local feedback described by sigma-functions is suggested. The use of a part of the state vector coordinates as fictitious controls in each block leads us to an analogy to a neural network structure, namely, the quantity of blocks corresponds to the quantity of layers, and fictitious controls dimensionality in each block is equal to the quantity of neurons in the corresponding layer, and, at the same time, the control vector dimensionality matches the quantity of the input layer neurons, and the dimensionality of the output variables vector matches the quantity of the output layer neurons. The above considerations may be viewed as an attempt to connect the poor formulization problem of choosing a controlling neural network structure and a controller's structure based on the mathematic model of dynamic systems.

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