Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Consensus in Networks with Diverse Input and Communication Delays ⋆

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Abstract: This paper studies the consensus problem for multi-agent systems with diverse input and communication delays. Decentralized consensus conditions are obtained based on the frequency-domain analysis and matrix theory. By these conditions, to achieve consensus under large input delays, one should use small interconnection gains or have small numbers of neighbors when the graph is kept connected. For systems with diverse communication delays, a consensus protocol with unified self-delay is proposed. The obtained consensus conditions are dependent on the self-introduced delay but independent of communication delays when the digraph contains a globally reachable node.

Keywords: Consensus, Multi-agent system, Communication delays, Input delays, Group cooperation.

1. INTRODUCTION

It is very important to study the delay effect on convergence of consensus protocols. Basically, there are two kinds of delays in multi-agent systems. One is related to process or connection time of each agent, which will be called input delay in this note. The other is related to communication from one agent to another. Input delays have been extensively studied in classic control problems (see, e.g., Niculescu (2001)). However, up to now there is no report on the consensus problem with input delays, to our knowledge. For systems with communication delays, it is a natural idea to introduce self-delays in the consensus protocol and the self-delays are usually chosen to be equal to the communication delays (see, e.g., Saber and Murray (2004)). But such a protocol cannot be robust because the measurement of communication delays always contain some uncertainty. Moreover, the analysis of convergence of the protocol is very difficult. Some convergence results were obtained only for multi-agent systems with an identical communication delay (Moreau, 2004; Saber and Murray, 2004). Based on the contraction theory and wave variable method, Wang and Slotine studied the consensus problem for the system with multi-variable agents under diverse communication delays (Wang and Slotine, 2006). They proposed a simple consensus protocol with zero self-delay, which is robust to arbitrary communication delays. However, the interconnection graph in their study is undirected and connected or unidirectional formed in closed rings.

In this paper, we first consider the consensus problem with diverse input delays. Using some early results obtained for the analysis of stability of congestion control algorithms (Tian and Yang, 2004), we get decentralized consensus conditions, which uses only local information of each agent. According to these conditions, to achieve consensus under large input delays, one should use small interconnection gains or have small numbers of neighbors when the graph is kept connected. Then, we propose a consensus protocol with unified self-introduced delay to solve the consensus problem with diverse communication delays. With the help of the frequency-domain method, we get decentralized consensus conditions for systems with diverse communication delays. Our results can be applied to networks with directed graphs and nonsymmetric weights. The obtained consensus conditions are dependent on the self-introduced delay but independent of communication delays when the digraph contains a globally reachable node. Under the proposed protocol the communication delays do not influence the convergence, but they prolong the converging time.

2. PROBLEM FORMULATION AND PRELIMINARIES

2.1 Some notions of graph theory

The notation used in this note for graph theory is quite standard. A weighted directed graph (digraph) $G = (V, E, A)$ consists of a set of vertices $V = \{v_1, ..., v_n\}$, a set of edges $E \subseteq V \times V$ and a weighted adjacency matrix $A = [a_{ij}] \in R^{n\times n}$. The node indexes belong to a finite index set $\mathcal{I} = \{1, 2, ..., n\}$. We assume that the adjacency elements associated with the edges of the digraph are positive, i.e., $a_{ij} > 0 \Leftrightarrow e_{ij} \in E$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. The set of neighbors of node v_i is denoted by $N_i = \{v_j \in V : (v_i, v_j) \in E\}$.

In the weighted digraph $G = (V, E, A)$, the out-degree of node i is defined as $\deg_{\text{out}}(v_i) = \sum_{j=1}^{n} a_{ij}$. Let D be the degree matrix of G, which is defined as a diagonal matrix with the out-degree of each node along its diagonal. The

⋆ This work was supported by National Natural Science Foundation of China (under grants 60425308 and 60673058), National "863" Programme of China (under grant 2006AA04Z263), and Natural Science Foundation of Jiangsu Province of China (under grant BK2006097).

Laplacian matrix of the weighted digraph is defined as $L = D - A$.

2.2 Consensus problem under diverse delays

In a multi-agent system, each agent can be considered as a node in a digraph, and the information flow between two agents can be regarded as a directed path between the nodes in the digraph. Thus, the interconnection topology in a multi-agent system can be described by a diagraph $G = (V, E, A)$.

Consider a discrete-time model of a network with integrator agents

$$x_i(k+1) = x_i(k) + u_i(k),\ i \in \mathcal{I}, \tag{1}$$

where $x_i \in R$ and $u_i \in R$ denote the state and the control input of agent i, respectively. The following consensus protocol for the multi-agent system (1) has been extensively studied in the literature (see, e.g., Saber and Murray (2004))

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k) - x_i(k)), \tag{2}$$

where N_i denotes the neighbors of agent i, and $a_{ij} > 0$ is the adjacency element of A in the digraph $G = (V, E, A)$.

When each agent is subject to an input delay D_i, system (1) becomes

$$x_i(k+1) = x_i(k) + u_i(k - D_i),\ i \in \mathcal{I}. \tag{3}$$

Under diverse communication delays, the consensus protocol (2) can be generally extended to the following form

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k - T_{ij}) - x_i(k - \tau_{ij})), \tag{4}$$

where T_{ij} represents the communication delay from agent j to agent i, τ_{ij} represents the self delays artificially introduced by agent i for agent j.

Under protocol (4), multi-agent system (3) is said to have an asymptotic consensus if

$$\lim_{k\to\infty} x_i(k) = c,\ \forall i \in \mathcal{I},$$

where c is a constant.

The closed-loop system of (3) and (4) is

$$\begin{aligned} x_i(k+1) &= x_i(k) \\ &+ \sum_{v_j \in N_i} a_{ij}(x_j(k - T_{ij} - D_i) - x_i(k - \tau_{ij} - D_i)), \\ &\quad i \in \mathcal{I}. \end{aligned} \tag{5}$$

Let

$$\begin{aligned} x(k) &= [x_1(k), \cdots, x_n(k)]^T, \\ d_{m1} &= T_{ij} + D_i,\ m_1 = 1, \cdots, n \times n, \\ d_{m2} &= \tau_{ij} + D_i,\ m_2 = n \times n + 1, \cdots, 2 \times n \times n. \end{aligned}$$

Then, equation (5) can be rewritten as a time-delayed system in vector form

$$x(k+1) = x(k) + \sum_{i=1}^{n_d} A_i x(k - d_i), \tag{6}$$

where $A_i \in R^{n\times n}$, and $n_d = 2 \times n \times n$. Obviously, $\sum_{i=1}^{n_d} A_i = L$, which is the Laplacian matrix of the multi-agent system.

The equilibrium set of system (6) is defined by

$$F = \{x \in R^n : Lx = 0\}. \tag{7}$$

When L is singular, F is a continuum of equilibrium points.

Assume the interconnection topology of the system is described by a connected undirected graph or a digraph containing a globally reachable node. Then the Laplacian matrix L has a simple eigenvalue 0, i.e., $\det(L) = 0$ and $\mathrm{rank}(L) = n - 1$ (see, e.g., Lin et al. (2005)). By the definition L we have $L[1, 1, \cdots, 1]^T = 0$. So, all the elements in F can be represented as $c[1, 1, ..., 1]^T$ where c is any constant. Therefore, in order to show the asymptotic consensus of the multi-agent system (6) with a connected undirected graph or a digraph containing a globally reachable node, it suffices to prove that the solution of the system starting from any given initial states $x(-k) \in R^n, k = 0, 1, \cdots, n_d - 1$, will asymptotically converge to an element in F.

Taking the z-transformation, we get the characteristic equation of system (6) as

$$\det((z-1)I - \sum_{i}^{n_d} A_i z^{-d_i}) = 0. \tag{8}$$

Lemma 1. If the roots of equation (8) have modulus less than unity except for a root at $z = 1$, then the system (6) with a connected undirected graph or a digraph containing a globally reachable node has an asymptotic consensus.

The proof is obvious and thus omitted due to the page limitation.

2.3 Other useful lemmas

The following lemmas are needed in the proof of our main results.

Lemma 2. The following inequality

$$\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} \le 2D + 1$$

holds for all nonnegative integers D and all $\omega \in [-\pi, \pi]$.

Proof. First of all, we claim that

$$\sin(\frac{\pi}{2(2D+1)}) \ge \frac{1}{2D+1} \tag{9}$$

holds for any nonnegative integer D. Indeed, by denoting $x = \frac{1}{2D+1}$, we have $x \in (0, 1]$ for any nonnegative integer D. Thus, inequality (9) is equivalent to the well-known inequality $\sin(\frac{\pi}{2}x) \ge x$, $x \in (0, 1]$.

Now, we note that

$$\lim_{\omega\to 0} \frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} = 2D + 1.$$

In the following we will prove

$$\frac{\sin(\frac{2D+1}{2}\omega)}{\sin\frac{\omega}{2}} \le 2D + 1 \tag{10}$$

holds for all nonnegative integers D and $\omega \in [-\pi, 0) \cup (0, \pi]$. Since $\frac{\sin(-\frac{2D+1}{2}\omega)}{\sin(-\frac{\omega}{2})} = \frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})}$, we just need to prove (10) for all $\omega \in (0, \pi]$.

When $\omega \in (0, \frac{\pi}{2D+1}]$, let $h(\omega) = \sin(\frac{2D+1}{2}\omega) - (2D+1)\sin(\frac{\omega}{2})$. Calculating the derivative of $h(\omega)$ on ω yields

$$\dot{h}(\omega) = \frac{2D+1}{2}(\cos(\frac{2D+1}{2}\omega) - \cos(\frac{\omega}{2})).$$

Obviously, we have $\dot{h}(\omega) \leq 0$, i.e., $h(\omega)$ is not increasing for all $\omega \in (0, \frac{\pi}{2D+1}]$. Since $h(0) = 0$, we have $h(\omega) \leq 0$, i.e., $\sin(\frac{2D+1}{2}\omega) \leq (2D+1)\sin(\frac{\omega}{2})$ for all $\omega \in (0, \frac{\pi}{2D+1}]$. Since $\sin(\frac{\omega}{2}) > 0$, we get $\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} \leq 2D+1$.

When $\omega \in (\frac{\pi}{2D+1}, \pi]$, we have

$$\sin(\frac{\omega}{2}) > \sin(\frac{\pi}{2(2D+1)}) > 0$$

for all nonnegative integers D. So, from (9), we get

$$\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} \leq \frac{1}{\sin(\frac{\omega}{2})} < \frac{1}{\sin(\frac{\pi}{2(2D+1)})} \leq 2D+1$$

for all $\omega \in (\frac{\pi}{2D+1}, \pi]$ and all nonnegative integers D. Lemma 2 is proved. □

Lemma 3. Vinnicombe (2000) Let $Q = Q^* > 0$ and $T = \mathrm{diag}(t_i, t_i \in C)$ be given. Then,

$$\lambda(TQ) \in \rho(Q)\mathrm{Co}(0 \cup \{t_i\}),$$

where $\rho(\cdot)$ denotes the matrix spectral radius, and $\mathrm{Co}(\cdot)$ denotes convex hull.

Lemma 4. Tian and Yang (2004) Given any real number $0 \leq \kappa < 1$ and natural number $m \geq 2$. Then, the convex hull $\kappa\mathrm{Co}(0 \cup \{G^i(\omega), i = 1 \cdots, m\})$ does not contain the point $(-1, \mathrm{j}0)$ for all $\omega \in [-\pi, \pi]$, where

$$G^i(\omega) = k_i \frac{\exp(-\mathrm{j}D_i\omega)}{\exp(\mathrm{j}\omega) - 1},\ i = 1, \cdots, m$$
$$k_i = 2\sin(\frac{\pi}{2(2D_i+1)}).$$

3. CONSENSUS WITH INPUT DELAYS

In this section we consider the consensus problem for multi-agent systems with input delays only. The closed-loop form of the time-delayed multi-agent system (3) with consensus protocol (2) is

$$x_i(k+1) = x_i(k) + \sum_{v_j \in N_i} a_{ij}(x_j(k-D_i) - x_i(k-D_i)),\quad i \in \mathcal{I}. \tag{11}$$

We assume the interconnection topology studied in this section is described by an undirected graph with symmetric weights, i.e., $a_{ij} = a_{ji}$.

Theorem 1. Consider a system of n coupled agents (11) based on an undirected and connected graph $G = (V, E, A)$ with symmetric weights. System (11) has an asymptotic consensus if

$$\sum_{v_j \in N_i} a_{ij} < \sin\frac{\pi}{2(2D_i+1)}, \forall\, i \in \mathcal{I}. \tag{12}$$

Proof. Taking the z-transformation of (11), we get

$$(z-1)X(z) = -\mathrm{diag}\{z^{-D_i}, i = 1 \cdots, n\}LX(z). \tag{13}$$

Note that L in (13) is a positive semi-definite matrix since an undirected graph is considered. The characteristic equation is

$$\det((z-1)I + \mathrm{diag}\{z^{-D_i}, i = 1 \cdots, n\}L) = 0. \tag{14}$$

Define $p(z) = \det((z-1)I + \mathrm{diag}\{z^{-D_i}, i = 1 \cdots, n\}L)$. Then, we will prove that all the zeros of $p(z)$ have modulus less than unity except for a zero at $z = 1$.

Let $z = 1$, then $p(1) = \det(L)$. Since $G = (V, E, A)$ is connected, 0 is a simple eigenvalue of L, i.e., $\det(L) = 0$ and $\mathrm{rank}(L) = n - 1$ (see, e.g., Lin et al. (2005)). Thus, $p(z)$ indeed has a simple zero at $z = 1$.

Now, we prove that the zeros of $\det(I + \mathrm{diag}\{\frac{z^{-D_i}}{z-1}, i = 1 \cdots, n\}L)$ have modulus less than unity. By the general Nyquist stability criterion, this is the case if the eigenloci of

$$F(\mathrm{j}\omega) = \mathrm{diag}\{\frac{\exp(-\mathrm{j}\omega D_i)}{\exp(\mathrm{j}\omega) - 1}, i = 1 \cdots, n\}L$$

does not enclose the point $(-1, \mathrm{j}0)$ for all $\omega \in [-\pi, \pi]$. To show this we rewrite $F(\mathrm{j}\omega)$ as

$$F(\mathrm{j}\omega) = \mathrm{diag}\{k_i\frac{\exp(-\mathrm{j}\omega D_i)}{\exp(\mathrm{j}\omega) - 1}\}\mathrm{diag}\{k_i^{-1}\}L,$$

where $k_i = 2\sin(\frac{\pi}{2(2D_i+1)})$. Now, using Lemma 3 we have

$$\begin{aligned}&\lambda(\mathrm{diag}\{k_i\frac{\exp(-\mathrm{j}\omega D_i)}{\exp(\mathrm{j}\omega) - 1}\}\mathrm{diag}\{k_i^{-1}\}L)\\ &\in \rho(\mathrm{diag}\{k_i^{-\frac{1}{2}}\}L\mathrm{diag}\{k_i^{-\frac{1}{2}}\})\times\\ &\mathrm{Co}(0 \cup \{\mathrm{diag}\{k_i\frac{\exp(-\mathrm{j}\omega D_i)}{\exp(\mathrm{j}\omega) - 1}\}\}).\end{aligned}$$

Since the spectral radius of any matrix is bounded by its maximum absolute row sum, it follows from the condition (12) that

$$\rho(\mathrm{diag}\{k_i^{-\frac{1}{2}}\}L\mathrm{diag}\{k_i^{-\frac{1}{2}}\}) < 1.$$

Therefore, from Lemma 4 we conclude that eigenloci of $F(\mathrm{j}\omega)$ does not enclose the point $(-1, \mathrm{j}0)$ for all $\omega \in [-\pi, \pi]$, which implies that the zeros of $p(z)$ have modulus less than unity except for a zero at $z = 1$. Theorem 1 is thus proved by Lemma 1. □

Now, we apply Theorem 1 to the linearized Vicsek's Model Jadbabaie, et al. (2003) with input delays

$$x_i(k+1) = x_i(k) + \frac{\varepsilon_i}{1+n_i}(\sum_{v_j \in N_i}(x_j(k-D_i) - x_i(k-D_i)), \tag{15}$$

where n_i denotes the number of the neighbors of agent i, $\varepsilon_i > 0$ is an adjustable interconnection gain.

From Theorem 1, we get the following corollary.

Corollary 1. Consider a system of n coupled agents (15) with an undirected graph that is connected. System (15) has an asymptotic consensus if

$$\frac{n_i \varepsilon_i}{1+n_i} < \sin \frac{\pi}{2(2D_i+1)}, \forall\, i \in \mathcal{I}. \tag{16}$$

Remark 1. It is easy to check that when $D_i \geq 1$, inequality (16) holds only for $\varepsilon_i < 1$.

Remark 2. Corollary 1 clearly shows the relationship among the input delay, interconnection gain and number of neighbors: for large input delay one should use small interconnection gain or have small number of neighbors when the graph is kept connected.

Similarly, we can apply Theorem 1 to Moreau's Model Moreau (2005) with input delays

$$x_i(k+1) = x_i(k) + \frac{1}{1+\sum_{v_j \in N_i} w_{ij}} \times \\ (\sum_{v_j \in N_i} w_{ij}(x_j(k-D_i) - x_i(k-D_i))), \tag{17}$$

where w_{ij} denotes the positive weight corresponding to the edge e_{ij} in the diagraph G.

Corollary 2. Consider a system of n coupled agents (17) with an undirected graph that is connected and has symmetric weights. System (17) has an asymptotic consensus if

$$\frac{\sum_{v_j \in N_i} w_{ij}}{1+\sum_{v_j \in N_i} w_{ij}} < \sin \frac{\pi}{2(2D_i+1)}, \forall\, i \in \mathcal{I}. \tag{18}$$

4. CONSENSUS UNDER COMMUNICATION DELAYS

In this section we consider networks with communication delays, in which each agent can get only time-delayed information from other agents. Denote by $T_{ij} > 0$ the delay of the information flow from agent j to agent i. For simplicity of discussion, we assume that there is no input delay, i.e., $D_i = 0, i = 1, \cdots, n$.

The following time-delayed consensus protocol was adapted in Saber and Murray (2004)

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k-T_{ij}) - x_i(k-T_{ij})). \tag{19}$$

However, the consensus condition was obtained only for the identical communication delays, i.e., $T_{ij} = T$. Reference Wang and Slotine (2006) proposed another consensus protocol without self-delay

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k-T_{ij}) - x_i(k)). \tag{20}$$

But they only analyzed the connected and undirected topology graph with symmetric weights and the unidirectional graph formed in closed rings with identical weights.

To solve the consensus problem of the multi-agent system (1) with diverse communication delays based on a digraph, we propose a consensus protocol given by

$$u_i(k) = \sum_{v_j \in N_i} a_{ij}(x_j(k-T_{ij}) - x_i(k-D)), \tag{21}$$

where $D \geq 0$ is the self-delay which is uniform for all the agents. Obviously, protocol (21) is a compromise between (19) and (20).

With the consensus protocol (21), the closed-loop form of the multi-agent system (1) is

$$x_i(k+1) = x_i(k) + \\ \sum_{v_j \in N_i} a_{ij}(x_j(k-T_{ij}) - x_i(k-D)). \tag{22}$$

Theorem 2. Consider a system of n coupled agents (22) with a static interconnection topology $G = (V, E, A)$ that has a globally reachable node. System (22) has an asymptotic consensus if

$$\sum_{v_j \in N_i} a_{ij} < \frac{1}{2D+1}, \forall\, i \in \mathcal{I}. \tag{23}$$

Proof. Taking the z-transformation of the system (22), we get

$$zX_i(z) = X_i(z) + \sum_{v_j \in N_i} a_{ij}(X_j(z) z^{-T_{ij}} - X_i(z) z^{-D}), \tag{24}$$

where $X_i(z)$ is the z-transformation of $x_i(k)$. Define a $n \times n$ matrix $\tilde{L}(z) = \{\tilde{l}_{ij}(z)\}$ as follows:

$$\tilde{l}_{ij}(z) = \begin{cases} -a_{ij} z^{D-T_{ij}}, & v_j \in N_i; \\ \sum\limits_{v_j \in N_i} a_{ij}, & j = i; \\ 0, & \text{otherwise}. \end{cases}$$

and $\tilde{L}(1) = L$, which is the Laplacian matrix. Then, (24) can be written as

$$zX(z) = X(z) - z^{-D}\tilde{L}(z)X(z),$$

where $X(z) = [X_1(z), X_2(z), ..., X_n(z)]^T$. The characteristic equation is

$$\det((z-1)I + z^{-D}\tilde{L}(z)) = 0.$$

Define $p(z) = \det((z-1)I + z^{-D}\tilde{L}(z))$. Then, we will prove that all the zeros of $p(z)$ have modulus less than unity except for a zero at $z = 1$ in the following.

Let $z = 1$, $p(1) = \det(1^{-D}\tilde{L}(1)) = \det(L)$. Since $G = (V, E, A)$ has a globally reachable node, 0 is a simple eigenvalue of L, i.e., $\det(L) = 0$ and $\text{rank}(L) = n-1$ (see, e.g., Lin et al. (2005)). Thus, $p(z)$ indeed has a simple zero at $z = 1$.

Now, we prove that the zeros of $f(z) = \det(I + \frac{z^{-D}}{z-1}\tilde{L}(z))$ have modulus less than unity. Based on the general Nyquist stability criterion, the zeros of $f(z)$ have modulus less than unity if the eigenloci of $\frac{e^{-j\omega D}}{e^{j\omega}-1}\tilde{L}(j\omega)$ does not enclose the point $(-1, j0)$ for $\omega \in [-\pi, \pi]$.

By Greshgorin disk theorem, we have

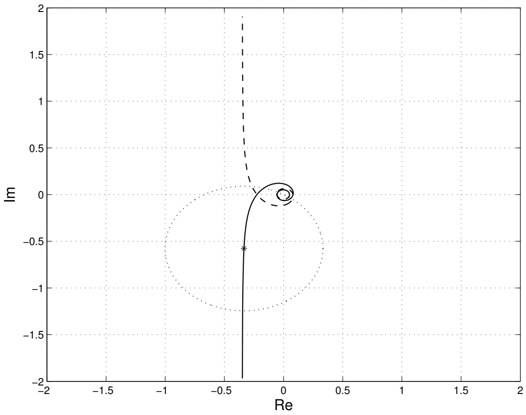

Fig. 1. Nyquist plot of $G(\omega)$.

$$\lambda(\frac{e^{-j\omega D}}{e^{j\omega}-1}\tilde{L}(j\omega)) \in \bigcup_{i\in\mathcal{I}} \{\zeta : \zeta \in C, \left. |\zeta - (\sum_{v_j\in N_i} a_{ij})\frac{e^{-j\omega D}}{e^{j\omega}-1}| \le |(\sum_{v_j\in N_i} a_{ij})\frac{e^{-j\omega D}}{e^{j\omega}-1}| \right\} \subseteq \left\{\zeta : |\zeta - K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}| \le |K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|\right\}$$

for all $\omega \in [-\pi,\pi]$, where $K_{\max} = \max_{i\in\mathcal{I}} \sum_{v_j\in N_i} a_{ij}$.

Now, define

$$G(\omega) = K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1} \tag{25}$$

and the Nyquist plot of $G(\omega)$ for $\omega \in [-\pi,\pi]$ is illustrated in Figure 1. Note that $G(\omega)$ is the center of the disc $\{\zeta : \zeta \in C, |\zeta - K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}| \le |K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|\}$. So, $\lambda(\frac{e^{-j\omega D}}{e^{j\omega}-1}\tilde{L}(j\omega))$ does not enclose the point $(-1, j0)$ for $\omega \in [-\pi,\pi]$ as long as we prove that $(-a, j0)$ with $a \ge 1$ dose not in the disc $\{\zeta : \zeta \in C, |\zeta - K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}| \le |K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|\}$ for all $\omega \in [-\pi,\pi]$.

From (25), we have

$$|-a + j0 - K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|^2 - |K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|^2 = a(a - K_{\max}\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})}).$$

Because $\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} \le 2D+1$ holds for $\omega \in [-\pi,\pi]$ by Lemma 3, using (23) we obtain

$$K_{\max}\frac{\sin(\frac{2D+1}{2}\omega)}{\sin(\frac{\omega}{2})} \le K_{\max}(2D+1) < 1 \le a.$$

Thus, $|a + j0 - K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|^2 - |K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|^2 > 0$, i.e., $|a + j0 - K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}| > |K_{\max}\frac{e^{-j\omega D}}{e^{j\omega}-1}|$ holds for $\omega \in [-\pi,\pi]$ with $a \ge 1$.

Now, we have proved that the zeros of $p(z)$ have modulus less than unity except for a zero at $z = 1$. Therefore, Theorem 2 is proved by Lemma 1. □

With communication delays, the linearized Vicsek's Model given in Jadbabaie, et al. (2003) becomes

$$x_i(k+1) = x_i(k) + \frac{1}{1+n_i}(\sum_{v_j\in N_i}(x_j(k-T_{ij}) - x_i(k-D)), \tag{26}$$

where n_i denotes the number of the neighbors of agent i.

From Theorem 2 we get the following corollary.

Corollary 3. Assume the interconnection topology of (26) has a globally reachable node. System (26) has an asymptotic consensus if

$$\frac{n_i}{1+n_i} < \frac{1}{2D+1}, \ \forall\, i \in \mathcal{I}. \tag{27}$$

Remark 3. Since n_i is no less than unity, inequality (27) holds if and only if $D = 0$.

Similarly, we can apply Theorem 2 to Moreau's Model Moreau (2005) with communication delays

$$x_i(k+1) = x_i(k) + \frac{1}{1+\sum_{v_j\in N_i} w_{ij}} \times (\sum_{v_j\in N_i} w_{ij}(x_j(k-T_{ij}) - x_i(k-D)). \tag{28}$$

Corollary 4. Assume the interconnection topology of (28) has a globally reachable node. System (28) has an asymptotic consensus if

$$\frac{\sum_{v_j\in N_i} w_{ij}}{1+\sum_{v_j\in N_i} w_{ij}} < \frac{1}{2D+1}, \ \forall\, i \in \mathcal{I}. \tag{29}$$

Remark 4. Inequality (29) always holds for $D = 0$.

5. SIMULATION STUDIES

We present some simulation examples to study the delay effect on consensus protocols.

5.1 Example 1: consensus under input delays

Consider a network of six agents described by the linearized Vicsek's Model (15). The interconnection topology is shown by Fig. 2, which is a connected graph. The input delays for agents are $D_1 = 1, D_2 = 3, D_3 = 2, D_4 = 2, D_5 = 4, D_6 = 2$. The interconnection gains are selected as $\varepsilon_1 = 0.2, \varepsilon_2 = 0.3, \varepsilon_3 = 0.1, \varepsilon_4 = 0.2, \varepsilon_5 = 0.3, \varepsilon_6 = 0.5$. The multi-agent system converges to a consensus as in Fig. 3.

Now, we increase the input delay, for example D_2, to study the convergence of the consensus protocol. Then, we find that the system has an asymptotic consensus until $D_2 = 8$. When $D_2 = 9$, no consensus can be achieved. When we increase the interconnection paths in the graph, the upper bound of permitted input delays become smaller. Simulation results validate the facts implied by Theorem 1 and Corollary 1.

5.2 Example 2: consensus under communication delays

Consider a multi-agent system with an interconnection digraph shown by Fig. 4. The digraph contains a globally reachable node but it is not strongly connected. The

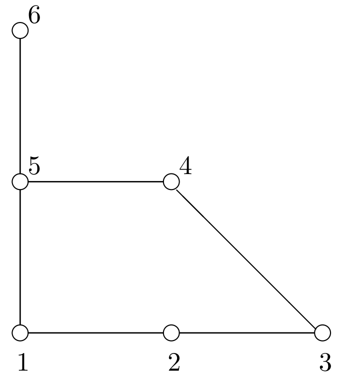

Fig. 2. Undirected graph of a group of 6 agents

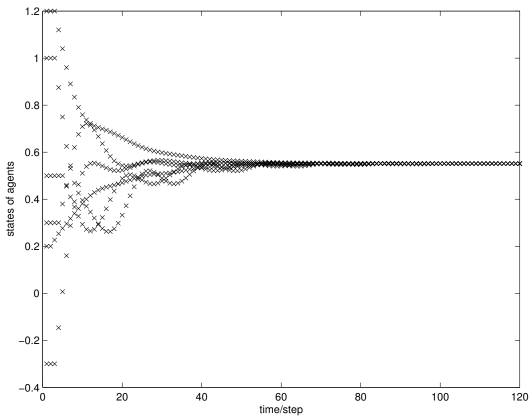

Fig. 3. Consensus with input delays.

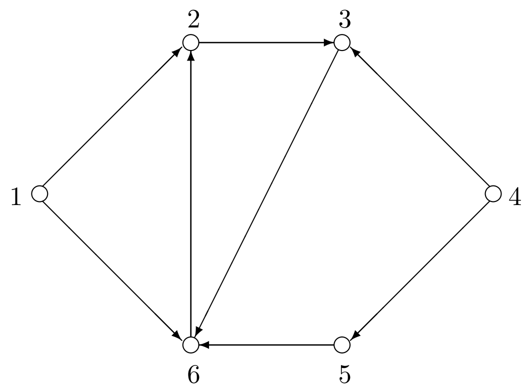

Fig. 4. Digraph of a group of 6 agents

weights of the directed paths are: $a_{12} = 0.1$, $a_{16} = 0.05$, $a_{23} = 0.15$, $a_{36} = 0.1$, $a_{43} = 0.05$, $a_{45} = 0.1$, $a_{56} = 0.15$, $a_{62} = 0.15$, and the corresponding communication delays are: $T_{12} = 5$, $T_{16} = 3$, $T_{23} = 4$, $T_{36} = 4$, $T_{43} = 4$, $T_{45} = 6$, $T_{56} = 6$, $T_{62} = 5$.

By Theorem 2, we get the self-delay $D \leq 2$ as the convergence condition. We choose $D = 2$ in the simulation. The initial states are generated randomly, and the multi-agent system converges to a consensus as in Fig. 5.

Theorem 2 implies that the convergence condition of system (22) is independent of communication delays. But these delays prolong the converging time.

6. CONCLUSION

We have consider the consensus problem with diverse input and communication delays. Decentralized consensus

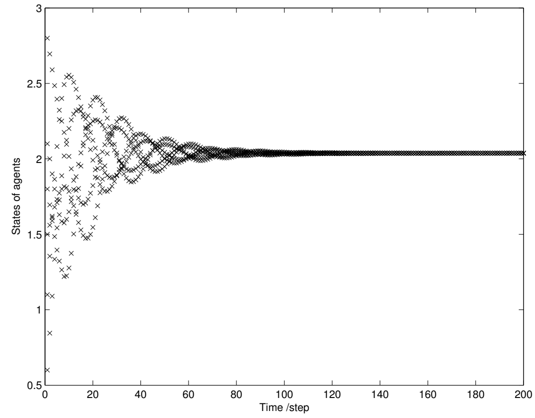

Fig. 5. Consensus with communication delays.

conditions are obtained. Then, we propose a consensus protocol with unified self-introduced delay to solve the consensus problem with diverse communication delays. The obtained consensus conditions are dependent on the self-introduced delay but independent of communication delays when the digraph is globally reachable.

REFERENCES

A. Jadbabaie, J. Lin, A.S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Autom. Control*, 48:988–1001, 2003.

Z. Lin, B. Francis, M. Maggiore. Necessary and sufficient graphical conditions for formation control of unicysle. *IEEE Trans. Autom. Control*, 50:121–127, 2005.

L. Moreau. Stability of continuous-time distributed consensus algorithms. *Proceeding of the 43rd IEEE Conference on Decision and Control*, Atlantis, Paradise island, Bahamas, pages 3998-4003, 2004.

L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Trans. Autom. Control*, 50:169–182, 2005.

S.-I.Niculescu, editor. *Delay Effects on Stability: A Robust Control Approach.* Lecture Notes in Control and Information Sciences, **269**, Springer, Berlin, 2001.

R. Olfati-Saber, R. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Trans. Autom. Control*, 49:1520–1533, 2004.

Y.-P. Tian, H.-Y.Yang. Stability of the Internet congestion control with diverse delays. *Automatica*, 40:1533-1541, 2004.

G. Vinnicombe. *On the stability of end-to-end congestion control for the Internet.* Technical report CUED/F-INFENG/TR. No.398, 2000. [on-line]. Available: http://www-control.eng.cam.ac.uk/gv/internet

W. Wang, J.J.E. Slotine. Contraction analysis of time-delayed communication delays. *IEEE Trans. Autom. Control*, 51:712–717, 2006.