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Observer-based synchronization of discrete-time chaotic systems under communication constraints ⋆

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Abstract: The paper is devoted to the synchronization problem of the discrete-time chaotic systems, coupled by the link ("*the communication channel*") with the limited bit-per-step rate. The observer-based full-order coder is designed, ensuring decay of the synchronization error asymptotically for the case when channel imperfections and computation errors are neglected. It is shown that if the computations in the both master and slave nodes of the channel are identical, the synchronization error can be made close to the maximum achievable accuracy of the given computer depending only on the number of digits in the computer (*practical synchronization*). If the calculations in the coder and decoder are not identical (e.g., if the computers on these nodes have different number of digits), after the some time interval of decreasing the synchronization error, the mis-synchronization occurs due to unstable properties of the chaotic systems. For this case, the practical synchronization may be ensured applying the fixed-point arithmetic calculations.
The result is illustrated numerically on the example of synchronization chaotic Hénon systems.

Keywords: chaotic behavior; communication constraints; synchronization; practical stability

1. INTRODUCTION

Chaotic synchronization has attracted the attention of researchers since the 1980s and is still an area of active research (Boccaletti et al., 2002; Fradkov and Pogromsky, 1998; Freitas et al., 2005; Pecora and Carroll, 1998; Pikovsky et al., 2001). During the last decade, the information-theoretic concepts were applied to analyze and quantify synchronization (Baptista and Kurths, 2005; Paluš et al., 2001; Pethel et al., 2003; Shabunin et al., 2002; Stojanovski et al., 1997). In (Paluš et al., 2001; Shabunin et al., 2002) mutual information measures were introduced for evaluating the degree of chaotic synchronization. In (Pethel et al., 2003; Stojanovski et al., 1997) the methods of symbolic dynamics were used to relate synchronization precision to capacity of the information channel and to the entropy of the drive system. Baptista and Kurths (2005) introduced the concept of a chaotic channel as a medium formed by a network of chaotic systems that enables information from a source to pass from one system (transmitter) to another system (receiver). They characterized a chaotic channel by the mutual information (difference between the sum of the positive Lyapunov exponents corresponding to the synchronization manifold and the sum of positive exponents corresponding to the transverse manifold). However, in existing papers limit possibilities for the precision of controlled synchronization have not been analyzed.

Recently the limitations of control under constraints imposed by a finite capacity information channel have been investigated in detail in the control theoretic literature, see (Nair et al., 2007) and references therein. It was shown that stabilization under information constraints is possible if and only if the capacity of the information channel exceeds the entropy production of the system at the equilibrium (Nair and Evans, 2003, 2004; Nair et al., 2004). In (Touchette and Lloyd, 2004) a general statement was proposed, claiming that the difference between the entropies of the open loop and the closed loop systems cannot exceed the information introduced by the controller, including the transmission rate of the information channel.

However, results of the mentioned works on control system analysis and design under information constraints do not apply to synchronization systems since in a synchronization problem trajectories in the phase space converge to a set (a manifold) rather than to a point, i.e. in the general case the problem cannot be reduced to simple stabilization. The problem is still more complicated for nonlinear systems, for incomplete state measurements and in the presence of uncertainty. Specifically, almost nothing is known about limit possibilities of estimation and control under information constraints for the partial stabilization, or set stabilization problem. Such a problem arises if one needs to stabilize a limit cycle or a chaotic attractor, which is important for the control of oscillatory modes in engineering systems (Andrievsky and Fradkov, 2003; Fradkov and Evans, 2005; Fradkov and Pogromsky, 1998). However, analytical performance estimates of chaotic control systems are known only for a few cases, even without information constraints, see, e.g. (Fradkov and Khryaschev, 2005; Khryashchev, 2004); their development requires a sophisticated mathematical apparatus.

Observer-based synchronization systems are used in the case of incomplete measurements, when all phase variables are not available for measurement and coupling. Such systems are well studied without information constraints (Morgül and Solak, 1996; Nijmeijer, 2001; Nijmeijer and Mareels, 1997).

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Observer-based synchronization of continuous-time chaotic systems under information constraints is studied in (Andrievsky et al., 2006; Fradkov et al., 2006b), where limit possibilities are established. Similar results for *adaptive* synchronization were recently obtained in (Fradkov et al., 2006a). The papers (Andrievsky et al., 2006; Fradkov et al., 2006a,b) deal with synchronization of continuous-time chaotic systems over the digital communication link with finite capacity. The overall system can be naturally viewed as a *hybrid* one, i.e., system described by a coupling between continuous and discrete dynamics (Liberzon, 2003). In the mentioned works the sampling rate was considered as a design parameter and the *one-step-memory* coder was used. Based on these conditions, it was established in (Andrievsky et al., 2006; Fradkov et al., 2006a,b) that the *binary* coding procedure minimizes the bit-per-second data rate over the channel, and also the ratio between the optimal sample time and the upper bound of the limit synchronization error was found.

The present paper is devoted to the synchronization problem of the *discrete-time* chaotic systems. In this case, the sampling time is not considered as a system parameter, and the *bit-per-step* rate is used as a measure of the channel capacity. Besides, the overall system for the discrete case is not a hybrid one. Therefore, the errors of modeling continuous-time systems by difference equations do not occur. This makes possible to use the *full-order* coders, ensuring decay of the synchronization error asymptotically. This result complies with the statement, that that if the capacity of the channel is larger than the Kolmogorov–Sinai entropy of the driving system, then the synchronization error can be made arbitrarily small (Stojanovski et al., 1997).

The paper is organized as follows. General form of the system model and problem statement are presented in Sec. 2. Coding procedure is described in Sec. 3. The example of synchronization of discrete-time chaotic *Hénon systems* over the limited-band communication link is given in Sec. 4. Concluding remarks are given in Sec. 5.

2. DESCRIPTION OF OBSERVED-BASED SYNCHRONIZATION SYSTEM

Consider the n-dimensional discrete-time unidirectionally coupled drive–response systems. A block-diagram for implementing drive–response synchronization of two unidirectionally coupled systems via a discrete communication channel is shown in Fig. 1. To simplify exposition we will consider drive (master, entraining) system in so-called *Lurie form*: right-hand sides are split into a linear part and a nonlinearity vector depending only on the measured output. Then the drive system is modeled as follows:

$$x_{k+1} = Ax_k + B\varphi(y_k), \; y_k = Cx_k, \tag{1}$$

where $k \in \mathbb{Z}$ is discrete time, $k = 0, 1, \ldots$; $x_k \in \mathbb{R}^n$ is the vector of state variables, y_k is the scalar output (coupling) variable, A is an $(n \times n)$-matrix, B is $(n \times 1)$-matrix C is $(1 \times n)$-matrix, $\varphi(y)$ is a continuous nonlinearity. We assume that all the trajectories of the system (1) belong to a bounded set Ω (e.g. attractor of a chaotic system). Such an assumption is typical for chaotic systems.

The response (slave, entrained) system is described as a *nonlinear observer*

$$\hat{x}_{k+1} = A\hat{x}_k + B\varphi(y_k) + L\varepsilon_k, \;\; \varepsilon_k = y_k - \hat{y}_k, \; \hat{y}_k = C\hat{x}_k, \tag{2}$$

where L is the vector of the observer parameters (gain). Apparently, the dynamics of the state error vector $e_k = x_k - \hat{x}_k$ is described by a linear equation

$$e_{k+1} = A_L e_k, \tag{3}$$

where $A_L = A - LC$.

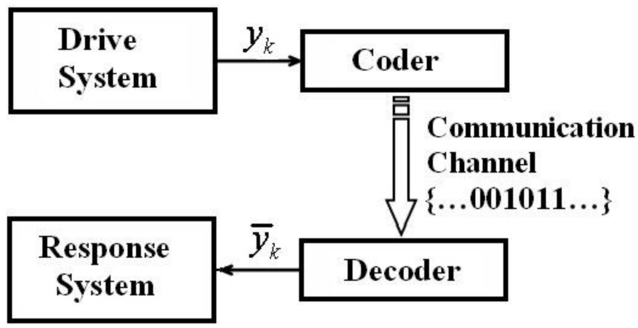

Fig. 1. Block-diagram for drive–response synchronization using a discrete communication channel.

For any observable pair (A, C) there exists L providing the matrix A_L with any given eigenvalues. Particularly, if all eigenvalues of A_L lie inside the unit circle on the complex plane, the system (3) is asymptotically stable [1] and $e_k \to 0$ as $k \to \infty$. The gain vector L may be found using standard pole locus placement technique, or applying H_2/H_∞ optimization procedure. Therefore, in the absence of measurement and transmission errors the synchronization error e_k decays to zero.

Let us turn to synchronization over the communication channel with a finite capacity. To simplify analysis, we assume that the observations are not corrupted by observation noise, transmissions delay and transmission channel distortions may be neglected. Therefore, it is assumed that the coded symbols are available at the receiver side at the same instant k, as they are generated by the coder.

Our goal is to demonstrate that the full-order (of order n) time-varying coder makes possible to ensure asymptotically arbitrarily small synchronization error if the channel capacity is sufficiently large. This result complies with those obtained for a linear case in (Nair and Evans, 1997, 1998, 1999; Savkin and Petersen, 2003). The idea is to use the "zooming" strategy to increase coder accuracy as the estimation error decreases, and, at the same time, to prevent coder saturation at the beginning of the process (Brockett and Liberzon, 2000; Liberzon, 2003; Nair and Evans, 2003; Tatikonda and Mitter, 2004a). Since the quantizer range decreases from step to step, the *prediction* procedure is used at the coder/decoder pair. To accomplish this procedure, the full-order coder is used, where the state estimation algorithm based on (2) is realized.

3. CODING PROCEDURE

3.1 Static coder.

At first, consider the memoryless (static) coder with uniform discretization and constant range. For given real number $M > 0$ and natural number $\nu \in \mathbb{Z}_+$ define a uniform scaled coder to be a discretized map $q_{\nu,M} : \mathbb{R} \to \mathbb{R}$ as follows. Introduce the *range*

[1] The finite n-step transient time may be obtained if all eigenvalues of A_L are zeros.

interval $\mathscr{I} = [-M, M]$ of length $2M$ and the *discretization interval* of length $\delta = M/\nu$. Apparently, 2ν is the number of different levels of the quantizer output in the interval $\mathscr{I}$. Define the coder function $q_{\nu,M}(y)$ as

$$q_{\nu,M}(y) = \min\left(\left\langle \delta^{-1}|y| \right\rangle \delta, M\right) \cdot \mathrm{sign}(y), \qquad (4)$$

where $\langle\cdot\rangle$ rounds the argument towards nearest integer, $\mathrm{sign}(\cdot)$ is the signum function:

$$\mathrm{sign}(y) = \begin{cases} 1 & \text{if } y \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

For illustration, the plot of function $q_{\nu,M}(y)$ for $\nu = 3, M = 1$ is given in Fig. 2.

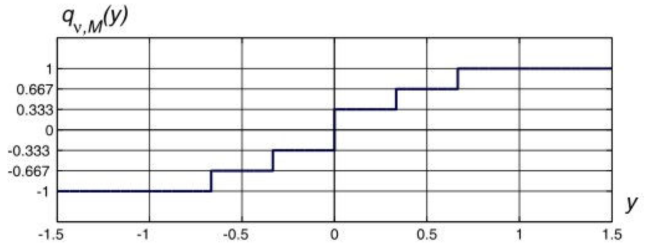

Fig. 2. Plot of the quantizer function (4) $q_{\nu,M}(y)$, $\nu = 3, M = 1$.

Notice that the interval $\mathscr{I}$ is split into 2ν equal parts. Therefore, the cardinality of the mapping $q_{\nu,M}$ image is equal to 2ν, and each codeword symbol contains $R = 1 + \log_2 \nu$ bits of information. [1] Thus, the discretized output of the considered coder is $\bar{y} = q_{\nu,M}(y)$. We assume that the coder and decoder make decisions based on the same information.

Expression (4) describes a simple ("primitive") static coder. More sophisticated encoding schemes utilize time-varying coders with memory, see, e.g., (Brockett and Liberzon, 2000; Liberzon, 2003; Nair and Evans, 2003; Tatikonda and Mitter, 2004a). The underlying idea for coders of this kind is to reduce the range parameter M, replacing the symmetric range interval $\mathscr{I}$ by the interval $\mathscr{Y}_{k+1}$, centerd at the predicted value for the $(k+1)$th observation y_{k+1}, $y_{k+1} \in \mathscr{Y}_{k+1}$. If the length of $\mathscr{Y}_{k+1}$ is small compared with the full range of possible measured output values y, then there is an opportunity to reduce the range parameter M and, consequently, to decrease the coding interval δ, preserving the bit-rate of transmission. To realize this scheme, memory should be introduced into the coder. Using such a "zooming" strategy allows to increase coder accuracy in the steady-state mode, and, at the same time, to prevent coder saturation at the beginning of the process. This means that the quantizer range M is updated at each sampling interval and a time-varying quantizer (with time-varying $M = M_k$) is used. The values of M_k may be precomputed (the *time-based* zooming), or, alternatively, current quantized measurements may be used at each step to update M_k (the *event-based* zooming).

3.2 Observer-based coder of the full order.

In this paper we use an nth-order coder with time-based zooming. Assume that the observation signal y_k is coded with symbols from a finite alphabet $\mathscr{S} = \{s_1, s_2, \ldots s_{\aleph}\}$ at time instants $k = 0, 1, 2, \ldots$. Due to the finite length of the codeword, the *transmission error*

$$\delta_{y,k} = y_k - \bar{y}_k, \qquad (5)$$

where $\bar{y}_k$ is a coded symbol transmitted over a digital communication channel, appears in the system. In this case, (3) does not describe the estimation error dynamics, and should be replaced with the following error model:

$$e_{k+1} = A_L e_k + B\big(\varphi(y_k) - \varphi(y_k + \delta_{y,k})\big) - L\delta_{y,k}. \qquad (6)$$

Equation (6) is a heterogeneous one and the asymptotic convergence e_k to zero can not be ensured.

To exclude the effect of quantization errors, retaining at the same time the data rate limitations, let us introduce the following coding/decoding procedure, based on implementation of the full-order coder.

Let the following observer-based quantizer at the side of a drive system be used:

$$\begin{cases} \hat{x}_{k+1} = A\hat{x}_k + B\varphi(\hat{y}_k + \bar{\varepsilon}_k) + L\bar{\varepsilon}_k, \;\; \hat{y}_k = C\hat{x}_k, \\ \varepsilon_k = y_k - \hat{y}_k, \;\; \bar{\varepsilon}_k = q_{\nu,M_k}(\varepsilon_k), \\ M_k = \max(M_0\rho^k, \mu), \end{cases} \qquad (7)$$

where $q_{(\cdot)}$ is the static coder function (4); M_k is the variable range of the quantizer, $M_0 > 0$ is its initial condition, the constant $0 < \rho < 1$ is the decay parameter, $\mu > 0$ stands for the limit value of M_k. The initial value M_0 should be large enough to capture all the region of possible values of ε_0.

The quantized *deviation signal* $\bar{\varepsilon}_k$ is represented as an R-bit information symbol from the coding alphabet $\mathscr{S}$ and transmitted over the communication channel to the decoder.

Under the foregoing assumptions on the channel properties, the signal $\bar{\varepsilon}_k$ is precisely decoded at the receiver end at the same instant t_k as it is generated by the coder. The similar to (7) procedure is realized at the receiver end, namely:

$$\hat{x}^r_{k+1} = A\hat{x}^r_k + B\varphi(\hat{y}^r_k + \bar{\varepsilon}_k) + L\bar{\varepsilon}_k, \;\; \hat{y}^r_k = C\hat{x}^r_k, \qquad (8)$$

where $x^r_k \in \mathbb{R}^n$ is the vector of the receiver state variables, y^r_k is the receiver output. Equation (8) describes the observer-based decoder/receiver of nth-order.

The proposed synchronization scheme makes possible to ensure convergence of the synchronization error to the μ-neighborhood of zero for any $\mu > 0$ (recall, that μ is the limit value of M_k). Such a property is called *practical synchronization*. For an ideal case $\mu = 0$ is taken, ensuring asymptotical convergence of the error e_k to zero. In the real systems the computation errors occur and μ should be positive. The numerical results are presented in next Section.

Proposition 1. In the system (1), (7), (8) the practical synchronization occurs.

Proof.

The key point of the proof is comparison of the discrete-time system in question with an auxiliary continuous-time system possessing useful stability and passivity properties (Derevitsky and Fradkov, 1974, 1981).

To apply this method introduce the following continuous-time system ("the *continuous model*")

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = (A - \mathbf{I})x(t) + B\varphi(y), \; y = Cx, \qquad (9)$$

[1] Traditionally, the information content of the codeword is defined according to on the probability distribution of the source (Cover and Thomas, 1991). Since we are interested in the guaranteed bounds, we evaluate the information content of the codeword as $R = 1 + \log_2 \nu$ bits, which is an exact upper bound on the information content of a codeword, realized if the source has a uniform distribution over the interval $[-M, M]$. In other words, we use a *combinatorial* definition of information (Kolmogorov, 1965), rather than a *probabilistic* one.

where $x \in \mathbb{R}^n$ is the model state vector, $\mathbf{I}$ is $n \times n$ identity matrix. Substituting $m_t \tau$ for t in (9), where m_t is a scaling factor, we obtain

$$\frac{\mathrm{d}x(t)}{\mathrm{d}\tau} = m_t(A-\mathbf{I})x(\tau) + m_t B\varphi(y), \; y = Cx. \qquad (10)$$

The discrete-time Euler model for the continuous-time system (10) is as follows:

$$x_{k+1} = x_k + \Delta t m_t (A-\mathbf{I})x_k + \Delta t m_t B\varphi(y_k), \; y_k = Cx_k. \qquad (11)$$

where $\Delta t > 0$ is sampling time.

Suppose that the nonlinearity $\varphi(\cdot)$ in (1) is locally Lipschitz and that all trajectories of (1) are bounded, $\|x(t)\| \le C$. Then, as shown in (Derevitsky and Fradkov, 1974, 1981), there exists sufficiently small Δt such that solutions to (10) lie in the given ε-vicinity of the trajectories (11). Choose $m_t = \Delta t^{-1}$. Then (11) read as

$$x_{k+1} = Ax_k + B\varphi(y_k), \; y_k = Cx_k. \qquad (12)$$

Therefore, (9) may be considered as a continuous-time model of the discrete system (1), where the sample time and time scale are appropriately chosen. Now results of (Fradkov et al., 2007) may be directly applied implying that proper choice of the continuous model (9) ensures desired behavior of the discrete-time system (1), (7), (8) within the accuracy $\varepsilon > 0$.

It should be stressed that the signal $\bar{\varepsilon}_k$ in this synchronization scheme is an *outer* one for the receiver (8) and an *inner* signal (a feedback signal) for the coder (7). Therefore, there is no stabilization feedback loop in the decoder (receiver), and divergence of the receiver variables may occur if the computations at the coder and decoder sides are not identical. For the authors' best knowledge, this circumstance is missed in the works devoted to the problems of estimation and control under communication constraints, see e.g. Baillieul (2002); Brockett and Liberzon (2000); Cheng and Savkin (2006); Delchamps (1990); Elia et al. (2001); Fagnani and Zampieri (2004); Liberzon (2003); Nair and Evans (2003); Tatikonda and Mitter (2004b). For preventing the divergence effect, the fixed-point computations are recommended to use at the both coder (7) and decoder (8) sides.

4. EXAMPLE. SYNCHRONIZATION OF CHAOTIC HÉNON SYSTEMS

Let us apply the above approach to synchronization of two chaotic Hénon systems coupled via a channel with limited capacity.

System equations. Consider the following chaotic *Hénon system* (Andrievsky and Fradkov, 2003; Moon, 1992):

$$\begin{cases} \xi_{1,k+1} = 1 - a\xi_{1,k}^2 + \xi_{2,k}, \\ \xi_{2,k+1} = bx_{1,k}, \end{cases} \qquad (13)$$

where $\xi = [\xi_1, \xi_2]^{\mathrm{T}} \in \mathbb{R}^2$ is the state vector, a, b are system parameters (in the sequel, $a = 1.4$, $b = 0.3$ are taken).

Applying the state-space transform, rewrite (13) in the following form:

$$\begin{cases} x_{1,k+1} = x_{2,k}, \\ x_{2,k+1} = bx_{1,k} - ax_{2,k}^2 + 1, \end{cases} \qquad (14)$$

where $x = [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$ is the state vector. Assume that $y(t) = x_{2,k}$ is the system output signal to be transmitted over the communication channel. The vector of initial conditions $x_0 = x(0)$ is assumed to be unknown at the side of the receiver.

Observer design. Since the drive system (14) has a Lurie form (1), let us use the coder (7) at the drive side and decoder (8) at the receiver end.

For the considered case, the matrices A, B, C and the function $\varphi(\cdot)$ in (2) are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = [0,1], \quad \varphi(z) = 1 - az^2. \qquad (15)$$

The coder (7) at the side of the drive system (14) has a following form:

$$\begin{cases} \hat{x}_{1,k+1} = \hat{x}_{2,k} + l_1\bar{\varepsilon}_k, \quad \hat{y}_k = x_{2,k}, \\ \hat{x}_{2,k+1} = b\hat{x}_{1,k} - a\bar{y}_k^2 + 1 + l_2\bar{\varepsilon}_k, \\ \varepsilon_k = y_k - \hat{y}_k, \;\; \bar{\varepsilon}_k = q_{\nu,M_k}(\varepsilon_k), \\ M_k = \max(M_0\rho^k, \mu). \end{cases} \qquad (16)$$

The decoder/response systems (8) is described by equations:

$$\begin{cases} \hat{x}^r_{1,k+1} = \hat{x}^r_{2,k} + l_1\bar{\varepsilon}_k, \quad \hat{y}^r_k = x^r_{2,k}, \\ \hat{x}^r_{2,k+1} = b\hat{x}^r_{1,k}a(\bar{y}^r_k)^2 + 1 + l_2\bar{\varepsilon}_k, \end{cases} \qquad (17)$$

where $\bar{\varepsilon}_k$ is the output signal of the coder (17), transmitted over the communication channel in the form of the codeword.

The characteristic polynomial $A_L(\lambda) = \det(\lambda\mathbf{I}_2 - A_L)$ of the matrix

$$A_L = A - LC = \begin{bmatrix} 0 & 1-l_1 \\ b & -l_2 \end{bmatrix}$$

has a form $A_L(\lambda) = \lambda^2 + \lambda l_2 - b + bl_1$. Let the desired eigenvalues $\lambda_{1,2} \in \mathbb{C}$ of the matrix A_L be given. Then the desired characteristic polynomial $A_L^*(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 + d_1\lambda + d_2$, where $d_1 = -(\lambda_1 + \lambda_2)$, $\lambda_2 = \lambda_1\lambda_2$. Therefore, one obtains the following expressions for the observer (17) gains: $l_1 = (\lambda_1\lambda_2 + b)b^{-1}$, $l_2 = -\lambda_1 - \lambda_2$.

Simulation results. The system (13), (17) was studied numerically for the following parameter values and initial conditions:

$$a = 1.4, \; b = 0.3, \; l_1 = 1.0, \; l_2 = -1.2 \cdot 10^{-3},$$
$$\rho = 0.8; \; x_{1,0} = 0.75, \; x_{2,0} = -0.5, \; \hat{x}_{1,0} = \hat{x}_{2,0} = 0.$$

The number of quantizer levels $\nu = 3$ was chosen. This number corresponds the channel bit-rate $R = 1 + \log_2(\nu) = 2.585$ bit per step. The minimal bound for the decay parameter ρ in (7), $\rho_{\min} \approx 0.75$, was also found. For the less values of ν and ρ the synchronization process failed. This result confirms the general statement, claiming that the difference between the entropies of the open loop and the closed loop systems cannot exceed the information introduced by the controller, including the transmission rate of the information channel (Nair and Evans, 2003).

Different values of the parameter μ in (7) were taken to evaluate the minimal possible limit synchronization error $\overline{\lim_{k\to\infty}}\|e_k\|$, obtained by means of available computer platform and simulation environment. [1] During the simulations it was found that the minimal admissible μ is $\mu_{\min} = 2 \cdot 10^{-15}$.

Some simulation results are presented in Figs. 3–6.

Simulation results show that the limit synchronization error can be made close to the maximum achievable accuracy of the given computer (*computer epsilon*, Forsythe et al. (1977)) depending

[1] The 32-bit AMD Athlon™ processor and MATLAB software with the floating point relative accuracy (*computer epsilon*) $\varepsilon_c = 2.2204 \cdot 10^{-16}$ were used.

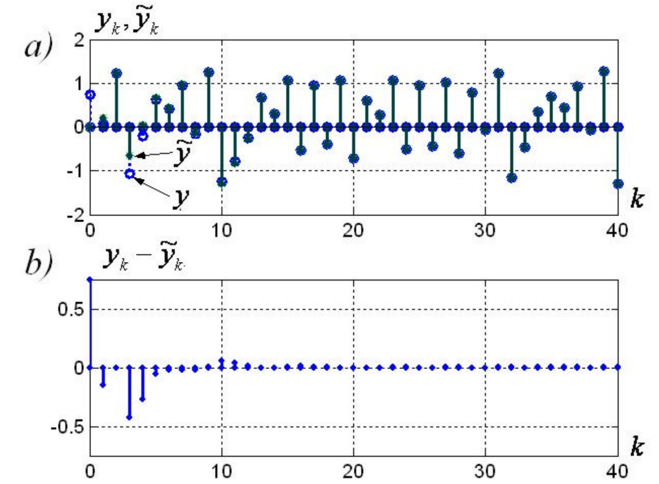

Fig. 3. Time histories of the drive and response systems outputs (*a*) and of the synchronization error (*b*).

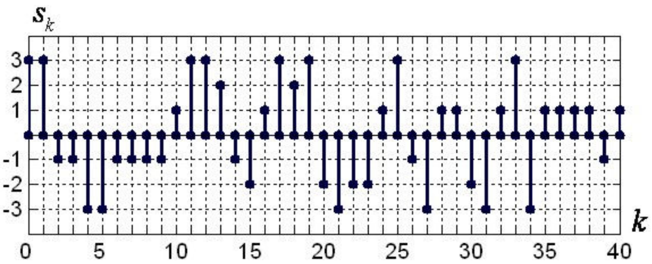

Fig. 4. Sequence of the codewords, transmitted over the channel, ($\mathscr{S} = \{-1, -2, -1, 1, 2, 3\}$).

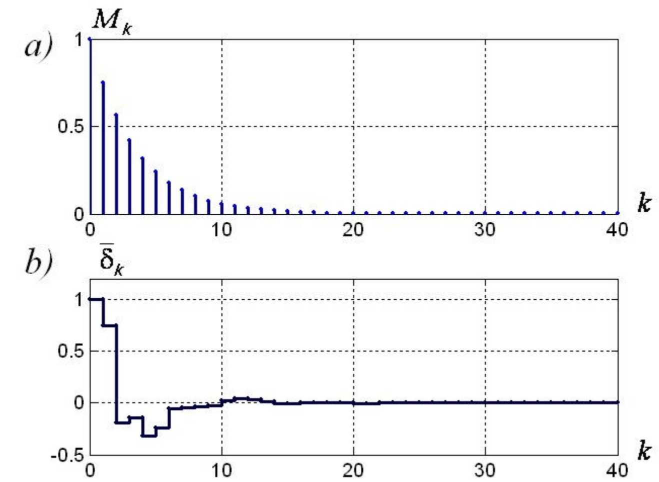

Fig. 5. Time histories of the quantizer range M_k (*a*) and the quantizer output (*b*).

only on the number of digits in the computer. Such a phenomenon can be called *practical synchronization*, by analogy with practical stability (Moreau and Aeyels, 2000): the limit absolute value of synchronization error decreases unlimitedly if the accuracy of the computation increases unlimitedly.

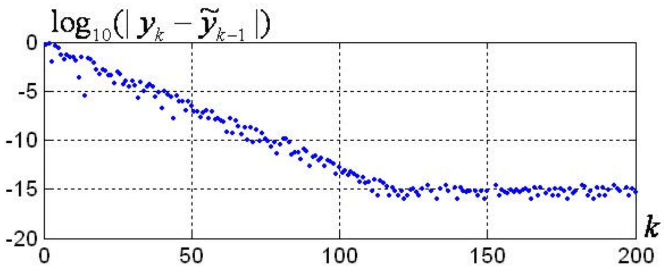

Fig. 6. Logarithmically scaled synchronization error.

5. CONCLUSIONS

We have studied synchronization over the limited bandwidth communication link for a class of discrete-time chaotic systems. Solution to the problem of observer-based synchronization over the limited bandwidth communication link for a class of discrete-time chaotic systems is presented. The result is demonstrated on the synchronization of chaotic Hénon systems via a channel with limited capacity.

In the paper the concept of *practical synchronization* is introduced and its occurrence for the proposed coding-decoding scheme is established.

It is shown that if a real computer is used for computation, then an effect of practical synchronization rather then asymptotical synchronization is observed. Namely, an absolute value of synchronization error is bounded by a value Δ rather then tends to zero. If transmission rate is sufficiently large, the value of Δ depends solely on the number of digits of the computer and it is close to the maximum achievable accuracy of the given computer. If the accuracy of the computation increases unlimitedly, then the limit absolute value of synchronization error decreases unlimitedly.

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