

## Stabilization of the Inverted Pendulum with backlash using $H_\infty$ -LMI Technique

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**Abstract:** The rotary inverted pendulum, also named Furuta Pendulum, has been studied extensively for control performance evaluation in under-actuated mechanisms. The  $H_\infty$  control invoking Linear Matrix Inequality ( $H_\infty$ -LMI) has been also wide employed for linear control design. This paper deals with the feasibility of  $H_\infty$ -LMI technique to stabilize the rotary inverted pendulum around its unstable equilibrium point when there exists a backlash nonlinearity in the actuator. So, the  $H_\infty$ -LMI design faces the nonlinear effect in the actuator and the non-linear pendulum model. Experimental realization of the designed  $H_\infty$ -LMI control evidences the good performance of the controller subject to external perturbation too.

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### 1. INTRODUCTION

The rotary inverted pendulum is a sub-actuated mechanical system widely used to test performance controller [Awtar et al. 2002]. A wide variety of studies has been already reported, most of which focused on the balancing problem (see [Åström and Furuta 2000], [Acosta et al. 2002], [Iwashiro et al. 1996] and references in it). Although, the swing up is also an interesting control objective and it is possible to combine these two problems: after the swing up, a local controller can be activated to balance the pendulum [Awtar et al. 2002]. These approaches can significantly improve the control performance, however deterioration on the actuators, e.g., backlash effect, is still an open problem. Reviewing journals data base, it was not possible to find a report about stabilization of the Furuta Pendulum subject to backlash effect in the actuator. In this paper,  $H_\infty$ -LMI control design is postulated to the stabilization problem of the Furuta Pendulum with backlash nonlinearity in the actuator. Backlash is considered as “the play between adjacent movable parts”, and it is present in every mechanical systems with gears (see [Nordin and Gutman 2002] for a backlash survey). From the control point of view, there exist several models of backlash, and depending of the mechanical environment, an appropriate model selection is done; but, it is common to employ a dead-zone model for control design and simulation of mechanisms with backlash (see [Nordin and Gutman 2002]) due to its simplicity. Moreover, the dead-zone model is a schematization of diverse physical phenomena with negligible dynamics [Wang et al. 2004]. In this paper, the dead-zone model for backlash is employed to design a robust controller for the stabilization control problem.

Moreover, external disturbance rejection problem is considered. To deal with it,  $H_\infty$  theory is used. For linear systems,  $H_\infty$  control theory offers the possibility of including robustness considerations explicitly in the design and the opportunity to formulate physically meaningful performance objectives that can be expressed as  $H_\infty$  design specifications. In the state-space formulation, the

problem of reducing the  $H_\infty$  norm or, equivalently, its induced  $L_2$ -norm, of the closed-loop system is viewed as a solution of the algebraic Riccati inequalities arising in linear quadratic differential theory (see [Doyle et al. 1989] and [Khargonekar et al. 1990] for references on the  $H_\infty$  design). In order to solve these Riccati inequalities, they are expressed in terms of linear matrix inequalities (LMI). It has been acknowledged that many control problems can be formulated in terms of LMIs (see [Boyd et al. 1994], [Apkarian et al. 2000], [Oliveira and Peres 2006] and references in it).

In the present paper, the design of a local controller that stabilizes the inverted pendulum is studied, in the presence of model uncertainties, external disturbances and backlash effect. These uncertainties appear because inverted pendulum linear model is considered and, in the experiment, the pendulum location is modified. The main objective is to demonstrate how a solution to such a complex control problem can be obtained by applying the efficient tools of LMI. To show the robustness of the proposed controller, laboratory experiments are carried on. An Industrial Emulator with backlash, which is a mechanical system used for control performance evaluation [Model 220], was configured with the Inverted Pendulum to examine the performance of our  $H_\infty$ -LMI control technique showing good response when just position measurement is engaged to the controller. In the last years, the control problem of robots using only position measurements have attained an increasing interest (see [Berghuis and Nijmeijer 1993] and references in it). A straightforward approach to this problem goes along a two-step design procedure: a) construct an observer, driven by the available inputs and outputs, which estimates the lacking velocity signal, b) design a state-feedback controller and replace the actual velocity by the one reconstructed from the observer. This idea is employed in the present work and it agrees with experimental setting, where velocity observers are used because only position measurements are available.

The paper is structured as follows. Section two presents the nonlinear description, that is, the Furuta Pendulum with

backlash. Here, the static dead-zone model of backlash is employed. In Section three, the control objective is stated. Section four displays the  $H_\infty$  control solution for the inverted pendulum, in terms of an LMI. In section five, experimental results are shown, proven the effectiveness of the methodology. Finally, conclusions and acknowledgments are state in section six and seven, respectively. The notation throughout the paper is fairly standard:  $I$  denotes the identity matrix; in symmetric block matrices or long matrix expressions, we use  $*$  as an ellipsis for terms that are induced by symmetry, e.g.

$$\begin{pmatrix} S + (*) & * \\ M & Q \end{pmatrix} \equiv \begin{pmatrix} S + S^T & M^T \\ M & Q \end{pmatrix}.$$

## 2. MECHANICAL PROBLEM

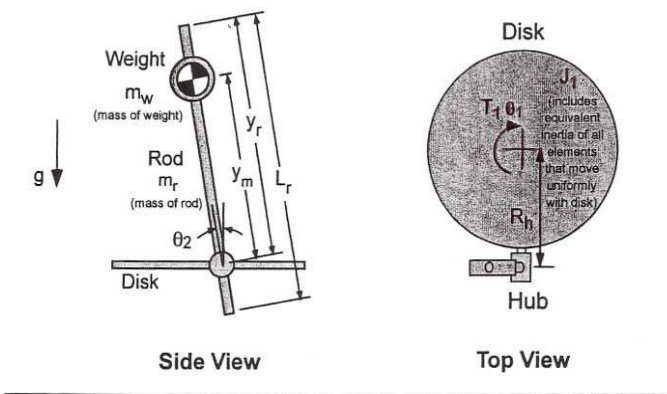


Fig. 1. Diagram representation of Furuta Pendulum with rotating base [Model 220].

In this section, the Furuta Pendulum with backlash problem is stated, where we consider the nonlinearity of the model as a disturbance in order to apply the  $H_\infty$  theory with LMI techniques.

The rotating base system shown in Figure 1 is representative of the pendulum attached to experimental Model 220 apparatus (see [Model 220]), where  $\theta_1$  is the motor angle position and  $\theta_2$  is the pendulum angle position. The classical representation considers the pendulum armed to the drive disk (without backlash). Then, the equations of motion are found via Lagrange's equations and may be linearized about  $\theta_1 = 0$  and  $\theta_2 = 0$  [Model 220]:

$$\ddot{\theta}_1 = \frac{1}{p} \left( -\dot{\theta}_1 \bar{J}_z - m^2 l_{cg}^2 R_h g \theta_2 + \bar{J}_z u_{ce} \right) \quad (1)$$

$$\ddot{\theta}_2 = \frac{1}{p} \left( m R_h l_{cg} \dot{\theta}_1 - m g l_{cg} (\bar{J}_1 + J_y) \theta_2 - m R_h l_{cg} u_{ce} \right) \quad (2)$$

where  $p = \bar{J}_z (\bar{J}_1 + J_y) - (m R_h l_{cg})^2$ ,  $\bar{J}_1 = J_1 + m R_h^2$  and  $\bar{J}_z = J_z + m l_{cg}$ ;  $J_1$  includes equivalent inertia of all elements that move uniformly with the motor disk;  $J_y$  and  $J_z$  are the moments of inertia for the pendulum relative to its center of mass;  $g$  is the gravity constant;  $u_{ce}(t)$  is the control effort to be designed. For more details on parameters and modelling, see [Model 220]. In this paper, we consider the state vector  $x(t) = [\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2]$ , where

$$\dot{\theta}_1 = \frac{d}{dt} \theta_1 \text{ and } \dot{\theta}_2 = \frac{d}{dt} \theta_2.$$

In order to incorporate a backlash effect in the actuator, from the experimental point of view, the pendulum is armed in the load disk. Taking into account this pendulum position change, it is important to note that the model (1)-(2) does not capture the variation in inertias and other effects. We can consider it as non-modeled uncertainties, which must have to be attenuated by our controller, as it will be appreciated in our experiments. In fact, we see the drive disk, the backlash and the load disk as unique active component. Moreover, only the load disk angle position ( $\theta_l$ ) and the pendulum angle position ( $\theta_p$ ) are measured by the experimental platform, but velocities will be required in the control design; so, we construct an observer to estimate the velocities  $\dot{\theta}_l$  and  $\dot{\theta}_p$  (see Section 5.2). From now on, we consider  $\theta_1 = \theta_l$  and  $\theta_2 = \theta_p$  in equations (1)-(2).

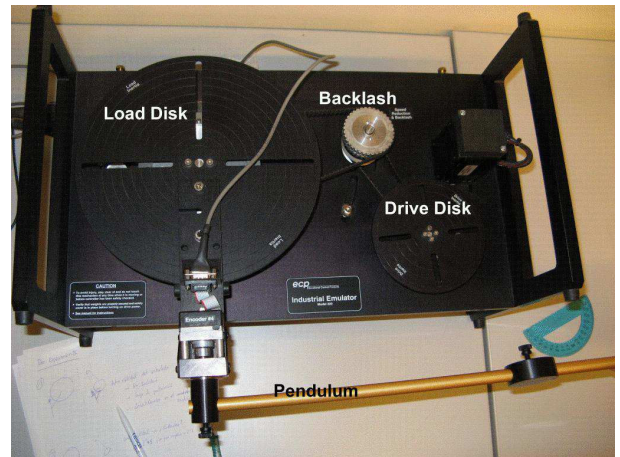


Fig. 2. Pendulum Assembly in Inverted closed-loop control with Backlash (Model M220) [Model 220].

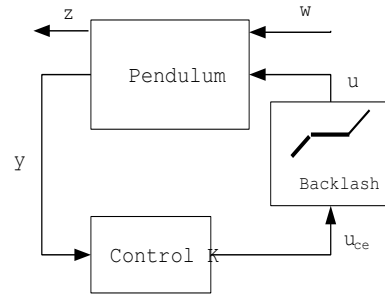


Fig. 3. Backlash in actuator:  $w$  external disturbance;  $u_{ce}$  control effort;  $u$  effective control;  $z$  virtual output;  $y$  measurable output.

Let define now the backlash model used in this paper. Consider the situation illustrated in Figure 3. That is, when backlash is present in the actuator, the effective control is not  $u_{ce}(t)$  but  $u(t)$ . The backlash nonlinearity is captured by an static dead-zone model ([Wang et al. 2004]):

$$u(t) = \begin{cases} m(u_{ce}(t) - \alpha) & u_{ce}(t) \geq \alpha \\ 0 & -\alpha < u_{ce}(t) < \alpha \\ m(u_{ce}(t) + \alpha) & u_{ce}(t) \leq -\alpha \end{cases} \quad (3)$$

where  $2\alpha$  is the width of the dead-zone. According with [Nordin and Gutman 2002], the dead-zone model for backlash is applicable when the shaft damping is zero. From [Wang et al. 2004], we consider  $m = 1$  and we can rewrite (3) as

$$u(t) = u_{ce}(t) + d_\alpha(u_{ce}(t)), \quad (4)$$

where

$$d_\alpha(s) = \begin{cases} -\alpha & \text{if } s \geq \alpha \\ -s & \text{if } |s| < \alpha. \\ \alpha & \text{if } s \leq -\alpha \end{cases} \quad (5)$$

From (5), we can appreciate that  $|d_\alpha(\cdot)| \leq \alpha$  for all  $u_{ce}(t)$ . Then, from (4), the dead-zone model can be considered as a linear part depending on control effort  $u_{ce}(t)$ , with a nonlinear bounded term  $d_\alpha(\cdot)$ .

### 3. CONTROL OBJECTIVE

The aim of this work is to design a control  $u_{ce} = Kx(t)$  verifying two properties. One is to ensure local stability when backlash is present in the actuator, where an static dead-zone model is considered. The other requirement imposed to the control design is the robustness in front of external  $\mathcal{L}_2$  disturbance. The problem of robust controller with guaranteed  $H_\infty$  performance is addressed to answer this question (see Figure 3): Does there exist a feedback control such that the  $H_\infty$  norm of the closed-loop system from input disturbance named  $w$  to output  $z$  is less than some prescribed value  $\gamma$ ? ([Doyle et al. 1989]). In order to solve this problem, we use the linear matrix inequalities (LMI) techniques.

Let consider system (1)-(2) where the control effort is the effective control  $u(t)$ , with dead-zone model (4)-(5). By model assumptions stated in Section 2, the vector state is defined by  $x^T(t) = [\theta_l(t) \ \dot{\theta}_l(t) \ \theta_p(t) \ \dot{\theta}_p(t)]$ . We consider the perturbation  $w(t) \in \mathcal{L}_2$  and  $z(t) = [\theta_l \ \theta_p \ u_{ce}]$  the virtual output to be compared to  $w(t)$  (see Figure 3). Then, the state-space representation of system (1)-(2) and dead-zone (4)-(5) yields

$$\begin{cases} \dot{x}(t) = Ax(t) + B_2u(t) + B_1w(t) \\ z(t) = C_1x(t) + D_{12}u(t) \end{cases} \quad (6)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\bar{J}_z & -m^2l_{cg}^2R_hg & 0 \\ 0 & 0 & 0 & 1 \\ 0 & mR_hl_{cg} & ml_{cg}g(\bar{J}_1 + J_y) & 0 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ \bar{J}_z \\ 0 \\ -mR_hl_{cg} \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Furthermore, the above dynamic systems satisfies the controllability and observability properties [Doyle et al. 1989].

The control objective is to design a gain matrix  $K$  such that the controller

$$u_{ce}(t) = Kx(t) \quad (7)$$

stabilizes the system (6) under  $\mathcal{L}_2$  disturbances, employing  $H_\infty$ -LMI theory. This case has been named full information problem [Doyle et al. 1989]. Classical way to solve this problem is to consider a Lyapunov function  $V(x(t))$  such that for any nonzero  $x(t)$  and input  $w(t) \in \mathcal{L}_2$ , the following condition holds ([Apkarian et al. 2000])

$$\frac{d}{dt}V(x(t)) + \gamma^{-1} z^T(t)z(t) - \gamma w^T(t)w(t) < 0. \quad (8)$$

Then, an  $H_\infty$  performance bound for the closed-loop system (6)-(7) is ensured (see [Apkarian et al. 2000] for details).

**Definition 1 ( $H_\infty$  Controller)** *If there exists a matrix  $K$  such that (8) holds, the control law  $u_{ce}(t) = Kx(t)$  is said to be an  $H_\infty$  controller for the system (6), that is, the system is internally stable with  $H_\infty$  norm less than  $\gamma$ .*

### 4. ROBUST CONTROL DESIGN WITH $H_\infty$ PERFORMANCE

The control design for solving the  $H_\infty$  problem (8) involves two steps. First, a sufficient condition is obtained for the existence of an  $H_\infty$  controller, in terms of a nonlinear matrix inequality. The second step is based on derive an LMI procedure for the effective construction of the controller. As instrumental tool, we need a result where an LMI characterization is stated in [Pujol 2007] and based on Projection Lemma ([Gahinet and Apkarian 1994]).

**Theorem 1. ( $H_\infty$ -LMI Control).** Consider the Furuta Pendulum system (6)-(7) with dead-zone model (4)-(5). If there exist  $\gamma > 0$ , matrices  $N, Y > 0$  symmetric and  $R$  regular such that

$$\begin{bmatrix} -(R^T + R) & * & * & * & * & * & * \\ AR + Y + B_2N & -Y & * & * & * & * & * \\ 0 & B_1^T & -\gamma & * & * & * & * \\ 0 & 0 & B_2^T & -\alpha^{-2} & * & * & * \\ C_1R & 0 & 0 & 0 & -\gamma & * & * \\ N & 0 & 0 & 0 & 0 & -\gamma & * \\ R & 0 & 0 & 0 & 0 & 0 & -Y \end{bmatrix} < 0 \quad (9)$$

is feasible, then the inequality (8) holds, taking the Lyapunov function  $V(x) = x^T P x$  with  $P := Y^{-1}$  and the control  $u_{ce}(t) = Kx(t)$  with  $K := NR^{-1}$ . Consequently,  $u_{ce}(t)$  is an  $H_\infty$  controller.

**Proof.** The condition (8) can be written as follows:

$$\begin{aligned} & \frac{d}{dt}V(x) + \gamma^{-1}z^Tz - \gamma w^T w = \\ & = (Ax + B_2u + B_1w)^T P x + x^T P (Ax + B_2u + B_1w) + \\ & \quad + \gamma^{-1}(C_1x + D_{12}u)^T (C_1x + D_{12}u) - \gamma w^T w < 0 \end{aligned} \quad (10)$$

From (4) and (7) and considering  $v^T = (x^T, w^T, d_\alpha^T)$ , inequality (10) is equivalent to impose the negative definite property to the matrix

$$\begin{bmatrix} (A + B_2K)^T P + (*) + \gamma^{-1} (C_1^T C_1 + K^T K) & * & * \\ & B_1^T P & -\gamma * \\ & B_2^T P & 0 \quad -\bar{\alpha} \end{bmatrix} \quad (11)$$

where  $\bar{\alpha} = \alpha^{-2}$ . Let's now demonstrate that the nonlinear matrix inequality (11) holds from (9). First, the Schur Complements [Apkarian et al. 2000] are applied to (11) in order to rewrite it. Since (9) satisfies item (ii) in Proposition 1 [Pujol 2007], we conclude thanks to this LMI characterization. Thus, condition (8) is verified and  $u_{ce}(t) = Kx(t)$  is an  $H_\infty$  controller.  $\square$

Theorem 1 gives an LMI characterization for an  $H_\infty$  controller, when an inverted pendulum with backlash is under study.

## 5. APPLICATION TO AN INDUSTRIAL EMULATOR

To test the obtained results, we proceed by studying experimentally the controller effectiveness.

### 5.1 Experimental setup

Experiments have been performed on an ECP Model 220 industrial emulator with Furuta Pendulum that includes a PC-based platform and DC brushless servo system ([Model 220]).

The mechatronic system includes two motors, one as servo actuator and the other as disturbance input (not used here), a power amplifier and two encoders which provide accurate position measurements; i.e., 4000 lines per revolution with 4X hardware interpolation giving 16000counts per revolution to each encoder; 1 count (equivalent to 0.000392 radians or 0.0225 degrees) is the lowest angular measurable ([Model 220]). The system was set up to incorporate inertia. The backlash is introduced at the idler pulley SR assembly between drive disk and load disk, by a screw adjustment. In this experiment, the screw is not clamp in order to obtain backlash effect, with approximately  $\alpha = 3$  degrees. The drive and load disks were connected via a 4 : 1 speed reduction.

A pentium 4, 2.80 GHz, 512 MB RAM computer running under Windows XP is programmed as controller together with the interface medium ECP MV Executive 5.1, a C-like programming language ([Model 220]). The system contains a data-acquisition board for digital to analog conversion and a counter board to read encoder outputs into the servo system. The minimum servo-loop closure sampling time  $T_s$  is 0.884 ms. The output voltage generated by the system is in the range of  $\pm 5V$  and is delivered to the motor drive via the DAC. The measurement feedback is a position signal (in counts or radians), measured at the shaft of each of the two disks by the optical rotary incremental position encoders, then it is read by the microcomputer by means of the counter board and delivered into the PC. In the present experiment,

the controller uses position sensor at the motor and load sides. A software interface has been built to easily transfer the raw data collected from the plant (by means of the ECP MV Executive program) to the Matlab workspace environment.

### 5.2 Experimental results

In the experiments, the pendulum is set to the following parameters:  $y_r = 42 \text{ cm}$ ,  $y_m = 32 \text{ cm}$  (see Figure 1). In order to study the backlash, the pendulum is posed in the load disk, implying some unmodeled uncertainties (see figure 4). System parameters in (6) were taken as [Model 220]:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.1379 & -28.769 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.7219 & 50.229 & 0 \end{bmatrix}$$

$$B_1 = B_2 = \begin{bmatrix} 0 \\ 318.7 \\ 0 \\ -202.2 \end{bmatrix}.$$

Using Theorem 1 and solving (9) with LMI Matlab Toolbox, the next  $H_\infty$  controller is obtained:

$$K = [0.38 \quad 0.43 \quad 6.38 \quad 1.09] \quad (12)$$

with  $\gamma = 8.2$ . This expression may be used directly for control modelling except that they must be scaled by appropriate system gains (amplifier and software gains and motor torque constants). In the experiments, the output controller in (12) was multiplied by 0.3 to compensate these gains.

Because velocity measurements are not available, we develop the following controller realization:

$$\begin{aligned} u_{ce} &= 0.3(0.38 \theta_l + 0.43 \dot{x}_l + 6.38 \theta_p + 1.09 \dot{x}_p) \\ \dot{x}_l &= -10x_l + 5\theta_l \\ \dot{x}_p &= -10x_p + 5\theta_p \end{aligned} \quad (13)$$

The above equations are a direct implementation of velocity observers given in [Berghuis and Nijmeijer 1993], where the parameter  $-10$  was set according to [Ogata 1998]. This consist to locate the observer poles from 5 to 10 times far away with respect to the vertical-axis-closest pole of  $(A + B_2K)$ . The gain 5 was adjusted experimentally. This approach goes along a two-step design procedure: a) design a state-feedback controller  $K$  as a complete access to state vector is available, b) construct a velocity observer. These steps are done independently. This design obeys the separation principle (see [Ogata 1998] and [Khalil 2000]).

In the experiments, the backlash angle was posed approximately at  $\alpha = 3$  degrees (133 counts). To prove the efficiency of the proposed method, the control action (Figure 5) and the controller variable (Figure 6) are evaluated. In Figure 5, the load disk position is pictured, showing the robustness of the proposed controller (13). Approximately, at 12s, 17s and 27s a perturbation by hand has been introduced. These perturbations consist in hitting the pendulum. Note that, without external perturbation, the load position stabilizes around of 300 counts (approx. 6.75 deg). This offset is a noise due to the position measurement feedback. Figure 6 displays the



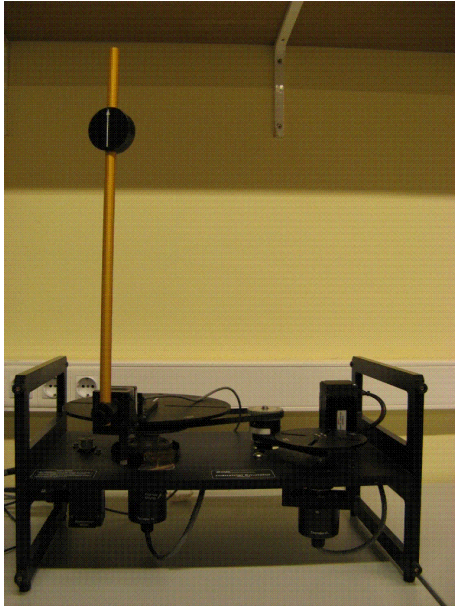


Fig. 4. Experimental performance display of Furuta Pendulum.

pendulum position. It has been stabilized around its upwards position by the robust controller in front of the external perturbations. Here, the maximal position pendulum deviation is about 60 counts (approx. 1.35 degrees). Also, there exists an offset noise. The backlash effect can be appreciated too. Experimentally, our  $H_\infty$ -LMI controller is robust in front of these additive noises. Visit

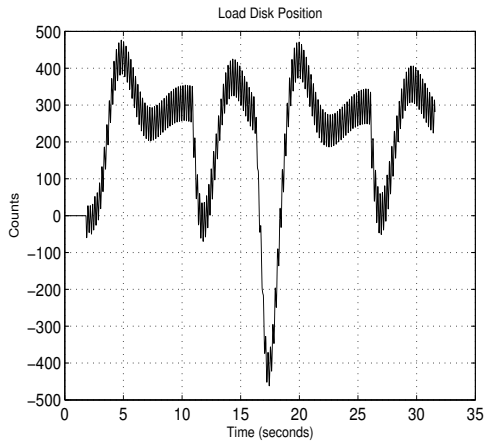


Fig. 5. Load Disk position (1 Count = 0.0225 degrees). Approximately, at 12s, 17s and 27s a perturbation by hand has been introduced. In absence of external perturbation, the load position moves around 300 counts, as can be appreciated here just before the first hitting perturbation.

[www-ma3.upc.es/codalab/lab.php?aaa=v7](http://www-ma3.upc.es/codalab/lab.php?aaa=v7) to see the experiments videos for different values of backlash including the non-backlash case (internal stability).

## 6. CONCLUSIONS

In this paper, the design of a local controller that stabilizes a mechanical system is studied, in the presence of

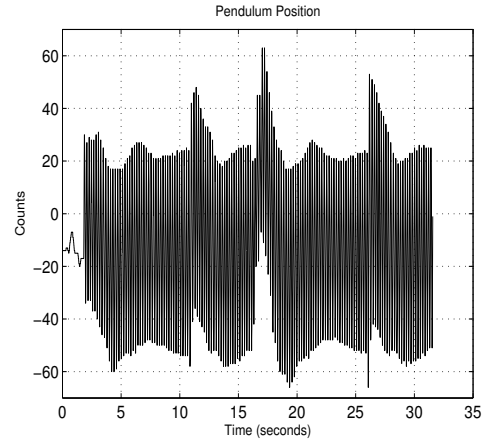


Fig. 6. Inverted Pendulum position (1 Count = 0.0225 degrees). Approximately, at 12s, 17s and 27s a perturbation by hand has been introduced.

model uncertainties, external disturbances and backlash effect. A sufficient condition in LMI terms was obtained. Effectiveness of LMI techniques to stabilize the nonlinear under-actuated mechanical systems with backlash effect in the actuator has been proved experimentally with Furuta Pendulum, when just position measurement is engaged to the controller. The LMI was stated in terms of  $H_\infty$  performance. Moreover, this kind of experiments for inverted pendulum with backlash effect is fresh.

## ACKNOWLEDGEMENTS

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