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Damping Angular Oscillations of a Pendulum under State Constraints

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Abstract: This paper considers the problem of damping the oscillation of a plane pendulum by moving the pivot in the vertical plane and the weight along the rod of the pendulum, with limited amplitudes. Conditions for stability of the motion of the pendulum are derived using energy-based methods, and based on them energy-based controls satisfying the amplitude constraints are developed, which move the pivot and weight sinusoidally and achieve an excellent control performance by controlling the variables with constant amplitudes. When the oscillation of the pendulum becomes sufficiently small, each of the energy-based controls is taken over by a linear or saturating control, which also satisfies the amplitude constraint, to regulate the entire state of the system to the nominal point. The saturating control is used to move the pivot horizontally, which also effectively damps the residual oscillation of the pendulum. Numerical and experimental results are given to demonstrate the effectiveness of the proposed control laws.

Keywords: Asymptotic stabilization; Nonlinear system control; Control of constrained systems; Lyapunov methods

1. INTRODUCTION

Pendulums are simple and familiar systems that are often taken as illustrative examples of oscillatory systems in textbooks on mechanics. On the other hand, variable-length pendulums with a movable pivot are known to be difficult to control due to the underactuatedness and nonlinearities. Since the equations of motion of such systems are similar to those of cranes, one-degree-of-freedom structure systems with an active mass damper, RTAC (rotational/translational actuator) systems, etc., control laws developed for such pendulums may also be applied to developing control laws for these systems. Therefore, the control problem of such pendulums is interesting from academic as well as engineering points of view.

The problem of controlling the motion of a variable-length pendulum has long been investigated; some of the work has been done to obtain or explain how to pump a swing. It is well known that an effective method for damping the oscillation of the pendulum is to lengthen it at the lowest point and shorten it at the highest points. This phenomenon has been explained in terms of *parameter excitation* or *energy variation*. Burns [1970] explained this by modeling the pendulum with an equation of motion in a parameter excitation problem, called the Mathieu equation, and obtaining its approximate solutions. Curry [1976] and Tea and Falk [1968] showed this through approximate computations of the energy of the pendulum. Stilling and Szyszkowski [2002] presented a way to obtain a rate of change of the length of the pendulum as a function of time that makes the damping ratio of the oscillation larger. Dimentberg [2002] gave a method for constructing feedback control laws for the length of the pendulum that decrease the energy of the pendulum. Also, there are studies using optimal control techniques. Lavrovskii and Formal'skii [1993] showed by a geometric approach that the bang-bang control of the length of the pendulum performed as mentioned above is the optimal control that maximizes the damping ratio of the oscillation. Piccoli and Kulkarni [2005] explained, using a geometric technique in optimal control theory or the maximum principle, that this control is time optimal.

A number of studies have been done on the problem of controlling the motion of a pendulum by moving its pivot horizontally, most of which were investigated as control problems for cranes. Generally, for crane systems the purpose of control is to transfer the suspended load swiftly to a specified point while suppressing the oscillation of the load. The challenge is to develop a control law such that the transfer of the load to a distant point and the suppression of the oscillation of the load are performed, keeping the input and state within allowable ranges, by a single input to the trolley. Some studies have investigated problems including hoisting or lowering of the load, which are generally introduced to avoid obstacles or to transfer the load to a prescribed point.

Mita and Kanai [1979] and Auernig and Troger [1987] have investigated time-optimal control problems for a crane without (Mita and Kanai [1979]) or with hoisting of the load (Auernig and Troger [1987]). Sakawa and Shindo [1982] have developed a computational algorithm for solving an optimal control problem for a crane with hoisting of the load, where the load swing is minimized. These studies used the maximum principle to obtain open-loop controls.

Fantoni and Lozano [2002], Chung and Hauser [1995], Fang et al. [2003], Yoshida and Kawabe [1992], and Yoshida [1998] have developed linear or nonlinear feedback control laws for controlling a crane, based on the Lyapunov stability theorem, the small gain theorem or energy-based methods. In Fantoni and Lozano [2002], it is proved, by constructing a Lyapunov function considering the passivity of the system, that a PD control for the trolley position asymptotically stabilizes the whole system. Chung and Hauser [1995] proposes a control law that regulates the energy of the pendulum at a prescribed value with the amplitude of the trolley being bounded, and proves the asymptotic stability around the periodic orbit using the small gain theorem; the control law consists of a small-gain PD control for the trolley position and a nonlinear control regulating the energy of the pendulum. In Fang et al. [2003], a combination of a PD control for the trolley position and a nonlinear control intensifying the coupling between the motions of the trolley and the load is used to obtain a better damping of the load swing. In Yoshida and Kawabe [1992], a saturating control law is developed that satisfies the input constraint and lowers the value of a quadratic cost function of the state. In Yoshida [1998], an energy-based control law is proposed to suppress the oscillation of the load by changing, with small amplitude, the position of the trolley and the length of the suspending rope.

Corriga et al. [1998], Giua et al. [1999], Bartolini et al. [2002], and Lee [2004] have proposed linear or nonlinear feedback control laws for a crane with hoisting or lowering of the load. In Corriga et al. [1998] and Giua et al. [1999], the crane is approximately modeled by a linear time-varying system where the time-varying parameter is the length of the suspending rope, and for this, linear time-varying feedback control laws are developed using a time-scaling technique and a stability theorem for time-varying systems Corriga et al. [1998] or using Wolovich's design procedure for time-varying systems Giua et al. [1999]. Bartolini et al. [2002] and Lee [2004] provide sliding mode controls, using a sliding surface coupling the motions of the trolley and the load, that effectively damp the load swing and are robust to modelling errors.

Burg et al. [1996] have proposed a nonlinear feedback control law for a crane that can transfer the load to a distant point keeping the load swing small, using the saturating control design technique developed by Teel [1993] .

Yu et al. [1995] have developed a feedback control law for a crane where the load is much heavier than the trolley, using singular perturbation methods; the control for the slow motion (the average motion of the transferred load) and the one for the fast motion (the load sway around the average motion) are separately designed and the sum of them is used as the control.

For a pendulum with a vertically movable pivot, it is known that when the motion of the pivot is harmonic, the equation of motion of the pendulum can be approximated by the Mathieu equation, whose solutions have been well analyzed, and the conditions for stability can be obtained through the results on the stability analysis (Meirovitch [1975]). There are however few studies on the problem of controlling the oscillation of a pendulum by actively changing both the length of the pendulum and the position of the pivot.

To find our control laws, energy-based methods are used, which are effective tools for controlling underactuated systems. In fact, for swing-up control problems of a cart-and-pendulum system or an Acrobot and a vibration suppression problem of an RTAC system, various energy-based controls have been proposed (Wei et al. [1995], Astrom and Furuta [2000], Chatterjee et al. [2002], Yoshida [1999], Andrievskii et al. [1996], Spong [1995], Matsumoto and Yoshida [2005]), some of which can consider the amplitude constraint of the pivot (Wei et al. [1995], Chatterjee et al. [2002], Yoshida [1999], Andrievskii et al. [1996]). Energy-based methods have also been applied to active stiffness control for structure systems (Chen [1984], Fujino et al. [1993]).

This paper addresses the problem of suppressing the oscillation of a plane pendulum using the three actuated variables, i.e., the horizontal and vertical positions of the pivot and the length of the pendulum, under the amplitude constraints. An energy-like function of the pendulum, called an energy function, is defined, and then energy-based controls are obtained that decrease the energy function effectively. They control the actuated variables sinusoidally with constant amplitudes. When the oscillation of the pendulum becomes sufficiently small, each of the energy-based controls is taken over by a linear or saturating control, which also satisfies the constraint. Also, the effectiveness of the control laws is examined by simulations and experiments.

2. PROBLEM STATEMENT

Figure 1 shows the pendulum considered in this study, which consists of a rigid rod with a pivot and a weight (a point mass) and moves in the vertical plane; the pivot can be moved in the vertical plane and the weight can be moved along the rod. For simplicity, it is assumed that

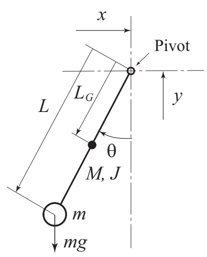

Fig. 1. Plane pendulum system.

there is no friction in the rotation of the pendulum. Let θ be the angular displacement of the pendulum, x and y the coordinates of the pivot, and L the distance from the pivot to the weight. The quantities x, y, and L are variables being able to be actuated; θ is not an actuated one. So the system is an underactuated one.

The equations of motion can be written as

$$(mL^2 + J)\ddot{\theta} + 2mL\dot{L}\dot{\theta} + (mL + ML_G)g\sin\theta$$
$$-(mL + ML_G)\ddot{x}\cos\theta + (mL + ML_G)\ddot{y}\sin\theta = 0 \tag{1}$$

$$\ddot{x} = u_x, \quad \ddot{y} = u_y, \quad \ddot{l} = u_l \tag{2}$$

where M is the mass of the rod, J the moment of inertia of the rod around the pivot, m the mass of the weight, L_G the distance from the pivot to the center of gravity of the rod, l the deflection of L from the nominal length, say L_0, i.e., $l = L - L_0$, and g the acceleration of gravity. The equations of motion for x, y, and l have been linearized by cancelling the nonlinear terms using each actuator as in (2), where u_x, u_y, and u_l are the inputs after the linearization.

Assume that x, y, and l are constrained as

$$|x| \leq a_x, \quad |y| \leq a_y, \quad |l| \leq a_l \tag{3}$$

where a_x, a_y, and a_l are the maximum amplitudes of x, y, and l, respectively. Suppose also that the initial state variables (the state consists of θ, x, y, l, and their time derivatives) are small in magnitude, and that

$$a_x, a_y, a_l \ll L_0. \tag{4}$$

Denote the peaks of $|\theta(t)|$ by $\theta_0, \theta_1, \ldots$ in the order of increasing time. Then the performance index is defined as the ratio between two adjacent ones of these amplitudes, i.e.,

$$\delta_i := \frac{\theta_{i+1}}{\theta_i}. \tag{5}$$

A smaller δ_i means a higher damping, i.e., a better performance of control. The problem is to find control laws for x, y, and l that make δ_i small under the conditions in (3).

Specifically, the performance of control will be evaluated by δ_0 or δ_1. Since each of the controls proposed below repeats a similar motion of the actuated variable over a half period of the oscillation of the pendulum, a smaller δ_i for some i means smaller δ_k for any k.

3. DESIGN METHODS

3.1 Conditions for stability

To derive conditions for stabilizing the pendulum, i.e., ones for making $(\theta, \dot{\theta}) \to 0$, we consider the function

$$V := (1 + \alpha L) \cdot$$
$$\left\{ \frac{1}{2}(mL^2 + J)\dot{\theta}^2 + (mL + ML_G)g(1 - \cos\theta) \right\} \tag{6}$$

where

$$\alpha = \frac{2mL_0\omega_n^2 - mg}{(ML_G + 2mL_0)g - (mL_0^2 - J)\omega_n^2} \tag{7}$$

which is chosed so that the approximation in (17) can be made. Here ω_n is the natural angular frequency of the pendulum with $L = L_0$, i.e.,

$$\omega_n = \sqrt{\frac{(mL_0 + ML_G)g}{mL_0^2 + J}}. \tag{8}$$

Then we see that for $0 \leq L \leq 2L_0$, $1 + \alpha L > 0$ and thus $V \geq 0$. Moreover, $V = 0$ holds only when the pendulum is at rest at the pendant position. Therefore, we see from LaSalle's invariance principle that if $\dot{V} \leq 0$ and moreover $\dot{V}$ is not identically zero, then V approaches 0 and thus the pendulum is asymptotically stabilized.

Taking the time derivative of V and using (1), we have

$$\dot{V} = \xi\dot{l} + (1 + \alpha L)(mL + ML_G)(\dot{\theta}\cos\theta\ddot{x} - \dot{\theta}\sin\theta\ddot{y}) \tag{9}$$

where

$$\xi := g(m + \alpha ML_G + 2\alpha mL)(1 - \cos\theta)$$
$$- \left(mL + \frac{1}{2}\alpha mL^2 - \frac{1}{2}\alpha J \right)\dot{\theta}^2. \tag{10}$$

From this we see that $\dot{V} \leq 0$ if the following relations hold:

$$\xi\dot{l} \leq 0 \tag{11}$$

$$\dot{\theta}\cos\theta\ddot{x} \leq 0 \tag{12}$$

$$-\dot{\theta}\sin\theta\ddot{y} \leq 0. \tag{13}$$

We shall obtain simplified conditions for (11), (12), and (13).

Define the angle φ as shown in Fig. 2 using the trajectory of the pendulum in the $(\theta, \dot{\theta}/\omega_n)$-plane, i.e.,

$$\varphi := -\tan^{-1}\frac{\dot{\theta}}{\omega_n\theta}. \tag{14}$$

Let

$$r = \sqrt{\theta^2 + \left(\frac{\dot{\theta}}{\omega_n}\right)^2} \tag{15}$$

and suppose $r > 0$. Then we see that

$$\theta = r\cos\varphi, \ \frac{\dot{\theta}}{\omega_n} = -r\sin\varphi. \tag{16}$$

From the assumption of small motions, the function ξ in (11) can be approximated as

$$\xi \simeq \xi_a := \beta\left\{\theta^2 - \left(\frac{\dot{\theta}}{\omega_n}\right)^2\right\} \tag{17}$$

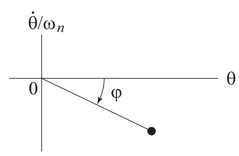

Fig. 2. Definition of the angle φ using the trajectory of the pendulum.

where

$$\beta := \frac{1}{2}g(m + \alpha M L_G + 2\alpha m L_0) \tag{18}$$

which can be shown to be positive. Here we take $L \simeq L_0$, $\cos\theta \simeq 1 - \theta^2/2$ in (10) to obtain (17). Substituting (16) into (17), we have

$$\xi_a = \beta r^2(\cos^2\varphi - \sin^2\varphi) = \beta r^2 \cos 2\varphi. \tag{19}$$

Since $\beta r^2 > 0$, $\cos 2\varphi$ has the same sign of ξ_a. Similarly, it can be shown that when $|\theta| < \pi/2$, $-\sin\varphi$ and $\sin 2\varphi$ have the same signs, respectively, as those of $\dot\theta\cos\theta$ and $-\dot\theta\sin\theta$.

We thus obtain the following conditions corresponding to (11), (12), and (13), respectively.

$$\cos 2\varphi\, \dot l \le 0 \tag{20}$$

$$-\sin\varphi\, \ddot x \le 0 \tag{21}$$

$$\sin 2\varphi\, \ddot y \le 0. \tag{22}$$

Condition (20) is an approximation of (11), while (21) and (22) are conditions equivalent to (12) and (13), respectively, as long as $|\theta| < \pi/2$.

Note that for a small free oscillation of the pendulum, the following approximation holds:

$$\varphi \simeq \omega_n t + \varphi_0 \tag{23}$$

where φ_0 is a constant determined from the initial conditions, $\theta(0)$ and $\dot\theta(0)$. We assume that the condition in (23) also holds while the pendulum is controlled.

Under condition (23), functions $\cos 2\varphi$, $-\sin\varphi$, and $\sin 2\varphi$, which are also in phase with ξ, $\dot\theta\cos\theta$, and $-\dot\theta\sin\theta$, respectively, become sinusoidal ones. Then we see that if l, x, and y are controlled sinusoidally so that $-\dot l$, $-\ddot x$, and $-\ddot y$ synchronize with these functions, conditions (20), (21), and (22) are satisfied and thus V approaches 0. Such controls will be developed in the following section. Although there are interactions between the motions of l, x, and y, each controller can be designed independently using each condition of (20), (21), and (22) under the assumption of (23).

3.2 Energy-based controls

Controls changing l, x, and y sinusoidally will be obtained based on (20), (21), and (22); this type of control will be called *an energy-based control*. Since each control is designed based on the same idea, only the case of l is explained. For x and y, just the results are given.

Control law for l Suppose that a servo system for l has been designed whose transfer function is $G_l(s)$, whose input is denoted by r_l, and whose output is l.

Theorem 1. The amplitude constraint $|l(t)| \le a_l, \forall t \ge 0$ is satisfied if the following three conditions hold (Yoshida and Matsumoto [2003]).

1) $\|G_l(s)\|_1 = \int_0^\infty |g_l(t)|dt \le 1.$

2) $|r_l(t)| \le a_l,\ \forall t \ge 0.$

3) The initial condition $[l(0), \dot l(0)]' \in \mathcal{R}_l$.

Here $g_l(t)$ is the impulse response of $G_l(s)$, and $\mathcal{R}_l$ is the set of all $[l, \dot l]'$ reachable from the origin by some input satisfying condition 2).

Construct the input r_l as

$$r_l = -a_l \sin(2\varphi - \angle G_l(j2\omega_n)). \tag{24}$$

Then it is obvious that r_l satisfies condition 2). Also, from (23) and (24) r_l is a sinusoidal function with a fundamental harmonic of frequency $2\omega_n$. Since $\varphi(t)$, computed from the phase trajectory of the pendulum, actually fluctuates about a linear function of t, i.e., $\omega_n t + \varphi_0$, due to the nonlinear oscillation of the pendulum and the observation noise, r_l must contain high-frequency components. Assume that $G_l(s)$ is designed to have a filtering property such that these high-frequency components are diminished. Then we have in the steady state, considering the gain and phase lag in the frequendy response at $2\omega_n$,

$$l \simeq -a_l|G_l(j2\omega_n)| \sin 2\varphi \tag{25}$$

and therefore

$$\cos 2\varphi\, \dot l \simeq -a_l|G_l(j2\omega_n)|2\omega_n \cos^2 2\varphi \le 0. \tag{26}$$

From this we see that (20) is approximately satisfied and that the function $\cos 2\varphi\, \dot l$ is not identically zero. Hence, the condition for stability is approximately satisfied. The stability analysis is not rigorous, but it would be natural to expect that synchronizing $\dot l$ with $-\cos 2\varphi$ leads to an effective damping, the validity of which will be shown by simulations and experiments.

A way of designing $G_l(s)$ satisfying condition 1) is to give

$$G_l(s) = \frac{1}{(T_l s + 1)^2} \tag{27}$$

where $T_l > 0$ is a design parameter determined from the magnitude of the fluctuation of $\varphi(t)$ from the linear function of t. We see from simulations that $G_l(s)$ has the desired filtering property if we choose T_l as

$$T_l = \frac{1}{n(2\omega_n)}, \quad n = 1 \sim 5. \tag{28}$$

While a larger n gives a larger value of the gain $|G_l(j2\omega_n)|$, i.e., a larger amplitude of l, achieving a better control performance, the control system becomes more sensitive to the nonlinearities and noise.

Specifically, from (27) the control law for l is written as

$$u_l = \frac{1}{T_l^2}(-l - 2T_l \dot l + r_l). \tag{29}$$

Control law for x The input and the transfer function of the servo system for x are, respectively,

$$r_x = -a_x \sin(\varphi - \angle G_x(j\omega_n)) \tag{30}$$

$$G_x(s) = \frac{1}{(T_x s + 1)^2} \tag{31}$$

with

$$T_x = \frac{1}{n\omega_n}, \quad n = 1 \sim 5 \tag{32}$$

where the initial condition $[x(0), \dot{x}(0)]'$ is required to be in $\mathcal{R}_x$, which is similarly defined as $\mathcal{R}_l$. Specifically, from (31) the control law for x is represented as

$$u_x = \frac{1}{T_x^2}(-x - 2T_x\dot{x} + r_x). \quad (33)$$

Control law for y The input and the transfer function of the servo system for y are, respectively,

$$r_y = a_y \sin(2\varphi - \angle G_y(j2\omega_n)) \quad (34)$$

$$G_y(s) = \frac{1}{(T_y s + 1)^2} \quad (35)$$

with

$$T_y = \frac{1}{n(2\omega_n)}, \quad n = 1 \sim 5 \quad (36)$$

where the initial condition $[y(0), \dot{y}(0)]'$ is required to be in $\mathcal{R}_y$, which is similarly defined as $\mathcal{R}_l$. Specifically, from (35) the control law for y is given as

$$u_y = \frac{1}{T_y^2}(-y - 2T_y\dot{y} + r_y). \quad (37)$$

3.3 Saturating control for x

We describe a saturating control law for x that satisfies the amplitude constraint on x and makes the control system asymptotically stable (globally asymptotically stable for the linearized model), which also uses the technique used in the designs of the energy-based controls to satisfy the amplitude constraints. For the derivation of it see Yoshida, Matsumoto and Ninomiya [2008].

The saturating control for x is designed based on the linearized model of the pendulum system with l and y being fixed to zero. The equations of motion are obtained, by letting in (1) and (2) $L = L_0$, $\dot{L} = 0$, $\ddot{y} = 0$, $\ddot{l} = 0$, $\sin\theta \simeq \theta$, and $\cos\theta \simeq 1$, as

$$\begin{cases} \ddot{x} = u_x \\ \ddot{\theta} = -\omega_n^2\theta + \dfrac{\ddot{x}}{L_e} \end{cases} \quad (38)$$

where

$$L_e := \frac{g}{\omega_n^2}.$$

Moreover, in the second equation of (38) we use in place of θ the new variable

$$z := x - L_e\theta.$$

Then (38) are rewritten as

$$\begin{cases} \ddot{x} = u_x \\ \ddot{z} = -\omega_n^2 z + \omega_n^2 x. \end{cases} \quad (39)$$

Assume that the servo system for x has been designed and let v be the input of the servo system of x, and $G(s)$ the transfer function from v to x. The constraint $|x(t)| \leq a_x, \forall t \geq 0$ is satisfied if the following three conditions hold.

1) $\|G(s)\|_1 \leq 1$.
2) $|v(t)| \leq a_x, \ \forall t \geq 0$.
3) The initial condition $[x(0), \dot{x}(0)]' \in \mathcal{R}$.

Here $\mathcal{R}$ is the set of all $[x, \dot{x}]'$ reachable from the origin by some input v satisfying condition 2).

To satisfy conditions 1) and 3), $G(s)$ is chosen as

$$G(s) = \frac{1}{(Ts + 1)^2} \quad (40)$$

where $T \leq T_x$, selected so that $\mathcal{R}_x \subset \mathcal{R}$.

The input u_x realizing (40) is given by

$$u_x = \frac{1}{T^2}(-x - 2T\dot{x} + v). \quad (41)$$

Substituting this into (39) and writing (39) in state equation form, we obtain

$$\dot{w} = Aw + Bv \quad (42)$$

where

$$w = [\, x \ \dot{x} \ z \ \dot{z} \,]'$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{1}{T^2} & -\dfrac{2}{T} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_n^2 & 0 & -\omega_n^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \dfrac{1}{T^2} \\ 0 \\ 0 \end{bmatrix}.$$

The saturating control that asymptotically stabilizes the system (42) is designed as

$$v = \mathrm{sat}(\tilde{F}Sw, a_x) \quad (43)$$

with

$$\tilde{F} = [\, 0 \ 0 \ 0 \ -2\zeta\omega_n \,]$$

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ T^2 & 0 & \dfrac{1}{\omega_n^2} - T^2 & \dfrac{2T}{\omega_n^2} \\ 2T & T^2 & -2T & \dfrac{1}{\omega_n^2} - T^2 \end{bmatrix}.$$

where $\mathrm{sat}(p, a_x)$ is the saturating function defined by

$$\mathrm{sat}(p, a_x) = \mathrm{sgn}(p)\min\{|p|, a_x\}.$$

Here ζ is a design parameter, chosen, in the range of $\zeta > 0$ (possibly $0 < \zeta \leq 1$), by simulations so that a good control performance is obtained. Of course, owing to conditions 1) through 3) the amplitude constraint $|x(t)| \leq a_x, \forall t \geq 0$ is satisfied.

Specifically, the control u_x is given by (41) with v of (43).

3.4 Control laws near the equilibrium point

Before (23) no longer holds, each energy-based control must be switched to another one. After the switching, for l and y we simply put $r_l = r_y = 0$ in the energy-based controls, and for x we use the saturating control.

As a switching criterion we adopt the following condition:

$$V < \epsilon_0. \quad (44)$$

That is, once (44) holds, the linear and saturating control laws take over permanently and thus the problem of chattering does not occur. Here ϵ_0, a small positive number, is

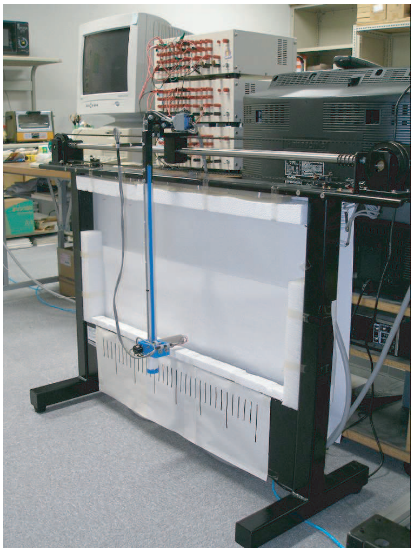

Fig. 3. View of the experimental apparatus.

a design parameter chosen by simulations so that a good control performance is obtained.

4. NUMERICAL AND EXPERIMENTAL RESULTS

Figure 3 shows a view of the experimental system. The pivot can be moved only horizontally, and the weight can be moved along the rod of the pendulum, both of which are driven by DC motors. The distance from the pivot to the end of the rod is 0.62m.

The parameters of the apparatus are as follows:

$$m = 0.268\text{kg},\ L_0 = 0.4\text{m},\ \omega_n = 4.91\text{rad/s}$$
$$ML_G = 0.0631\text{kgm}.$$

The acceleration of gravity is $g = 9.81\text{m/s}^2$.

In the experimental system the controller was a personal computer with a CPU of the MMX Pentium(166MHz) and with a 12-bit AD/DA board, where the control laws were programmed with the C language and the sampling period was 1ms. The quantities θ, l, and x were measured by potentiometers and their time derivatives were estimated from the difference between every two adjacent samplings. Data for making graphs were stored in the computer's memories at every ten samplings to spare the memories.

Due to the observation noise in the measurements of the potentiometers, a large amount of noise was included in the estimates of the time derivatives. These measurements and estimates were directly used to construct the servo systems for l and x and to compute the saturating control, where we could ignore the nonlinearities of the equations of motion, owing to the robustness of the servo systems. Specifically, each actuator, which consists of a DC motor and the driver, was in advance compensated with an approximate angular velocity feedback plus a precompensator of a first-order lag so that the transfer function from the input of the actuator to the output (the actuated variable) became

$$\frac{1}{s(1+T_0 s)} \quad \text{with } T_0 = 0.2\text{s}. \tag{45}$$

Then for example the equation of motion for x is

$$\ddot{x} = u_x = \frac{1}{T_0}(-\dot{x} + v_x) \tag{46}$$

where v_x is the scaled input of the actuator. When u_x is determined via the energy-based methods or the saturating control design technique, v_x can be obtained from (46) as $v_x = \dot{x} + T_0 u_x$.

As the value of the energy function V, we used the signal that was computed by (6) with the measurements and estimates and smoothed by the low-pass filter whose transfer function is

$$\frac{1}{1 + 0.04s}.$$

$\dot{\theta}/\omega_n$ is necessary to compute the angle φ, but for this we could not use the estimate of $\dot{\theta}$ because it was too noisy. So we estimated $\dot{\theta}/\omega_n$ with the signal obtained as the output of the following transfer function with the measurement of θ being applied as the input:

$$G_d(s) = -\frac{2\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}.$$

The reason why this can be done is as follows. Noting that $|G_d(j\omega_n)| = 1$ and $\angle G_d(j\omega_n) = -\pi/2$, we see that for $\theta = \theta_0 \sin(\omega_n t + \varphi_0)$, in the steady state the output is represented by

$$-\theta_0 \sin\left(\omega_n t + \varphi_0 - \frac{\pi}{2}\right) = \theta_0 \cos(\omega_n t + \varphi_0) = \frac{\dot{\theta}}{\omega_n}.$$

Also, $G_d(s)$ works as a low-pass filter with the break frequency ω_n, so that the noise in the measurement of θ can be diminished.

The following constraints were considered:

$$|l| \le 0.02\text{m},\ |x| \le 0.02\text{m}.$$

Hence, we set

$$a_l = a_x = 0.02\text{m}.$$

The parameters of control laws were given as follows. For the energy-based control:

$$T_l = \frac{1}{2n\omega_n},\ T_x = \frac{1}{n\omega_n} \quad \text{with } n = 2.$$

For the saturating control:

$$\zeta = 0.7,\ \ T = \frac{1}{n\omega_n} \quad \text{with } n = 2.$$

For the switching criterion:

$$\epsilon_0 = 0.02.$$

When n was set to be greater than 2, chatterings were likely to occur in the control of the servo systems due to the observation noise in the measurements. So we set n as above.

Figs. 4 through 8 show some of the numerical and experimental results. In all cases, the corresponding numerical results are also shown. In each case $(\theta(0), \dot{\theta}(0)) \simeq (0.5\text{rad}, 0\text{rad/s})$, $(x(0), \dot{x}(0)) \simeq (0\text{m}, 0\text{m/s})$, and $(l(0), \dot{l}(0)) \simeq (0\text{m}, 0\text{m/s})$. Of course, $(x(0), \dot{x}(0))' \in \mathcal{R}_x$ and $(l(0),$

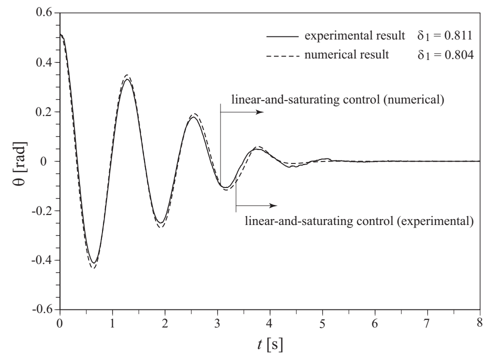

Fig. 4. Numerical and experimental results for $\theta(t)$ of the control system with $n = 2$, $\zeta = 0.7$, and $\epsilon_0 = 0.02$.

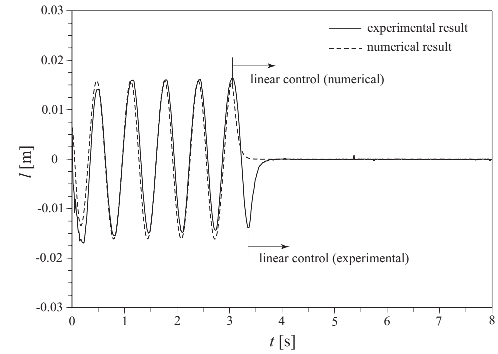

Fig. 5. Numerical and experimental results for $l(t)$ of the control system with $n = 2$, $\zeta = 0.7$, and $\epsilon_0 = 0.02$.

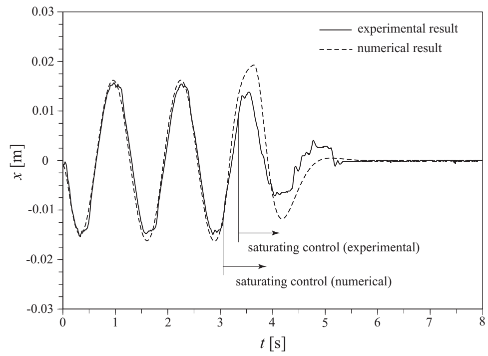

Fig. 6. Numerical and experimental results for $x(t)$ of the control system with $n = 2$, $\zeta = 0.7$, and $\epsilon_0 = 0.02$.

$\dot{l}(0))' \in \mathcal{R}_l$. It can be thought that the differences between the numerical and experimental results are due to the observation noise, the estimating errors of the state, and the mechanical frictions in the servo systems and other modelling errors. However, it is seen that, as a whole, the numerical and experimental results are in good agreement, and that a good control performance was obtained with the constraints on l and x being satisfied; we obtained $\delta_1 = 0.811$. Also, Fig.8 shows the estimate of φ for the energy-based control, which shows the assumption that $\varphi \simeq \omega_n t$ is valid.

Fig. 9 shows, for reference, the numerical results, under the conditions of the experiment, with an LQ optimal control for x, designed so that a highest damping of θ is obtained under the constraint on x, with $l = 0$. Specifically, for

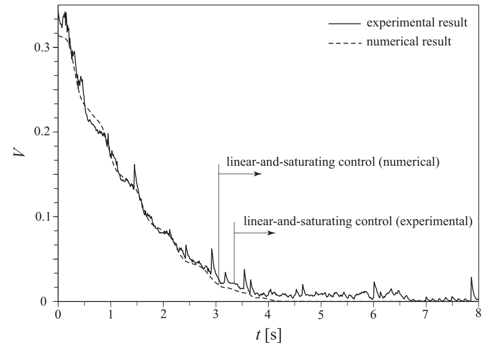

Fig. 7. Numerical and experimental results for the value of the energy function $V(t)$ of the control system with $n = 2$, $\zeta = 0.7$, $\epsilon_0 = 0.02$.

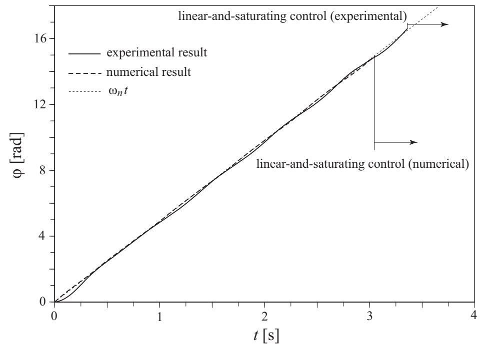

Fig. 8. Numerical and experimental results for the angle $\varphi(t)$ of the control system with $n = 2$, $\zeta = 0.7$, and $\epsilon_0 = 0.02$.

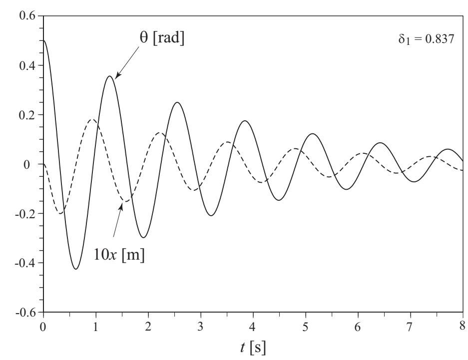

Fig. 9. Numerical results by the LQ optimal control satisfying the constraint on x.

the linearized system (38), the LQ optimal control was designed as

$$u_x = [\,-102.5\ \ -16.77\ \ -3.655\ \ 0.6677\,]\,\tilde{x}$$

using the performance index

$$J = \int_0^\infty \left(\tilde{x}'Q\tilde{x} + u_x^2\right) dt$$

where

$$\tilde{x} = \left[\, x\ \dot{x}\ \theta\ \dot{\theta}\,\right]', \quad Q = \mathrm{diag}\{10500, 0, 0, 1\}.$$

Since such a linear control cannot maintain the amplitude of x close to the maximal value, the control performance is much lower than that via the proposed one.

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