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Observer Based Controller Design for Linear Systems with Input Constraints ⋆

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Abstract: A systematic design method for observer based linear control of LTI systems with input constraints is introduced. The method allows to optimize the observer parameters with respect to the system's performance while at the same time the compliance with the constraints is guaranteed. To improve the results we further propose a method to design both the controller and the observer simultaneously. Since the presented methods are based on LMI techniques they are computationally very efficient. The proposed methods are demonstrated by means of an example.

Keywords: Linear Systems with Input Constraints, Observers, Actuator Saturation, LMIs

1. INTRODUCTION

A lot of research has been carried out on the control of linear systems with input constraints, as input constraints are probably the most important type of nonlinearity encountered in real world systems. Most research on this topic is focused on one of the following strategies:

- Model Predictive Control, where the input saturation is taken into account as a constraint of the optimization problem at each iteration, see e.g. Camacho and Bordons (2004),
- Anti-Windup, where an additional nonlinear controller is added to counteract the negative consequences of saturation, see e.g. Kothare et al. (1994) and Hippe (2006),
- the design of an appropriate state feedback controller such that stability is guaranteed even under saturation or such that saturation does not occur at all, as in Hu and Lin (2001), Hu et al. (2002), Saberi et al. (2000), and Adamy and Flemming (2004).

In this paper we deal with the design of observers for controllers of the latter type. The existing approaches for the design of observers for this kind of controls (Hu and Lin, 2001; Saberi et al., 2000; Shi et al., 2002) guarantee asymptotic stability. The usual strategy to obtain a good performance is to choose a very high observer gain. Using an observer with high gain results in a very fast decay of the estimation error, such that the system behaves like under state feedback after a short time span and thus reaches approximately the same performance as under state feedback.

Unfortunately, a system with a very fast observer is sensitive to noisy measurements and modeling inaccuracies. If on the other hand the observer gain is chosen lower to avoid these issues, there is no guarantee that the performance will be acceptable. Additionally, the existing methods offer no possibility to guarantee that the input constraint is not violated by the control law, i.e. that saturation does not occur. This may be a design specification, e.g. if using the maximum input during a longer time span might damage the actuator.

In this paper we develop a constructive method to design a classical Luenberger observer for the stabilization of systems with input constraints such that the performance is optimized and the constraints are not violated. The method is applicable to both stable and unstable systems of any order. Our method offers an effective design procedure and leads to a simple observer based control structure with reasonable observer gain.

2. PROBLEM STATEMENT

Consider the following linear time invariant system with a linear feedback control law:

$$\Sigma: \begin{cases} \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}, & \text{(1a)} \\ \boldsymbol{y} = \mathbf{C}\boldsymbol{x}, & \text{(1b)} \\ \boldsymbol{y}_M = \mathbf{C}_M\boldsymbol{x}, & \text{(1c)} \\ \boldsymbol{u} = -\mathbf{K}\boldsymbol{x}, & \text{(1d)} \\ |u_i| \le u_{\max,i}, \quad i = 1, \dots p, & \text{(1e)} \\ \boldsymbol{x}(0) \in \mathcal{X}_0, & \text{(1f)} \end{cases}$$

with the state vector $\boldsymbol{x} \in \mathbb{R}^n$, the plant input $\boldsymbol{u} \in \mathbb{R}^p$, and the output $\boldsymbol{y} \in \mathbb{R}^q$. By $\boldsymbol{y}_M$ we denote the vector of measured outputs. The absolute value $|u_i|$ of each input is constrained by $u_{\max,i}$.

The set of possible initial values, denoted by $\mathcal{X}_0$, is assumed to be a convex subset of the null controllable region $\mathcal{C}$. This region contains all states for which there exists a control law such that the trajectory $\boldsymbol{x}(t)$ of Σ reaches the origin in finite time (Hu and Lin, 2001). In case of stable plant dynamics $\mathcal{C}$ is the complete state space $\mathbb{R}^n$.

To allow the stabilization of Σ using measurement feedback, the system is extended by a Luenberger observer to

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$$\Omega : \begin{cases} \dot{\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{B}\boldsymbol{u}, & (2a)\\ \dot{\tilde{\boldsymbol{x}}} = \mathbf{A}\tilde{\boldsymbol{x}} + \mathbf{B}\boldsymbol{u} + \mathbf{L}\mathbf{C}_M(\boldsymbol{x} - \tilde{\boldsymbol{x}}), & (2b)\\ \boldsymbol{u} = -\mathbf{K}\tilde{\boldsymbol{x}}, & (2c) \end{cases}$$

where $\boldsymbol{x}$ is the state of the plant, $\tilde{\boldsymbol{x}}$ the state of the observer, and $\mathbf{L}$ the observer feedback matrix.

For ease of exposition a full observer is chosen, but the results can readily be extended to reduced observers. The composite system Ω can be written in the coordinates $\boldsymbol{x}$ and $\boldsymbol{e}$ as the familiar equations

$$\dot{\boldsymbol{x}} = (\mathbf{A} - \mathbf{BK})\,\boldsymbol{x} + \mathbf{BK}\boldsymbol{e}, \quad (3a)$$

$$\dot{\boldsymbol{e}} = (\mathbf{A} - \mathbf{LC}_M)\,\boldsymbol{e}, \quad (3b)$$

where $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$ is the observation error.

We address the following stabilization problem:

Problem 1. Find, if possible, an observer based linear control law (2b), (2c), i.e. suitable matrices $\mathbf{K}$ and $\mathbf{L}$, such that the system Ω is stabilized for all initial values $\boldsymbol{x}(0)$ and $\tilde{\boldsymbol{x}}(0)$ in given convex sets $\mathcal{X}_0$ and $\tilde{\mathcal{X}}_0$ with guaranteed performance while respecting the input constraint $|u_i| \leq u_{\max,i}$.

Remark 1. The input constraint is considered to be a *hard* constraint which must not be exceeded by the control law. It should be noted, however, that the approach of this paper can easily be extended in order to consider linear control laws that are allowed to saturate by incorporating invariance analysis methods from Hu and Lin (2001) or Alamo et al. (2005). For reasons of space and simpler presentation we omit these possible extensions.

Our approach to solve Problem 1 is as follows: first, we recall the design of a stabilizing linear state feedback matrix $\mathbf{K}$ for Σ in Section 3. In Section 4 we deal with the region where the control action is linear. Then, in Section 5, we propose a method to design an observer $\mathbf{L}$ such that Problem 1 is solved by $\mathbf{K}$ and $\mathbf{L}$. Finally, a method for the simultaneous design of controller and observer is proposed in Section 6. It is based on the previous methods and leads to improved results. In the last section we demonstrate the effectiveness of the proposed method on the basis of an example.

3. CONTROLLER DESIGN – STATE FEEDBACK

First we consider the design of a controller $\mathbf{K}$ for the linear feedback law (1d) of system Σ, i. e. using state feedback without an observer. The controller should exhibit good performance – which, of course, implies stability – for all initial values $\boldsymbol{x}_0$ in a given set $\mathcal{X}_0$ without any element of the input $\boldsymbol{u}$ saturating.

This design problem can be cast as a convex optimization problem with constraints in the form of linear matrix inequalities (LMIs). We will briefly introduce the optimization problem and the LMIs related to the requirements imposed on the controller. For a detailed derivation of the LMIs we refer to Boyd et al. (1994).

We use the following notation in the context of LMIs: By $\mathbf{X}\succ(\succeq)0$ we mean that the matrix $\mathbf{X}$ is symmetric and positive (semi)definite, analogously by $\mathbf{X}\prec(\preceq)0$ we mean that $\mathbf{X}$ is symmetric and negative (semi)definite. The convex hull of a set $\mathcal{A}$ is denoted by $\operatorname{conv}\mathcal{A}$.

The performance requirement can be formulated as: Find a set $\mathcal{G} \subset \mathbb{R}^n$ and a minimum value for γ such that $\mathcal{G}$ is positively invariant, $\mathcal{X}_0$ is a subset of $\mathcal{G}$, and γ is an upper bound on the output energy

$$J = \int_0^\infty \boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}\,\mathrm{d}t \quad (4)$$

over $\mathcal{G}$, i.e.

$$J \leq \gamma \quad \forall \boldsymbol{x}(0) \in \mathcal{G} \supseteq \mathcal{X}_0. \quad (5)$$

By defining the set $\mathcal{G}$ using a Lyapunov function,

$$\mathcal{G} = \{\boldsymbol{x}^{\mathrm{T}}\mathbf{R}\boldsymbol{x} \leq 1\}, \quad \mathbf{R} \succ 0, \quad (6)$$

Eq. (5) is guaranteed to be fulfilled if the matrix inequality

$$\begin{bmatrix} \mathbf{QA}^{\mathrm{T}} + \mathbf{AQ} - \mathbf{BW} - \mathbf{W}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} & \mathbf{QC}^{\mathrm{T}} \\ \mathbf{CQ} & -\gamma\mathbf{I}_q \end{bmatrix} \prec 0 \quad (7)$$

is met, where $\mathbf{Q} = \mathbf{R}^{-1}$ and the substitution

$$\mathbf{W} = \mathbf{KR}^{-1} \quad (8)$$

has been used in order to obtain a *linear* matrix inequality.

If $\mathcal{X}_0$ is a convex polytope given by

$$\mathcal{X}_0 = \operatorname{conv}\{\boldsymbol{x}_{0,1}, \ldots, \boldsymbol{x}_{0,k}\}, \quad (9)$$

we can express the requirement $\mathcal{X}_0 \subseteq \mathcal{G}$ as the set of LMIs

$$\begin{bmatrix} \mathbf{Q} & \boldsymbol{x}_{0,i} \\ \boldsymbol{x}_{0,i}^{\mathrm{T}} & 1 \end{bmatrix} \succeq 0, \quad i = 1, \ldots, k. \quad (10)$$

Finally, we demand that $|\mathbf{K}\boldsymbol{x}| \leq \boldsymbol{u}_{\max}$. This can be cast as the matrix inequalities

$$\begin{bmatrix} \mathbf{Q} & \mathbf{W}^{\mathrm{T}} \\ \mathbf{W} & \mathbf{X} \end{bmatrix} \succeq 0, \quad (11a)$$

$$\mathbf{X}_{ii} < u_{\max,i}^2, \quad (11b)$$

where $\mathbf{X}$ is a slack matrix variable (Boyd et al., 1994).

The final optimization problem is then

$$\min \gamma \quad (12a)$$

such that

$$\mathbf{Q} \succ 0, \quad (12b)$$

$$(7), (10), (11).$$

From the solution of this optimization problem the controller can then be found as $\mathbf{K} = \mathbf{WQ}^{-1}$.

4. LINEAR REGION

Before we turn to the design of the observer, i.e. the problem how to choose the observer matrix $\mathbf{L}$, we first deal with the problem of estimating the linear region for a given $\mathbf{L}$. The linear region $\mathcal{L}$ is the region where the input remains unsaturated at all times $t \geq t_0$:

$$\mathcal{L} = \left\{ \begin{bmatrix} \boldsymbol{x}(t_0) \\ \boldsymbol{e}(t_0) \end{bmatrix} \middle| \, |\mathbf{K}(\boldsymbol{x}(t) - \boldsymbol{e}(t))|_i \leq u_{\max,i}\, \forall t \geq t_0 \right\}. \quad (13)$$

The representation of the estimate set, $\mathcal{L}_e$, is particularly simple if we define it using a Lyapunov function $v(\boldsymbol{x}, \boldsymbol{e})$ as

$$\mathcal{L}_e = \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} \middle| \, v(\boldsymbol{x}, \boldsymbol{e}) = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix}^{\mathrm{T}} \mathbf{P} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} \leq 1 \right\}, \quad (14)$$

where $\mathbf{P} \succ 0$, $\mathbf{P} \in \mathbb{R}^{2n\times 2n}$.

To be a correct estimate of $\mathcal{L}$, $\mathcal{L}_e$ must be positively invariant and the input must be unsaturated over $\mathcal{L}_e$. First we show how these conditions can be cast as LMIs. In a second step, we deal with the optimization of $\mathbf{P}$ such that the volume of the region $\mathcal{L}_e$ is maximized.

For $\mathcal{L}_e$ to be an invariant set with respect to the system Ω, the LMIs

$$\mathbf{P} \succ 0, \quad \hat{\mathbf{A}}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\hat{\mathbf{A}} \prec 0, \tag{15}$$

with

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC}_M \end{bmatrix}, \tag{16}$$

have to be met by the matrix variable $\mathbf{P}$.

The input must meet the constraints (1e) for all states contained in the set $\mathcal{L}_e$. The input can be written as a function of $\boldsymbol{x}$ and $\boldsymbol{e}$ as

$$\boldsymbol{u} = -\mathbf{K}\tilde{\boldsymbol{x}} = -\left[\mathbf{K}\ -\mathbf{K}\right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix}, \tag{17}$$

resulting – analogously to (11) – in the LMIs

$$\begin{bmatrix} \mathbf{P} & \begin{matrix} \mathbf{K}^{\mathrm{T}} \\ -\mathbf{K}^{\mathrm{T}} \end{matrix} \\ \mathbf{K}\ -\mathbf{K} & \mathbf{X} \end{bmatrix} \succeq 0, \quad \mathbf{X}_{ii} < u^2_{\max,i}, \tag{18}$$

where $i = 1, \ldots, p$. These LMIs guarantee that control law (2c) never saturates over $\mathcal{L}_e$.

After deriving the LMIs for the constraints, we can now continue by maximizing the volume of $\mathcal{L}_e$. The volume of $\mathcal{L}_e$ is related to $\mathbf{P}$ as

$$\mathrm{vol}(\mathcal{L}_e) \sim \frac{1}{\det \mathbf{P}} = \det \mathbf{P}^{-1}. \tag{19}$$

The maximization of the volume of $\mathcal{L}_e$ can be cast as a convex optimization problem in $\mathbf{Q} = \mathbf{P}^{-1}$. By the fact that the function

$$\sqrt[2n]{\det \mathbf{Q}} \tag{20}$$

is convex in $\mathbf{Q}$ and that $\sqrt[2n]{\cdot}$ is strictly increasing, the maximization of (19) is equivalent to the maximization of (20) (Boyd and Vandenberghe, 2004).

Transforming the LMIs (15) and (18) to LMIs in terms of $\mathbf{Q}$ together with the convex objective function (20) leads to the optimization problem

$$\max \sqrt[2n]{\det \mathbf{Q}} \tag{21a}$$

such that

$$\mathbf{Q} \succ 0, \tag{21b}$$

$$\mathbf{Q}\hat{\mathbf{A}}^{\mathrm{T}} + \hat{\mathbf{A}}\mathbf{Q} \prec 0, \tag{21c}$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{Q}\begin{bmatrix} \mathbf{K}^{\mathrm{T}} \\ -\mathbf{K}^{\mathrm{T}} \end{bmatrix} \\ \begin{bmatrix} \mathbf{K}^{\mathrm{T}} \\ -\mathbf{K}^{\mathrm{T}} \end{bmatrix} \mathbf{Q} & \mathbf{X} \end{bmatrix} \succeq 0, \tag{21d}$$

$$\mathbf{X}_{ii} \leq u^2_{\max,i}. \tag{21e}$$

However, maximizing the volume of $\mathcal{L}_e$ may often lead to the phenomenon that $\mathcal{L}_e$ becomes large only in certain dimensions. This means that – despite of $\mathcal{L}_e$ having a large volume – in some coordinates only small initial states are guaranteed to be admissible, which is of course not desirable. By adding a shape constraint to the optimization, we can reach a better result. A set $\mathcal{Z} \subset \mathbb{R}^{2n}$ with a certain shape is chosen first. In a second step the size of this region is scaled by a factor α. The optimization problem is then to find the maximum value of α for which a suitable matrix $\mathbf{Q}$ can be found such that all constraints are met and $\alpha\mathcal{Z}$ is a subset of $\mathcal{L}_e$.

If $\mathcal{Z}$ is chosen as a convex polytope

$$\mathcal{Z} = \mathrm{conv}\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k\}, \quad \boldsymbol{z}_i \in \mathbb{R}^{2n}, \tag{22}$$

the condition $\alpha\mathcal{Z} \subseteq \mathcal{L}_e$ can be written as the set of LMIs

$$\begin{bmatrix} \mathbf{Q} & \alpha\boldsymbol{z}_i \\ \alpha\boldsymbol{z}_i^{\mathrm{T}} & 1 \end{bmatrix} \succeq 0, \quad i = 1, \ldots, k. \tag{23}$$

Retaining the other constraints of optimization problem (21), we arrive at the following optimization problem

$$\max \alpha \tag{24a}$$

such that

$$\mathbf{Q} \succ 0, \tag{24b}$$

$$\mathbf{Q}\hat{\mathbf{A}}^{\mathrm{T}} + \hat{\mathbf{A}}\mathbf{Q} \prec 0, \tag{24c}$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{Q}\begin{bmatrix} \mathbf{K}^{\mathrm{T}} \\ -\mathbf{K}^{\mathrm{T}} \end{bmatrix} \\ \left[\mathbf{K}\ -\mathbf{K}\right] \mathbf{Q} & \mathbf{X} \end{bmatrix} \succeq 0, \tag{24d}$$

$$\mathbf{X}_{ii} < u^2_{\max,i}, \tag{24e}$$

$$\begin{bmatrix} \mathbf{Q} & \alpha\boldsymbol{z}_i \\ \alpha\boldsymbol{z}_i^{\mathrm{T}} & 1 \end{bmatrix} \succeq 0. \tag{24f}$$

If the above optimization problem (24) yields

$$\mathcal{X}_0 \times \tilde{\mathcal{X}}_0 \subseteq \tilde{\mathcal{L}}_e = \left\{ \begin{bmatrix} \boldsymbol{x} \\ \tilde{\boldsymbol{x}} \end{bmatrix} \middle| \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{x} - \tilde{\boldsymbol{x}} \end{bmatrix} \in \mathcal{L}_e \right\}, \tag{25}$$

Problem 1 is solved by the given observer matrix $\mathbf{L}$. However, this case rarely occurs unless $\mathbf{L}$ is chosen with care. We therefore introduce a design method for the observer matrix $\mathbf{L}$ in the next section.

5. OBSERVER DESIGN

We now consider the design of an observer $\mathbf{L}$ which solves Problem 1 for a given controller $\mathbf{K}$. For reasons that will become clear below, we express the LMIs in terms of $\mathbf{P}$, not of its inverse $\mathbf{Q}$.

5.1 Stability and performance

The invariance condition (15) now depends on the variables $\mathbf{P}$ and $\mathbf{L}$. By partitioning $\mathbf{P}$ in four blocks each of size $n \times n$ as

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^{\mathrm{T}} & \mathbf{P}_{22} \end{bmatrix}, \tag{26}$$

we can separate the bilinear terms of Eq. (15) and obtain

$$\begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}^{\mathrm{T}} \mathbf{P} + \mathbf{P} \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{P}_{12}\mathbf{LC}_M \\ \mathbf{C}_M^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{P}_{12}^{\mathrm{T}} & \mathbf{C}_M^{\mathrm{T}}\mathbf{L}^{\mathrm{T}}\mathbf{P}_{22} + \mathbf{P}_{22}\mathbf{LC}_M \end{bmatrix} \prec 0. \tag{27}$$

Expression (27) contains terms that are products of matrix variables. It is not an LMI but a bilinear matrix inequality (BMI). In contrast to LMI problems, BMI problems are generally nonconvex and therefore are much harder to solve. This is mainly due to the fact that nonconvex optimization problems can possess local optima.

Unfortunately, Eq. (27) cannot be written as an LMI. Even using substitutions as we did in Eq. (8) in the control design case, say $\mathbf{V}_1 = \mathbf{P}_{12}\mathbf{L}$ and $\mathbf{V}_2 = \mathbf{P}_{22}\mathbf{L}$, fails, since then an additional rank constraint on $[\mathbf{V}_1 \ \mathbf{V}_2]$ is needed to guarantee a solution for $\mathbf{L}$. Such a rank constraint renders the optimization problem nonconvex (Orsi et al., 2006).

A restriction on the variable space enables us to obtain an LMI formulation of the problem nontheless. By setting

$$\mathbf{P}_{12} = \mathbf{0} \text{ and } \mathbf{V} = \mathbf{P}_{22}\mathbf{L} \tag{28}$$

we get

$$\begin{bmatrix} (\mathbf{A}-\mathbf{BK})^{\mathrm{T}}\mathbf{P}_{11} + \mathbf{P}_{11}(\mathbf{A}-\mathbf{BK}) & \mathbf{P}_{11}\mathbf{BK} \\ \mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{P}_{11} & \mathbf{A}^{\mathrm{T}}\mathbf{P}_{22} + \mathbf{P}_{22}\mathbf{A} - \mathbf{VC}_M - \mathbf{C}_M^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} \end{bmatrix} \prec 0, \tag{29}$$

which is an LMI in $\mathbf{P}$ and $\mathbf{V}$. At this point we can see that the above substitution would be impossible if the equations were written in terms of $\mathbf{Q}$.

Of course, restricting $\mathbf{P}$ by fixing $\mathbf{P}_{12} = \mathbf{0}$ will increase the conservativeness of the solution, but this is the price to pay for an LMI formulation of the problem.

5.2 Initial region

Another design objective is the inclusion of the set $\mathcal{Z}_0$ of possible initial conditions

$$\boldsymbol{z}_0 \hat{=} \begin{bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{e}_0 \end{bmatrix} \tag{30}$$

in the Lyapunov region $\mathcal{L}_e$. In case of a given convex polytopic set $\mathcal{Z}_0$, we obtain a set of LMIs for this constraint analogously to Eq. (23):

$$\begin{bmatrix} \mathbf{Q} & \boldsymbol{z}_{0,i} \\ \boldsymbol{z}_{0,i}^{\mathrm{T}} & 1 \end{bmatrix} \succeq 0 \Leftrightarrow \begin{bmatrix} \mathbf{P} & \mathbf{P}\boldsymbol{z}_{0,i} \\ \boldsymbol{z}_{0,i}^{\mathrm{T}}\mathbf{P} & 1 \end{bmatrix} \succeq 0, \tag{31}$$

where $\mathbf{P} = \mathbf{Q}^{-1}$.

Usually, however, only the set $\mathcal{X}_0$ is given a-priori. Because the initial state of the observer, $\tilde{\boldsymbol{x}}_0$, is a virtual state, it can be chosen freely. Of course, it must not depend on $\boldsymbol{x}_0$, since the latter is unknown. This means that the initial set in the coordinates $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ must be of the form $\mathcal{X}_0 \times \tilde{\mathcal{X}}_0$.

If we assume that $\mathcal{X}_0$ is a convex polytope that is symmetric about the origin, two options appear suggestive:

(1) We choose $\tilde{\mathcal{X}}_0 = \mathcal{X}_0$. In this configuration, $\boldsymbol{x}_0$ and $\tilde{\boldsymbol{x}}_0$ can take mutually independent values in the set $\mathcal{X}_0$. Then the result for $\mathcal{Z}_0$ in the coordinates $\boldsymbol{x}$ and $\boldsymbol{e}$ is a parallelepiped:

$$\mathcal{Z}_0 = \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} \middle| \, \boldsymbol{x} \in \mathcal{X}_0, \boldsymbol{x} - \boldsymbol{e} \in \mathcal{X}_0 \right\}. \tag{32}$$

(2) We can choose $\tilde{\boldsymbol{x}}_0$ such that the worst case initial estimation error $\boldsymbol{e}_0$ is minimized:

$$\tilde{\boldsymbol{x}}_0 = \arg\min \max_i \|\tilde{\boldsymbol{x}}_0 - \boldsymbol{x}_{0,i}\|, \tag{33}$$

where $\boldsymbol{x}_{0,i}$ are the vertices of $\mathcal{X}_0$.

Since we assumed $\mathcal{X}_0$ to be symmetric with respect to the origin[1], problem (33) is easily solved to $\tilde{\boldsymbol{x}}_0 = \mathbf{0}$, resulting in the initial set $\mathcal{X}_0 \times \{\mathbf{0}\}$. Eq. (32) then simplifies to

$$\mathcal{Z}_0 = \left\{ \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{bmatrix} \middle| \, \boldsymbol{x} = \boldsymbol{e} \in \mathcal{X}_0 \right\}. \tag{34}$$

[1] If this is not the case, problem (33) can be written as a set of LMIs and is easily solved numerically.

Remark 2. The set $\mathcal{Z}_0$ is a constraining factor in the optimization. A larger set will generally lead to a slower observer, and thus to poorer performance, or even render the optimization problem infeasible. It is therefore advisable to choose $\mathcal{Z}_0$ as small as the application allows.

5.3 Input constraint

We can simply use the LMIs (18) to consider the input constraint.

5.4 Optimization problem

LMI (29) can be extended by an upper limit γ on the output energy as in (7) to

$$\begin{bmatrix} (\mathbf{A}-\mathbf{BK})^{\mathrm{T}}\mathbf{P}_{11} + \mathbf{P}_{11}(\mathbf{A}-\mathbf{BK}) & \mathbf{P}_{11}\mathbf{BK} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{K}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{P}_{11} & \mathbf{A}^{\mathrm{T}}\mathbf{P}_{22} + \mathbf{P}_{22}\mathbf{A} - \mathbf{VC}_M - \mathbf{C}_M^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & -\gamma\mathbf{I}_q \end{bmatrix} \prec 0. \tag{35}$$

Combining the above LMIs, we arrive at the optimization problem

$$\min \gamma \tag{36a}$$

such that

$$\mathbf{P} \succ 0, \tag{36b}$$

$$(18), (35), (31),$$

in the matrix variables $\mathbf{P}$ and $\mathbf{V}$. Reversing the substitution (28) yields the observer matrix $\mathbf{L} = \mathbf{P}_{22}^{-1}\mathbf{V}$.

In some cases, the above procedure leads to an admissible observer. However, problem (36) is infeasible in many cases. Apart from the restriction $\mathbf{P}_{12} = \mathbf{0}$ this is due to the fact that the controller has been designed first without taking into account that an observer will be added afterwards. This limits the set of feasible $\mathbf{L}$ and $\mathbf{P}$. If a solution exists, the resulting observer based control may perform poorly.

However, we can extend the above results to a method for the simultaneous design of both the controller and the observer. This extended method solves the problems mentioned above, as is shown in the next section.

6. SIMULTANEOUS CONTROLLER AND OBSERVER DESIGN

6.1 Design method A: Input Constraint Adaptation

The observer we designed in Section 5 must be compatible with the original controller $\mathbf{K}$. The controller $\mathbf{K}$ has been designed before as in Section 3 under the assumption of state feedback. We can expect these parameters to be close to the boundary of the feasible set, which limits the set of feasible observer matrices $\mathbf{L}$ and Lyapunov matrices $\mathbf{P}$. These limitations – together with the restrictions on $\mathbf{P}_{12}$ – might result in bad performance or even render the observer design optimization problem infeasible.

If we choose a weaker controller we can expect that the set of feasible observer parameters enlarges and that the parameters found will lead to a better overall performance of the system.

In order to design a weaker controller the input constraint is tightened. We adapt the input constraint to

$$|u_i| \leq \xi u_{\max,i}, \tag{37}$$

with $\xi \in (0,1)$. If feasible, optimization problem (12) will then yield a weaker controller $\mathbf{K}_\xi$. We can then design an observer $\mathbf{L}_\xi$ using procedure (36).

The result, $\mathbf{K}_\xi$ and $\mathbf{L}_\xi$, of course depends on ξ. So does the result for γ of (36). The function $\gamma(\xi)$ can be evaluated as follows:

(1) Design the controller as in (12) with LMI (11b) being replaced by the LMI
$$\mathbf{X}_{ii} < \xi^2 u_{\max,i}^2. \tag{38}$$
(2) Design the observer using the above controller as in (36). The result for γ is $\gamma(\xi)$. The function $\gamma(\xi)$ is defined to be ∞ if one of the design steps is infeasible.

Figure 1 shows $\gamma(\xi)$ for the example in Section 7 below. The curve shown is typical. For small values of ξ, the controller becomes very slow, resulting in a bad performance. For large values of ξ, the observer becomes slow, also resulting in a bad performance. Somewhere in between there exists an optimal value. A local numerical optimization method, e.g. a hill climbing algorithm, can be used to find that optimal value for ξ. If the system is unstable, it may happen that no ξ exists for which γ is finite. This can have two reasons: Either there exists an optimal solution, but due to the conservativeness of the approach it cannot be found, or the problem simply does not have a feasible solution at all.

Remark 2. For systems with multiple inputs one could define different factors ξ_i for each input. In this case it is much harder to arrive to an intuitive assessment of the landscape of the objective function $\gamma(\xi_1, \ldots, \xi_p)$, as it depends on p variables. As a result the optimization will be computationally heavier.

6.2 Design Method B: BMI Solver

Alternatively to the method of the preceding section, the problem can be left in its original BMI formulation (15) and be solved with a dedicated BMI solver. The use of a BMI solver has two major advantages: we can simultaneously optimize $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{P}$ and the restriction $\mathbf{P}_{12} = \mathbf{0}$ can be dropped. However, there are also some serious drawbacks: The solver can only optimize locally and convergence is not guaranteed. Additionally, the solver is not always able to find a feasible solution by itself. Design method A can therefore be very helpful in order to find a feasible starting point for the BMI optimization.

6.3 Additional pole constraint

For some examples, including the double integrator of Section 7 below, both optimization methods A and B tend to shift the real part of one or several eigenvalues of the estimation error matrix $\mathbf{A} - \mathbf{LC}_M$ to very high negative values. This is not desirable, as it renders the observer prone to noise. To prevent this phenomenon, we add the constraint

$$\mathrm{Re}(p_i) > -\rho, \quad \rho > 0, \tag{39}$$

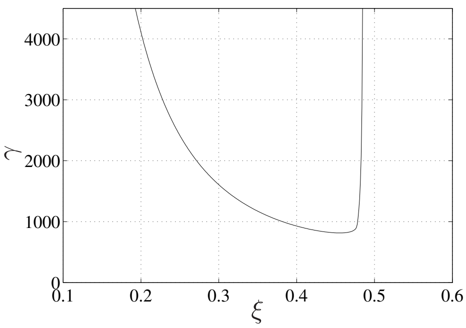

Fig. 1. $\gamma(\xi)$ for the double integrator

on the poles p_i of that matrix to the optimization problem. This constraint can be written as the LMI

$$2\rho\mathbf{P}_{22} + \mathbf{A}^{\mathrm{T}}\mathbf{P}_{22} + \mathbf{P}_{22}\mathbf{A} - \mathbf{V}\mathbf{C}_M - \mathbf{C}_M^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} \succ 0, \tag{40}$$

as shown in Chilali and Gahinet (1996). The value for ρ should be chosen depending on the plant. According to our experience, the optimal value for γ remains almost unchanged as long as ρ is large enough.

7. EXAMPLE

We will now demonstrate the above methods in the following example. For the computations we used the interface YALMIP (Löfberg, 2004) with the solver SeDuMi (Sturm, 1999) for the LMI optimizations. For the BMI optimization of method B we used the solver PENBMI (Kočvara and Stingl, 2006).

Consider the double integrator system

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = [1 \;\; 0], \tag{41}$$

with the input constraint $|u| \leq u_{\max} = 1$ and the set of possible initial conditions $\mathcal{X}_0 = \{\boldsymbol{x} \in \mathbb{R}^2 | \, |x_i| \leq 2\}$. The observer is assumed to be initialized at $\tilde{\boldsymbol{x}}_0 = \mathbf{0}$, i. e. $\tilde{\mathcal{X}}_0 = \{\mathbf{0}\}$.

Using the design method of Section 3, we obtain the controller

$$\mathbf{K} = [0.1745 \;\; 0.3135]. \tag{42}$$

If we design the observer heuristically by placing the poles ten times faster than those of the closed loop system, the observer matrix is given by

$$\mathbf{L} = \begin{bmatrix} 3.135 \\ 17.45 \end{bmatrix}. \tag{43}$$

We use $\mathcal{Z} = \mathcal{X}_0 \times \tilde{\mathcal{X}}_0$ as the shape constraint. Method (24) of Section 4 to compute an estimate of the linear region then yields an optimal value $\alpha_{\max} = 0.355$. Since $\mathcal{Z} = \mathcal{X}_0 \times \tilde{\mathcal{X}}_0$, a value $\alpha_{\max} \geq 1$ is needed to fulfill Eq. (25). Thus, Problem 1 has not been solved by this heuristic design.

The method of Section 5 turns out to be infeasible for this example.

The input constraint adaptation design method A of Section 6.1 yields $\xi = 0.4577$ and the parameters

$$\mathbf{K} = [0.048346\ 0.17176]\,, \tag{44a}$$

$$\mathbf{L} = \begin{bmatrix} 50.301 \\ 12.222 \end{bmatrix}, \tag{44b}$$

$$\mathbf{P} = 10^{-2} \begin{bmatrix} 0.6854 & 1.476 & 0 & 0 \\ 1.476 & 11.75 & 0 & 0 \\ 0 & 0 & 3.638 & -1.880 \\ 0 & 0 & -1.880 & 8.124 \end{bmatrix}, \tag{44c}$$

for $\rho = 100$.

Using the BMI design method B of Section 6.2 with the parameters (44) as initialization values we obtain

$$\mathbf{K} = [0.10149\ 0.091633]\,, \tag{45a}$$

$$\mathbf{L} = \begin{bmatrix} 100.04 \\ 5.6071 \end{bmatrix}, \tag{45b}$$

$$\mathbf{P} = 10^{-2} \begin{bmatrix} 1.505 & 1.253 & 0.03249 & -1.056 \\ 1.253 & 13.67 & -0.06157 & 1.476 \\ 0.03249 & -0.06157 & 3.803 & -0.1356 \\ -1.056 & 1.476 & -0.1356 & 3 \end{bmatrix}. \tag{45c}$$

A simulation with the parameters of each method for $\boldsymbol{x}_0 = [2\ 2]^{\mathrm{T}}$, an initial condition on the corner of $\mathcal{X}_0$, is shown in Fig. 2. A cut of the corresponding set $\mathcal{L}_e$ in the $\boldsymbol{x}$-plane is displayed in Fig. 3. The upper bound γ on the output energy J and the effective output energy for $\boldsymbol{x}_0$ are listed in Table 1.

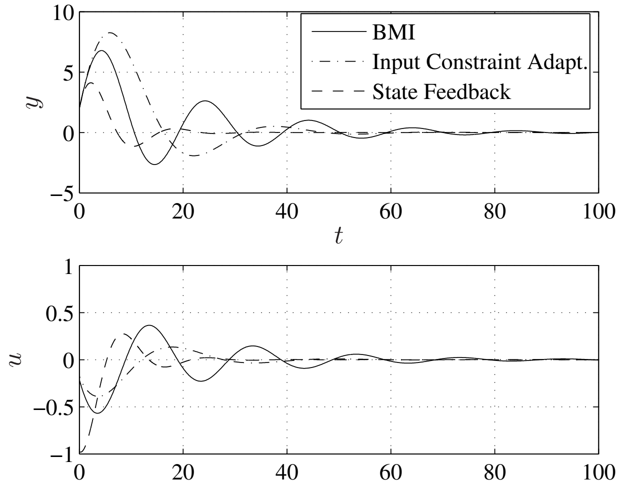

Fig. 2. Simulation results of the different observer design methods and the state feedback without observer.

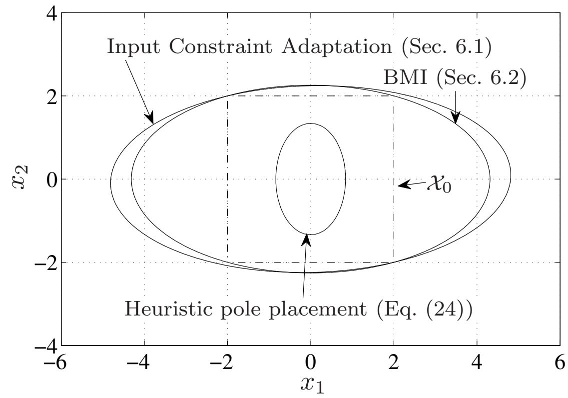

Fig. 3. Cut of $\mathcal{L}_e$, the positive invariant estimate of the linear region $\mathcal{L}$, in the $\boldsymbol{x}$-plane.

Method	γ	$J(\boldsymbol{x}_0)$
State feedback (Eq. (12))	69.44	69.44
Observer design (Eq. (36))	∞ (infeasible)	-
Input constraint		
adaptation (Section 6.1)	795.86	575.84
BMI Solver (Section 6.2)	409.95	338.62

Table 1. Performance comparison, $\boldsymbol{x}_0 = [2\ 2]^{\mathrm{T}}$.

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