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Stable $\mathcal{H}^\infty$ Controller Design for Systems with Multiple Time-Delays: The Case of Data-Communication Networks

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Abstract: Stable $\mathcal{H}^\infty$ controller design is considered for systems which involve multiple time-delays. Flow control problem in data-communication networks is chosen to present the proposed design approach. An algorithm, which produces a robust stable controller, is developed. An example is also presented, where the optimal controller, which is unstable, fails to produce a stable response due to nonlinear effects. The proposed controller, however, can robustly stabilize the system and produce desired response despite uncertain time-varying multiple time-delays.

1. INTRODUCTION

Many systems, biological, economical, or physical, include time-delays. These delays may be ignored for controller design, when they are sufficiently small. However, when they become significant, they may have a detrimental effect on the system. In such a case, time-delays should be taken into account during controller design. Existence of time-delays, however, challenges the controller design problem, since such systems are infinite dimensional. In the literature, numerous approaches have been proposed for controller design for time-delay systems (see Niculescu (2001) for a wide survey). Operator theoretical approaches have been used by Curtain and Zwart (1995) and by Foias et al. (1996) to design controllers for general infinite dimensional systems. Toker and Özbay (1995) used Hankel+Toeplitz operator theory to design an $\mathcal{H}^\infty$ controller for single-input single-output (SISO) infinite dimensional systems. The state-space techniques were used to design a controller for systems with a single time-delay by Meinsma and Zwart (2000). Later, by using the chain-scattering framework and J-spectral factorizations, an $\mathcal{H}^\infty$ controller design approach for multi-input multi-output (MIMO) systems with multiple time-delays was presented by Meinsma and Mirkin (2005).

When an optimization approach, such as $\mathcal{H}^\infty$, is used to design a controller for any system, the resulting controller may or may not be stable. When the resulting controller is unstable, although it theoretically stabilizes the overall system and optimizes a certain performance/robustness measure, the closed-loop system may become highly sensitive to sensor/actuator faults, numerical errors, and nonlinear effects. Such effects, may indeed cause an unstable behaviour in a practical implementation. To avoid such undesirable behaviour, stable controller design problem, which is also referred to as the *strong stabilization problem*, has been considered in the literature for a long time (e.g., Zeren and Özbay (2000)). Strong stabilization problem has also been considered for time-delay systems (e.g., Abedor and Poola (1989); Suyama (1991); Gümüşsoy (2004)). However, those studies have been limited to systems which involve a single delay. To the author's best knowledge, stable controller design problem for systems which involve multiple delays have not been considered in the literature up to date.

In the present paper, we consider the stable $\mathcal{H}^\infty$ controller design problem for systems which involve multiple time-delays. We take data-communication networks as an example case. In particular, we consider the flow control problem in data-communication networks. This problem has been widely studied, not only in the control literature, but in the computer and communication literatures as well (e.g., see Quet et al. (2002) and references therein). In the rate-based flow control, a controller is implemented at the bottleneck node, in order to adjust the rates of the sources that send data to that node so that congestion can be avoided. The existence of time-delays between the bottleneck node and the sources, however, makes this problem challenging. Furthermore, these delays are usually uncertain and time-varying. An $\mathcal{H}^\infty$ controller, which is robust to uncertain time-varying multiple time-delays in different channels was designed by Quet et al. (2002) using the techniques of Toker and Özbay (1995). However, since the results of Toker and Özbay (1995) were limited to SISO systems, the controller proposed in Quet et al. (2002) was obtained by defining seperate controller design problems for each channel. Later, Ünal et al. (2006) used the MIMO techniques of Meinsma and Mirkin (2005) to present an $\mathcal{H}^\infty$-optimal controller to solve the same problem. For technical reasons, however, Ünal et al. (2006) assumed that the uncertain parts of the time-delays were always non-negative. This assumption was later removed in Ünal et al. (2007), using the result of Ünal and İftar (2007). The approach was also presented in more detail in Ünal et al. (2007). In Ünal et al. (2007), it was also realized that when the controller, apart from the integral action, is not stable, an unstable behaviour may result in the closed-loop system due to nonlinear effects. Therefore, in the present paper, starting with the optimal controller of Ünal et al. (2007), we present an approach to find a stable controller (apart from the integral action), which also robustly stabilizes the overall system and produces a desired response.

1.1 Notation

For a time function $x(t)$, $\dot{x}(t)$ denotes its derivative with respect to the *time variable* t. For a matrix M, M^T denotes its transpose and M^{-1} denotes its inverse. For two symmetric matrices M and N, $N \leq M$ means that $M - N$ is non-negative definite. For a positive integer k, I_k denotes the $k \times k$ dimensional identity matrix. I and 0 respectively denote appropriately dimensioned identity and zero matrices. For positive integers k and l, $J_{k,l} := \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix}$ is a *signature matrix*. $\mathrm{diag}(m_1, \ldots, m_k)$ denotes a $k \times k$ dimensional diagonal matrix with $m_1, \ldots, m_k$ on its main diagonal. For appropriately dimensioned matrices A, B, C, and D, $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denotes a realization of the transfer function matrix (TFM) $G(s) = C(sI - A)^{-1}B + D$. $\|\cdot\|_\infty$ and $\|\cdot\|_2$ respectively denote the $\mathcal{H}^\infty$ and $\mathcal{L}^2$ norms (Zhou et al. (1996)). A TFM is said to be *stable* if it is in $\mathcal{H}^\infty$. A stable TFM is said to be *bistable* if its inverse exists in $\mathcal{H}^\infty$. A constant square matrix is said to be *Hurwitz* if all its eigenvalues have negative real parts. A TFM $Q \in \mathcal{H}^\infty$ is said to be *contractive* if $\|Q\|_\infty < 1$. For TFMs G and K, where $G =: \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ and G_{11} is $k \times k$, G_{22} is $l \times l$, and K is $k \times l$ dimensional, $HM(G, K)$ denotes the *homographic transformation* (Kimura (1996)), which is defined as

$$HM(G, K) = (G_{11}K + G_{12})(G_{21}K + G_{22})^{-1} .$$

Two important properties of the homographic transformation are the *chain property*:

$$HM(\Psi, HM(G, K)) = HM(\Psi G, K) , \qquad (1)$$

where Ψ is a TFM which has the same dimensions as G, and the *inverse property*: if $Q = HM(G, K)$, where G is invertible, then

$$K = HM(G^{-1}, Q) . \qquad (2)$$

2. PROBLEM STATEMENT

In this section, we consider the flow control problem for data-communication networks to illustrate stable $\mathcal{H}^\infty$ controller design for systems with multiple time-delays.

2.1 Network Model

A data communication network with n sources feeding a single bottleneck node is taken as a model. The flow controller, which is to be designed, is implemented at the bottleneck node. The controller calculates a rate command for each source to adjust the rate of data it sends to the bottleneck node in order to regulate the queue length at the bottleneck node so that congestion is avoided. The dynamics of the queue length are given as (Quet et al. (2002)) $\dot{q}(t) = \sum_{i=1}^{n} r_i^b(t) - c(t)$, where

$q(t)$ is the queue length at the bottleneck node at time t,

$r_i^b(t)$ is the rate of data received by the bottleneck node at time t from the i^{th} source, $i = 1, \ldots, n$,

$c(t)$ is the outgoing rate of data from the bottleneck node at time t, which equals to the capacity of the outgoing link assuming that $q(t)$ is positive.

The rate of data received by the bottleneck node, $r_i^b(t)$, is given in terms of the rate command at time t, $r_i(t)$, issued by the controller as follows (Quet et al. (2002)):

$$r_i^b(t) = \begin{cases} (1 - \dot{\delta}_i^f(t)) r_i(t - \tau_i(t)), & t - \tau_i^f(t) \geq 0 \\ 0, & t - \tau_i^f(t) < 0 \end{cases} . \qquad (3)$$

Here, $\tau_i(t) = \tau_i^b(t) + \tau_i^f(t)$ is the *round-trip time-delay* at time t in channel i, where

$\tau_i^b(t) = h_i^b + \delta_i^b(t)$ is the *backward time-delay* at time t, which is the time required for the rate command to reach the i^{th} source. Here, h_i^b is the nominal time-invariant known backward time-delay, and $\delta_i^b(t)$ is the time-varying backward time-delay uncertainty,

$\tau_i^f(t) = h_i^f + \delta_i^f(t)$ is the *forward time-delay* at time t, which is the time required for the data sent from the i^{th} source to reach the bottleneck node. Here, h_i^f is the nominal time-invariant known forward time-delay, and $\delta_i^f(t)$ is the time-varying forward time-delay uncertainty.

The nominal round-trip time-delay for the i^{th} channel is then given as $h_i = h_i^b + h_i^f$ and the round-trip time-delay uncertainty is given as $\delta_i(t) = \delta_i^b(t) + \delta_i^f(t)$. It is assumed that the uncertainties are bounded as follows:

$$|\delta_i(t)| < \delta_i^+, \quad |\dot{\delta}_i(t)| < \beta_i, \quad |\dot{\delta}_i^f(t)| < \beta_i^f \qquad (4)$$

for some bounds $\delta_i^+ > 0$ and $0 < \beta_i^f \leq \beta_i < 1$. It is further assumed that, $\delta_i(t)$ is such that $\tau_i(t) \geq 0$ at all times. In a real application, there also exist some *hard constraints*, such as non-negativity constraints and upper bounds on the queue length and data rates. In this work, however, we assume that such constraints are always satisfied for the purpose of controller design. We will, however, explicitly take these constraints into account while doing simulations in Section 4.

2.2 Control Problem

It is desired to design a controller for the above described system to regulate the queue length, $q(t)$. The controlled system must be robustly stable against all time-varying uncertainties in the time-delays which satisfy (4). Moreover, assuming that $\lim_{t\to\infty} c(t) =: c_\infty$ exists, the nominal system must satisfy the *tracking* requirement:

$$\lim_{t\to\infty} q(t) = q_d \qquad (5)$$

and the *weighted fairness* requirement (Quet et al. (2002)):

$$\lim_{t\to\infty} r_i(t) = \alpha_i c_\infty , \qquad i = 1, \ldots, n . \qquad (6)$$

Here, q_d is the *desired queue length*, which is chosen as some positive number (typically half the buffer size) and $\alpha_i > 0$, $i = 1, \ldots, n$, are the *fairness weights*, which satisfy $\sum_{i=1}^{n} \alpha_i = 1$.

Now, we can describe the overall system as shown in Fig. 1, where

$P_o(s) := \frac{1}{s}\left[\, 1 \; \cdots \; 1 \,\right]$ is the *nominal plant*,

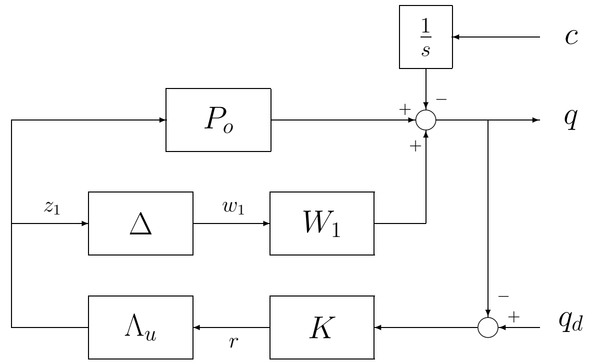

Fig. 1. Overall system (Ünal et al. (2007)).

K is the controller to be designed,
$r := [r_1 \ \cdots \ r_n]^T$,
$\Lambda_u(s) := \text{diag}\left(e^{-h_1 s}, \ldots, e^{-h_n s}\right)$ represents the nominal time-delays, which are taken outside the plant and ordered as $h_1 \geq h_2 \geq \ldots > h_n \geq 0$, in order to apply the approach of Meinsma and Mirkin (2005),
Δ is a linear time-varying block, whose $\mathcal{L}^2$-induced norm is less than 1 and which represents the uncertainties in the time-delays, and
$W_1(s) := \left[\overline{W}_1(s) \ \cdots \ \overline{W}_n(s)\right]$, where, for $i = 1, \ldots, n$, $\overline{W}_i(s) := \sqrt{2}\left[\frac{\beta_i + \beta_i^f}{s\sqrt{1-\beta_i}} \ \ 2\delta_i^+\right]$.

The structure of Δ and the derivation of W_1 can be found in Ünal et al. (2007).

To solve the control problem defined above, the mixed sensitivity minimization problem shown in Fig. 2, which was first defined by Ataşlar (2004), was considered in Ünal et al. (2007). Here, $W_2(s) := \frac{1}{s}$, $W_3(s) := \frac{\sigma_1}{s}$, and

$$W_4(s) := \frac{\sigma_2}{s}\begin{bmatrix} \frac{\alpha_2}{\alpha_1} & -1 & 0 & & 0 \\ \frac{\alpha_3}{\alpha_1} & 0 & -1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \frac{\alpha_n}{\alpha_1} & 0 & 0 & & -1 \end{bmatrix},$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$ are design parameters. Furthermore, $d := \dot{q}_d - c$, e_1 is the integral of the error, $y := q_d - q$, and is introduced to achieve tracking (5), and e_2 is introduced to achieve the weighted fairness requirement (6). The problem now is to determine a controller K which minimizes the $\mathcal{H}^\infty$ norm of the TFM from $w := [w_1^T \ \ d]^T$ to $z := [z_1^T \ \ e_1 \ \ e_2^T]^T$ in Fig. 2 with Δ block removed.

2.3 $\mathcal{H}^\infty$ Controller Design

To solve the problem defined at the end of the previous subsection, a co-prime factorization, $P_o(s) = \widetilde{M}^{-1}(s)\widetilde{N}(s)$, in $\mathcal{H}^\infty$ is necessary. Here, as in Ünal et al. (2007), we take $\widetilde{N}(s) = \frac{1}{s+\epsilon}\left[\,1 \cdots 1\,\right]$ and $\widetilde{M}(s) = \frac{s}{s+\epsilon}$ for an arbitrary $\epsilon > 0$. Then the system in Fig. 2 can be shown as in Fig. 3, where $\hat{y} := \widetilde{M}^{-1}y$ and

$$\widehat{K}(s) := K(s)\widetilde{M}(s) = \frac{s}{s+\epsilon}K(s)\,. \qquad (7)$$

Then the problem is to design a controller $\widehat{K}$ so that $\|F_l(\widehat{P}, \Lambda_u\widehat{K})\|_\infty < \gamma$, for minimum possible γ, where $F_l(\widehat{P}, \Lambda_u\widehat{K})$ is the closed-loop TFM from w to z in Fig. 3.

For a satisfactorily large given sensitivity level, γ, a controller $\widehat{K}$, which satisfies $\|F_l(\widehat{P}, \Lambda_u\widehat{K})\|_\infty < \gamma$, can be obtained by applying the approach proposed by Meinsma and Mirkin (2005), as shown in Ünal et al. (2007). The corresponding controller, K, can then be found from (7) as $K(s) = \frac{s+\epsilon}{s}\widehat{K}(s)$. This controller has the structure shown in Fig. 4, where $\kappa := \frac{\gamma}{2\sqrt{2\sum_{i=1}^{n}(\delta_i^+)^2}}$ is a constant, F_1 and F_2 are blocks which consist of delays and finite impulse response (FIR) filters, G_Λ is a bistable finite dimensional TFM, and Q_Λ is a contractive, but otherwise arbitrary, TFM (see Ünal et al. (2007) for details).

The optimal controller, K_{opt}, can be found by an iterative procedure on γ as proposed in Ünal et al. (2007). The minimum γ, for which there exists a solution to $\|F_l(\widehat{P}, \Lambda_u\widehat{K})\|_\infty < \gamma$, is denoted by γ^{opt}. Then, $K_{opt}(s) = \frac{s+\epsilon}{s}\widehat{K}_{opt}(s)$, where $\widehat{K}_{opt}$ satisfies $\|F_l(\widehat{P}, \Lambda_u\widehat{K}_{opt})\|_\infty < \gamma^{opt}$.

Any controller K, including K_{opt}, obtained as above is unstable due to the integral term (see Fig. 4). This is in

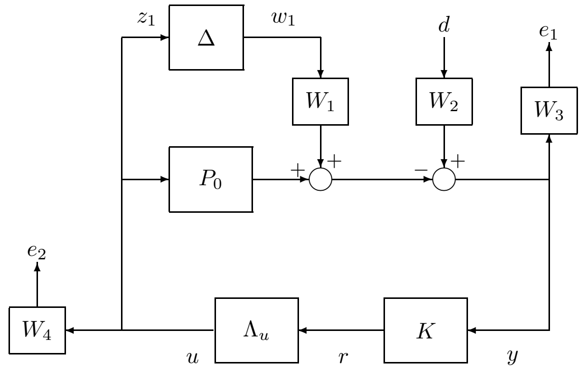

Fig. 2. System for the mixed sensitivity minimization problem (Ataşlar (2004)).

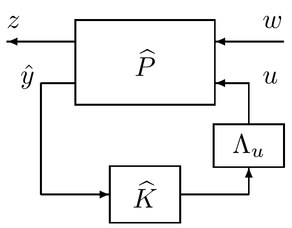

Fig. 3. Equivalent four-block problem (Ünal et al. (2007)).

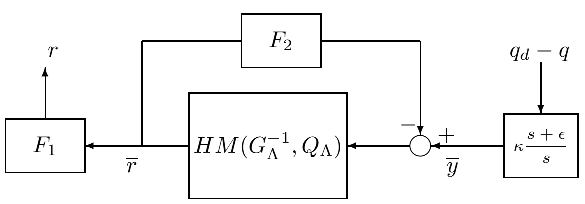

Fig. 4. Structure of the controller K.

fact a desirable property which ensures tracking. The rest of the controller, i.e., the part from $\bar{y}$ to r in Fig. 4 (which differs from $\widehat{K}$ only by a constant), may or may not be stable. When this part (equivalently $\widehat{K}$) is unstable, due to nonlinearities in the system (i.e., the hard constraints), an unstable behaviour may be observed, at least for certain actual delays and/or initial conditions (see Section 4). In order to avoid such undesirable behaviour, in the next section we will propose a design methodology which ensures that the TFM from $\bar{y}$ to r in Fig. 4 is stable.

3. PROPOSED DESIGN METHODOLOGY

In this section, we will propose a design methodology to obtain a stable (apart from the integral action) controller to solve the problem presented in the previous section. For this, the TFM from $\bar{y}$ to r in Fig. 4 must be stable. Note that both F_1 and F_2 in Fig. 4 are stable, since they contain only delays and FIR filters. Therefore, it is sufficient to ensure that the TFM from $\bar{y}$ to $\bar{r}$ in Fig. 4 is stable. Furthermore, $\gamma_{F_2} := \|F_2\|_\infty$ is finite. Therefore, by the small gain theorem (e.g., see Zhou et al. (1996)), the mapping from $\bar{y}$ to $\bar{r}$ is stable if

$$\|HM(G_\Lambda^{-1}, Q_\Lambda)\|_\infty < 1/\gamma_{F_2} \, . \tag{8}$$

Therefore, to solve our problem, we will try to find a controller which satisfies (8) in addition to the condition $\|Q_\Lambda\|_\infty < 1$, which is required for the robust stability of the overall system (see Subsection 2.3).

Let us partition G_Λ^{-1} as $G_\Lambda^{-1} =: \begin{bmatrix} \widehat{G}_{11} & \widehat{G}_{12} \\ \widehat{G}_{21} & \widehat{G}_{22} \end{bmatrix}$, where $\widehat{G}_{11}$ is $n \times n$ and $\widehat{G}_{22}$ is 1×1. Then, $HM(G_\Lambda^{-1}, Q_\Lambda) = (\widehat{G}_{11}Q_\Lambda + \widehat{G}_{12})(\widehat{G}_{21}Q_\Lambda + \widehat{G}_{22})^{-1}$. Thus, (8) is satisfied if

$$\|(\widehat{G}_{11}Q_\Lambda + \widehat{G}_{12})(\gamma_{F_2}^{-1}\widehat{G}_{21}Q_\Lambda + \gamma_{F_2}^{-1}\widehat{G}_{22})^{-1}\|_\infty < 1 \, . \tag{9}$$

By defining

$$G_{\Lambda_\gamma}^{-1} := \begin{bmatrix} I_n & 0 \\ 0 & \gamma_{F_2}^{-1} \end{bmatrix} G_\Lambda^{-1} = \begin{bmatrix} \widehat{G}_{11} & \widehat{G}_{12} \\ \gamma_{F_2}^{-1}\widehat{G}_{21} & \gamma_{F_2}^{-1}\widehat{G}_{22} \end{bmatrix} , \tag{10}$$

(9) is equivalent to that

$$S := HM(G_{\Lambda_\gamma}^{-1}, Q_\Lambda) \tag{11}$$

is contractive.

To find a condition to guarantee contractiveness of Q_Λ, let us also partition G_{Λ_γ} as $G_{\Lambda_\gamma} = \begin{bmatrix} \overline{G}_{11} & \overline{G}_{12} \\ \overline{G}_{21} & \overline{G}_{22} \end{bmatrix}$, where $\overline{G}_{11}$ is $n \times n$ and $\overline{G}_{22}$ is 1×1. Then, using (11) and (2), we obtain

$$Q_\Lambda = HM(G_{\Lambda_\gamma}, S) = (\overline{G}_{11}S + \overline{G}_{12})(\overline{G}_{21}S + \overline{G}_{22})^{-1} . \tag{12}$$

Following Lee and Soh (2005), let us introduce a nonzero *tuning parameter* λ, so that

$$Q_\Lambda = (\lambda\overline{G}_{11}S + \lambda\overline{G}_{12})(\lambda\overline{G}_{21}S + \lambda\overline{G}_{22})^{-1} \, . \tag{13}$$

Let us also define $U := \lambda(\overline{G}_{11}S + \overline{G}_{12})$, $V := 1 - \lambda(\overline{G}_{21}S + \overline{G}_{22})$, and $\Gamma := \sqrt{2}\begin{bmatrix} U \\ V \end{bmatrix}$. Then

$$Q_\Lambda = U(1 - V)^{-1} = HM(G_\Gamma, \Gamma) \, , \tag{14}$$

where $G_\Gamma := \begin{bmatrix} [\, I_n \;\; 0 \,] & 0 \\ [\, 0 \;\; -1 \,] & \sqrt{2} \end{bmatrix}$ and it satisfies $G_\Gamma^T J_{n,1} G_\Gamma \leq J_{(n+1),1}$. This property of G_Γ can be used to present a sufficient condition for the contractiveness of Q_Λ. Let us define z_Γ, w_Γ, u_Γ and y_Γ such that $\begin{bmatrix} z_\Gamma \\ w_\Gamma \end{bmatrix} = G_\Gamma \begin{bmatrix} u_\Gamma \\ y_\Gamma \end{bmatrix}$ and $u_\Gamma = \Gamma y_\Gamma$. Then, by (14), $z_\Gamma = Q_\Lambda w_\Gamma$. Since $G_\Gamma^T J_{n,1} G_\Gamma \leq J_{(n+1),1}$, then

$$\begin{aligned} \begin{bmatrix} z_\Gamma \\ w_\Gamma \end{bmatrix}^T J_{n,1} \begin{bmatrix} z_\Gamma \\ w_\Gamma \end{bmatrix} &= \begin{bmatrix} u_\Gamma \\ y_\Gamma \end{bmatrix}^T G_\Gamma^T J_{n,1} G_\Gamma \begin{bmatrix} u_\Gamma \\ y_\Gamma \end{bmatrix} \\ &\leq \begin{bmatrix} u_\Gamma \\ y_\Gamma \end{bmatrix}^T J_{(n+1),1} \begin{bmatrix} u_\Gamma \\ y_\Gamma \end{bmatrix} \end{aligned}$$

which gives $z_\Gamma^T z_\Gamma - w_\Gamma^T w_\Gamma \leq u_\Gamma^T u_\Gamma - y_\Gamma^T y_\Gamma$, which implies

$$\|z_\Gamma\|_2^2 - \|w_\Gamma\|_2^2 \leq \|u_\Gamma\|_2^2 - \|y_\Gamma\|_2^2 \, . \tag{15}$$

Since $u_\Gamma = \Gamma y_\Gamma$, if Γ is contractive, $\|u_\Gamma\|_2 < \|y_\Gamma\|_2$. Then, from (15), $\|z_\Gamma\|_2 < \|w_\Gamma\|_2$. However, since $z_\Gamma = Q_\Lambda w_\Gamma$, this implies contractiveness of Q_Λ. Therefore, Q_Λ is contractive if Γ is contractive. Furthermore, recall that S must also be contractive. These two conditions are simultaneously satisfied if

$$\left\| \begin{bmatrix} \Gamma \\ S \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} \sqrt{2} \begin{bmatrix} \lambda(\overline{G}_{11}S + \overline{G}_{12}) \\ 1 - \lambda(\overline{G}_{21}S + \overline{G}_{22}) \end{bmatrix} \\ S \end{bmatrix} \right\|_\infty < 1 \, . \tag{16}$$

Let us define $G_G := \begin{bmatrix} \sqrt{2}\lambda\overline{G}_{11} & \vdots & \sqrt{2}\lambda\overline{G}_{12} \\ -\sqrt{2}\lambda\overline{G}_{21} & \vdots & \sqrt{2}(1 - \lambda\overline{G}_{22}) \\ I_n & \vdots & 0 \\ \cdots & \cdot & \cdots \\ 0 & \vdots & 1 \end{bmatrix}$. Then, $HM(G_G, S) = \begin{bmatrix} \Gamma \\ S \end{bmatrix}$. Therefore, condition (16) is satisfied if and only if $HM(G_G, S)$ is contractive. As a result, the problem of finding a contractive Q_Λ which satisfies (8) is solved if there exists $\lambda > 0$ and a contractive S such that $\|HM(G_G, S)\|_\infty < 1$. This condition is equivalent to that there exists a $(\bar{J}, \hat{J})$-lossless factorization of G_G such as $G_G = \Theta_G \Phi_G$, where $\bar{J} := J_{(n+1+n),1}$, $\hat{J} := J_{n,1}$, Θ_G is $(\bar{J}, \hat{J})$-lossless and Φ_G is bistable (Kimura (1996)). Since, by (1), $HM(G_G, S) = HM(\Theta_G, HM(\Phi_G, S))$, $(\bar{J}, \hat{J})$-lossless property of Θ_G implies that $\|HM(G_G, S)\|_\infty < 1$ if and only if $\|HM(\Phi_G, S)\|_\infty < 1$ (Kimura (1996)). Therefore, if an arbitrary but contractive $Q_G := HM(\Phi_G, S)$ is chosen, then, using (2), contractive S is obtained as

$$S = HM(\Phi_G^{-1}, Q_G) \, . \tag{17}$$

Then, a contractive Q_Λ which satisfies (8) is obtained by (12).

To find the state-space solution of $(\bar{J}, \hat{J})$-lossless factorization of G_G, note that $\lim_{s\to\infty} G_\Lambda^{-1}(s) = I_{n+1}$ (Ünal et al. (2007)) and let a minimal realization of G_Λ^{-1} be given as

$$G_\Lambda^{-1} = \left[\begin{array}{c|cc} \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline \widehat{C}_1 & I_n & 0 \\ \widehat{C}_2 & 0 & 1 \end{array} \right] . \tag{18}$$

Then a minimal realization of $G_{\Lambda_\gamma}^{-1}$ is given as

$$G_{\Lambda_\gamma}^{-1} = \left[\begin{array}{c|cc} \widehat{A} & \widehat{B}_1 & \widehat{B}_2 \\ \hline \widehat{C}_1 & I_n & 0 \\ \gamma_{F_2}^{-1}\widehat{C}_2 & 0 & \gamma_{F_2}^{-1} \end{array}\right] . \quad (19)$$

Thus, a minimal realization of G_{Λ_γ} is given as

$$G_{\Lambda_\gamma} = \left[\begin{array}{c|cc} \widehat{A} - \widehat{B}_1\widehat{C}_1 - \widehat{B}_2\widehat{C}_2 & \widehat{B}_1 & \gamma_{F_2}\widehat{B}_2 \\ \hline -\widehat{C}_1 & I_n & 0 \\ -\widehat{C}_2 & 0 & \gamma_{F2} \end{array}\right] . \quad (20)$$

Therefore, a minimal realization of G_G can be obtained as

$$G_G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array}\right] , \quad (21)$$

where $A_G := \widehat{A} - \widehat{B}_1\widehat{C}_1 - \widehat{B}_2\widehat{C}_2$, $B_G := \left[\, \widehat{B}_1 \ \gamma_{F_2}\widehat{B}_2 \,\right]$,

$$C_G := \begin{bmatrix} -\sqrt{2}\lambda\widehat{C}_1 \\ \sqrt{2}\lambda\widehat{C}_2 \\ 0 \\ 0 \end{bmatrix}, \text{ and } D_G := \begin{bmatrix} \sqrt{2}\lambda I_n & 0 \\ 0 & \sqrt{2}(1-\lambda\gamma_{F_2}) \\ I_n & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that, since G_Λ is bistable, from (18), both $\widehat{A}$ and $\widehat{A} - \widehat{B}_1\widehat{C}_1 - \widehat{B}_2\widehat{C}_2 = A_G$ are Hurwitz. Thus, G_G is stable.

The state-space solution of $(\bar{J}, \hat{J})$-lossless factorization of G_G is given in the following theorem, which is taken from Kimura (1996).

Theorem 1. Consider the stable system G_G and its minimal realization given in (21). G_G has a $(\bar{J}, \hat{J})$-lossless factorization, $G_G = \Theta_G\Phi_G$, if and only if there exists a nonsingular matrix E_G, such that

$$D_G^T\bar{J}D_G = E_G^T\hat{J}E_G , \quad (22)$$

and a solution $X_G \geq 0$ to the following Riccati equation

$$X_GA_G + A_G^TX_G - Y_G^T(D_G^T\bar{J}D_G)^{-1}Y_G + C_G^T\bar{J}C_G = 0 \quad (23)$$

such that $A_\pi := A_G + B_GF_G$ is Hurwitz, where $F_G := -(D_G^T\bar{J}D_G)^{-1}Y_G$ and $Y_G := D_G^T\bar{J}C_G + B_G^TX_G$. In that case, $\Theta_G = \left[\begin{array}{c|c} A_\pi & B_G \\ \hline C_G + D_GF_G & D_G \end{array}\right] E_G^{-1}$ and

$$\Phi_G = E_G \left[\begin{array}{c|c} A_\pi & -B_G \\ \hline F_G & I_{n+1} \end{array}\right] . \quad (24)$$

Note that, $D_G^T\bar{J}D_G = \begin{bmatrix} (2\lambda^2+1)I_n & 0 \\ 0 & -d \end{bmatrix}$, where $d := 4\lambda\gamma_{F_2} - 2\lambda^2\gamma_{F_2}^2 - 1$. Suppose E_G is nonsingular and let $V := E_G^{-1}$. Then, from (22), $V^TD_G^T\bar{J}D_GV = \hat{J}$. Let $y := Vx$, where $x := [\,0 \cdots 0 \ 1\,]^T$. Then, $y^TD_G^T\bar{J}D_Gy = x^T\hat{J}x = -1$, which implies that $D_G^T\bar{J}D_G$ must have at least one negative eigenvalue. However, since $D_G^T\bar{J}D_G = \begin{bmatrix} (2\lambda^2+1)I_n & 0 \\ 0 & -d \end{bmatrix}$, and $2\lambda^2+1 > 0$, we must have $-d < 0$ if E_G is nonsingular. Equivalently, E_G is nonsingular only if $d > 0$. On the other hand, if $d > 0$, a nonsingular E_G can be obtained as $E_G = \begin{bmatrix} \sqrt{2\lambda^2+1}I_n & 0 \\ 0 & \sqrt{d} \end{bmatrix}$. Therefore, a nonsingular E_G satisfying (22) exists if and only if $d > 0$. However, note that, $d > 0$ if and only if λ is chosen in the interval $\left(\frac{\sqrt{2}-1}{\sqrt{2}\gamma_{F_2}}, \frac{\sqrt{2}+1}{\sqrt{2}\gamma_{F_2}}\right)$.

Therefore, a controller which solves the problem of Subsection 2.2 and which is stable apart from the integral action can be obtained by the following algorithm.

Algorithm 1:

1. Find the optimal sensitivity level γ^{opt} (see Subsection 2.3) and let $\gamma = \gamma^{opt}$.
2. Find F_1, F_2, and G_Λ (see Subsection 2.3) for the current sensitivity level γ. Also compute $\gamma_{F_2} := \|F_2\|_\infty$. Choose a sufficiently large l and equally spaced values $\lambda_1, \lambda_2, \ldots, \lambda_l$ within the interval $\left(\frac{\sqrt{2}-1}{\sqrt{2}\gamma_{F_2}}, \frac{\sqrt{2}+1}{\sqrt{2}\gamma_{F_2}}\right)$. Let $i = 1$.
3. For $\lambda = \lambda_i$, if there exists a solution $X_G \geq 0$ to the Riccati equation (23), go to step 6. Otherwise, continue with step 4.
4. If $i = l$, go to step 5. Otherwise, set $i = i + 1$ and go to step 3.
5. Increase γ by a small amount and go to step 2.
6. Let $Q_\Lambda = HM(G_{\Lambda_\gamma}, S)$, where, by (10), $G_{\Lambda_\gamma} = G_\Lambda \begin{bmatrix} I_n & 0 \\ 0 & \gamma_{F_2} \end{bmatrix}$ and, by (17), $S = HM(\Phi_G^{-1}, Q_G)$, where Φ_G is given by (24) and Q_G is contractive but otherwise arbitrary. The desired controller is then given by (see Fig. 4)

$$K(s) = F_1(s)H(s)\frac{\kappa(s+\epsilon)}{s(1+F_2(s)H(s))} , \quad (25)$$

where $H := HM(G_\Lambda^{-1}, Q_\Lambda)$ and $\kappa := \frac{\gamma}{2\sqrt{2\sum_{i=1}^n(\delta_i^+)^2}}$.

4. ILLUSTRATIVE EXAMPLE

In this section, a network with two sources is considered as an example to demonstrate the time-domain performance of the controllers obtained by the proposed design approach. Simulations are done using MATLAB Simulink, where non-linear effects (hard constraints) are also taken into account. We consider two cases. In the first one the nominal delays in both channels are the same; in the second case they are different. In both cases, we take $h_i^f = h_i^b = \frac{1}{2}h_i$, $i = 1, 2$. Moreover, the desired queue length, q_d, is taken as 30 packets and the buffer size (maximum queue length) is taken as 60 packets. The capacity of the outgoing link is taken as 90 packets/second. The rate limits for the sources are taken as 150 packets/second. In both cases the optimal controller excluding the integral action (i.e., $\widehat{K}_{opt}$) turns out to be unstable. We present the results for both the optimal and the proposed controller. For the optimal controller, we take $Q_\Lambda = 0$ and for the proposed controller, we take $Q_G = 0$. In Figures 5–11, q (whose scale is on the right) is the queue length and r_1^s and r_2^s (whose scale is on the left) are the actual flow rates at source 1 and 2, respectively ($r_i^s(t) = r_i(t - \tau_i^b(t))$, $i = 1, 2$).

Case 1: We let $h_1 = h_2 = 2$, $\delta_1^+ = 0.5$, $\delta_2^+ = 1$, $\beta_1 = \beta_2 = 0.6$, $\beta_1^f = \beta_2^f = 0.3$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$, and $\sigma_1 = \sigma_2 = 0.25$. Applying the approach of Subsection 2.3, the optimal sensitivity level is obtained as $\gamma^{opt} = 6.5623$. We apply the optimal controller to the system for two different actual

Table 1. Uncertain part of the actual time-delays.

Case	i	$\delta_i^b(t)$	$\delta_i^f(t)$
1a	1,2	$0.3{+}0.3\sin(\frac{2\pi}{50}t)$	$0.1{+}0.1\sin(\frac{2\pi}{100}t)$
1b	1,2	$1.3{+}0.3\sin(\frac{2\pi}{50}t)$	$0.1{+}0.1\sin(\frac{2\pi}{100}t)$
1c	1,2	$1.3{+}0.3\sin(\frac{2\pi}{50}t)$	0.1
2	1	$1.3{+}0.3\sin(\frac{2\pi}{30}t)$	$0.1{+}0.1\sin(\frac{2\pi}{100}t)$
	2	$0.3{+}0.3\sin(\frac{2\pi}{50}t)$	$0.1{+}0.1\sin(\frac{2\pi}{100}t)$

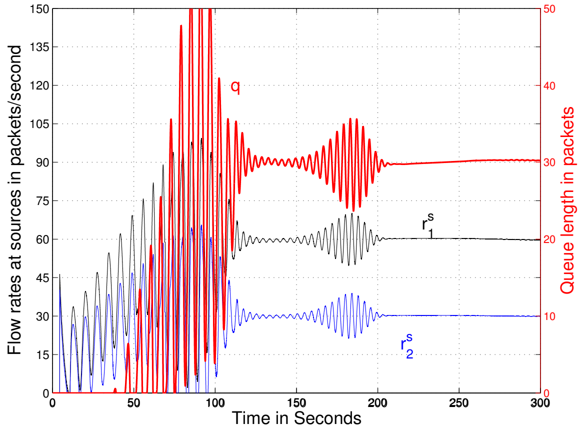

Fig. 5. Results for the optimal controller for Case 1a.

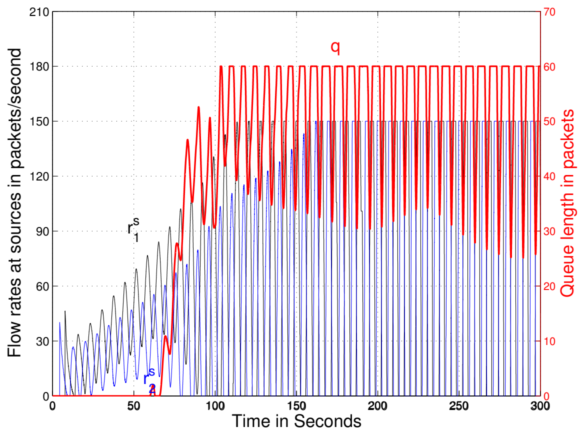

Fig. 6. Results for the optimal controller for Case 1b.

time-delays, whose uncertain parts are given in Table 1 as cases 1a and 1b. The results are shown in Figures 5 and 6, respectively. In case 1a, where the uncertain parts of the actual delays are relatively smaller, the optimal controller can recover following some transient oscillations. However, in case 1b, although the linear closed-loop system is stable, a totally unstable response is obtained due to the nonlinear effects.

We also design a controller which is stable apart from the integral action, using Algorithm 1. The sensitivity level obtained for this controller is $\gamma = 12.2123$. We then apply this controller to the system for the same actual time-delays as in cases 1a and 1b. The results are shown in Figures 7 and 8, respectively. It is seen that, unlike the optimal controller, the proposed controller produces a smooth response in both cases. Both the queue length and the flow rates oscillate around their desired values (given by (5) and (6), respectively) at steady-state. The reason for these oscillations is the time-varying forward delays, and they are unavoidable, unless the actual delays

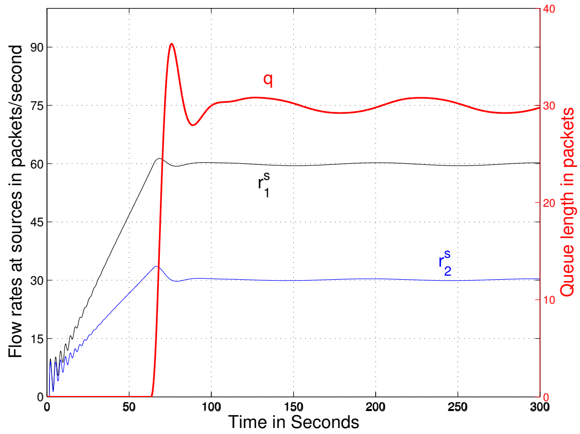

Fig. 7. Results for the proposed controller for Case 1a.

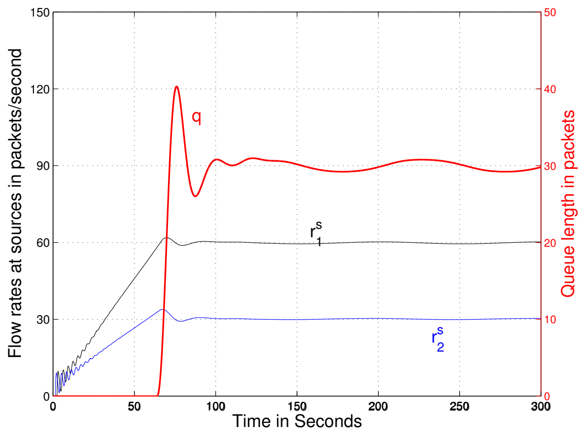

Fig. 8. Results for the proposed controller for Case 1b.

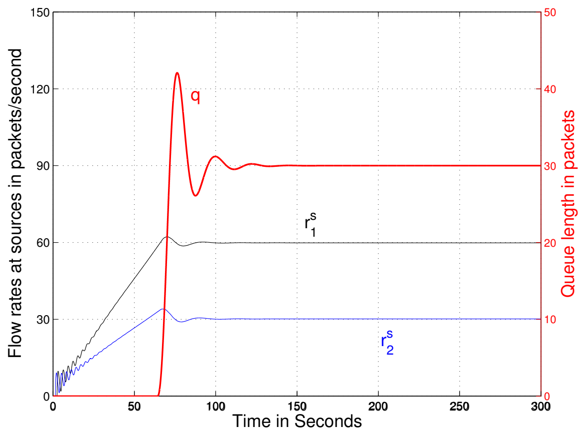

Fig. 9. Results for the proposed controller for Case 1c.

are known by the controller (see Quet et al. (2002)). If the forward delays are taken time-invariant, as in case 1c in Table 1, for example, these oscillations disappear as shown in Fig. 9. In this case, the proposed controller robustly stabilizes the overall system and satisfies both the tracking (5) and fairness (6) requirements exactly, despite time-varying uncertain backward time-delays and time-invariant uncertain forward time-delays.

Case 2: We let $h_1 = 4$, $h_2 = 2$, $\delta_1^+ = 0.5$, $\delta_2^+ = 1$, $\beta_1 = 0.6$, $\beta_2 = 0.4$, $\beta_1^f = \beta_2^f = 0.3$, $\alpha_1 = \frac{2}{3}$, $\alpha_2 = \frac{1}{3}$, and

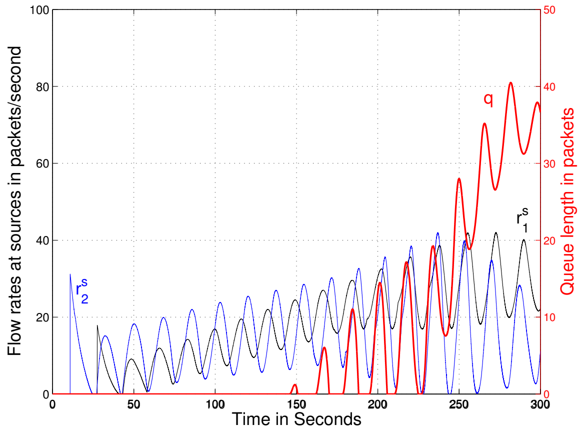

Fig. 10. Results for the optimal controller for Case 2.

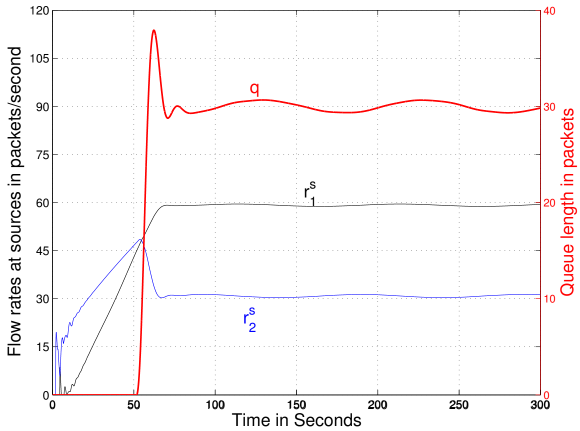

Fig. 11. Results for the proposed controller for Case 2.

$\sigma_1 = \sigma_2 = 0.25$. Applying the approach of Subsection 2.3, the optimal sensitivity level is obtained as $\gamma^{opt} = 7.6090$. We apply the optimal controller to the system for actual time-delays, whose uncertain parts are given in Table 1 as case 2. The results are shown in Fig. 10, where an unstable behaviour is observed. For the controller designed using Algorithm 1, the sensitivity level obtained is $\gamma = 12.2090$. The results for this controller are shown in Fig. 11, for the same actual time-delays. As in case 1, a smooth response is obtained, where the queue length and the flow rates oscillate around their desired values at steady-state.

5. CONCLUSION

Stable $\mathcal{H}^\infty$ controller design has been considered for systems which involve multiple time-delays. Flow control problem in data-communication networks has been chosen to present the proposed design approach. The main result, which is summarized as Algorithm 1, however, can be applied to any system which involve multiple time-delays, once the problem is stated as in Subsection 2.2, and the steps of Subsection 2.3 are applied.

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