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Necessary and Sufficient Conditions for Perfect Command Following and Disturbance Rejection in Fractional Order Systems

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Abstract: The aim of this paper is to present a modified explanation of the classic internal model principle for certain class of finite-dimensional, time-invariant, deterministic fractional-order systems commonly known as fractional systems of commensurate order. The necessary and sufficient conditions for perfect command tracking and disturbance rejection are provided. The difficulty of applying the classic internal model principle to fractional-order systems is due to the difference between integer-order and fractional-order systems from the zero-pole cancellation point of view. The notion of zero-pole cancellation is discussed for the systems under consideration in a well posed mathematical framework. It is also shown that fractional elements can be used for command tracking and disturbance rejection purposes which provides more flexibility for controller design applications. Two illustrative examples confirm the applicability of the proposed theorems.

Keywords: Fractional systems; Controller constraints and structure; Analytic design; Internal model principle.

1. INTRODUCTION

The idea of internal model principle was first introduced in the work of Francis and Wonham (1976) which dealt with the regulator problem for linear, time-invariant, finite-dimensional systems with deterministic disturbance and reference signals. The main result of that work, for the closed-loop system shown in Fig. 1, is that the controller $C(s)$ must incorporate in the feedback path a suitable model of the dynamic structure of the disturbance and reference signal in order to achieve perfect asymptotic disturbance rejection and command tracking. That is why an integrator must be provided in the forward path of a given stable closed-loop system for tracking the step input without steady-state error.

In recent years there has been an increasing attention to fractional-order systems. These systems are of interest for both modelling and controller-design purposes. In the field of continuous-time modelling, fractional derivatives have proved to be useful in linear viscoelasticity, acoustics, rheology, polymeric chemistry, biophysics, etc (Oldham and Spanier, 1974; Hilfer, 2000). In general, fractional-order systems are useful to model various stable physical phenomena (commonly diffusive type systems) with anomalous decay, say those that are not of an exponential type. For example, Miller and Ross (1993) introduced a real-world system with impulse response

$$h(t) = \frac{\sqrt{2g\pi}}{\Gamma(3/2)} t_+^{1/2}, \tag{1}$$

which corresponds to the transfer function

$$H(s) = \frac{\sqrt{2g\pi}}{s^{3/2}}. \tag{2}$$

As an example of using fractional derivatives for modelling, Beyer and Kempfle (1995) studied the generalized damping equation

$$(D^2 + aD^q + b)x(t) = f(t), \quad q \in (0, 2) \tag{3}$$

and discussed the advantages of fractional modelling. The transfer function of the above system is easily found to be

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + as^q + b}. \tag{4}$$

In the field of linear viscoelasticity, Glöckle et al. (1991) used fractional calculus to generalize the Zener model. They proposed the fractional (integral) equation of unknown orders β and μ:

$$\frac{1}{\tau_0^\beta} {}_0D_t^{-\beta}\sigma(t) + \sigma(t) - \sigma_0 = \frac{G_e}{\tau_0^\beta} {}_0D_t^{-\mu}\varepsilon(t) + G_0[\varepsilon(t) - \varepsilon_0], \tag{5}$$

where σ and ε are stress and strain, respectively, and τ_0, G_m, η_m, and G_e are real physical constants. Equation (5) corresponds to the transfer function

$$H(s) = \frac{\widetilde{\sigma}(s)}{\widetilde{\varepsilon}(s)} = \frac{G_0 + G_e(s\tau_0)^{-\mu}}{1 + (s\tau_0)^{-\beta}}, \tag{6}$$

where the initial values are chosen such that $\sigma_0 = G_0\varepsilon_0$. The transfer functions (2), (4), and (6) represent practical systems with non-integer powers of the Laplace variable.

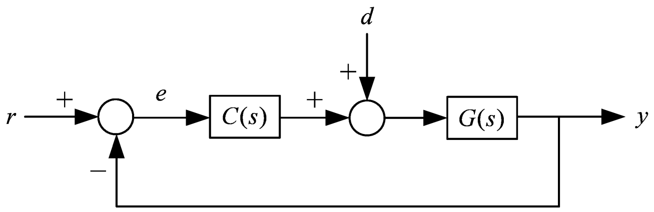

Fig. 1. The standard closed-loop system

An interesting study of fractional differential systems appeared in (Viano et al., 1994) using a stochastic framework. The idea of fractional powers is also used for identification purposes in order to reach more accurate models. Tsao et al. (1989) and Poinot and Trigeassou (2004), clarify the identification method when the members of model set are of fractional order. Two applications of such identifications can be found in (Vinagre et al., 1998) and (Chauchois et al., 2003). Fractional differential systems are also used in control field. Podlubny (1999) and Valério and Costa (2006) discussed methods of designing $PI^{\lambda}D^{\mu}$ controllers, Raynaud and ZergaInoh (2000) studied fractional-order lead-lag compensators and Oustaloup et al. (1995, 1996) introduced the so-called CRONE controllers.

Systems of *commensurate order* of derivatives are the systems that have been described by fractional differential equations of commensurate order. Such systems lend themselves well to some algebraic tools (Miller and Ross, 1993; Beyer and Kempfle, 1995). For instance, $H(s)$ as defined in (2) is a transfer function for a system of commensurate order. More examples of practical fractional differential systems of commensurate order can be found in (Beyer and Kempfle, 1995; Vinagre et al., 1998; Chauchois et al., 2003). The inverse Laplace transform of such systems involve special functions (for definition and notations see Miller and Ross, 1993).

It was shown in (Francis and Wonham, 1976) that the purpose of the internal model is to supply closed-loop transmission zeros which cancel the unstable poles of the disturbance and reference signals. But unfortunately the notion of zero-pole cancellation in fractional case (e.g., in dealing with transfer functions like (2), (4), or (6)) is much more different from the integer case. Note that unlike the integer case, if $A(s)$ and $B(s)$ are two fractional-order polynomials (see Definition 1) with the same zeros, then in general we cannot conclude that $A(s)/B(s)$ is equal to a constant value, i.e. a zero does not necessarily cancel the same pole. For example, consider $A(s) = s^{1/2} - 1$ and $B(s) = s^{1/3} - 1$. Both A and B have only one zero at $s = 1$ (see Proposition 3), but

$$\begin{aligned}\frac{A(s)}{B(s)} &= \frac{s^{1/2}-1}{s^{1/3}-1}\\ &= \frac{(s^{1/6}-1)(s^{1/3}+s^{1/6}+1)}{(s^{1/6}-1)(s^{1/6}+1)}\\ &= \frac{s^{1/3}+s^{1/6}+1}{s^{1/6}+1} \neq \text{constant}.\end{aligned}$$

This example shows the need for a modified explanation of the existing internal model principle which is discussed in this paper. The aim of this brief is not to propose a controller synthesis algorithm but only to provide the necessary and sufficient conditions needed for perfect command tracking and disturbance rejection in fractional case.

The rest of this paper is divided to four sections. Problem preliminaries are presented in Section 2. Theorems 8 and 9 are the main results of this paper which provide the necessary and sufficient conditions for perfect command tracking and disturbance rejection for fractional systems under consideration. These two theorems are studied in Section 3. Two illustrative examples are presented in Section 4 and finally, Section 5 contains the conclusion.

2. PRELIMINARIES

2.1 Problem Prerequisites

Before introducing the main problem, some definitions and notations are provided. For simplicity, the "fractional system of commensurate order" will be addressed by "fractional system" in the rest of this paper.

Definition 1. The function

$$Q(s) = a_1 s^{q_1} + a_2 s^{q_2} + \ldots + a_n s^{q_n}, \tag{7}$$

is a fractional-order polynomial if and only if $q_i \in \mathbb{Q}^+ \cup \{0\}, a_i \in \mathbb{R}, i = 1 \ldots n$, where $\mathbb{Q}^+$ and $\mathbb{R}$ stand for the sets of positive rational numbers and real numbers, respectively.

Definition 2. Consider the fractional-order polynomial

$$Q(s) = a_1 s^{\frac{\alpha_1}{\beta_1}} + a_2 s^{\frac{\alpha_2}{\beta_2}} + \ldots + a_n s^{\frac{\alpha_n}{\beta_n}}, \tag{8}$$

where

$$a_i \in \mathbb{R}, \quad \alpha_i \in \mathbb{N} \cup \{0\}, \quad \beta_i \in \mathbb{N},$$

and α_i, β_i are relatively prime for $i = 1, \ldots, n$ and $\mathbb{N}$ is the set of natural numbers. (If for some i, $\alpha_i = 0$ then by definition $\beta_i = 1$.) Let λ be the least common multiple (lcm) of $\beta_1, \beta_2, \ldots, \beta_n$ denoted as $\lambda = \text{lcm}\{\beta_1, \beta_2, \ldots, \beta_n\}$. Then

$$\begin{aligned}Q(s) &= a_1 s^{\frac{\lambda_1}{\lambda}} + a_2 s^{\frac{\lambda_2}{\lambda}} + \ldots + a_n s^{\frac{\lambda_n}{\lambda}} & (9)\\ &= a_1 (s^{\frac{1}{\lambda}})^{\lambda_1} + a_2 (s^{\frac{1}{\lambda}})^{\lambda_2} + \ldots + a_n (s^{\frac{1}{\lambda}})^{\lambda_n}. & (10)\end{aligned}$$

Now the fractional degree (fdeg) of $Q(s)$ is defined as $\text{fdeg}\{Q(s)\} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

The domain of definition for (10) is a Riemann surface with finite number of Riemann sheets (λ sheets here) where origin is a branch point (of order $\lambda - 1$) and the branch-cut is assumed at $\mathbb{R}^-$ (LePage, 1961). Note that the fractional-order polynomial and the fractional degree as defined above reduce to the conventional concepts of polynomial and the degree of a polynomial when $\lambda = 1$. The following proposition gives the roots number for a fractional algebraic equation.

Proposition 3. Let $Q(s)$ be a fractional-order polynomial with $\text{fdeg}\{Q(s)\} = n$. Then the equation $Q(s) = 0$ has exactly n roots on the Riemann surface.

Proof. Consider

$$Q(s) = a_1 (s^{\frac{1}{v}})^n + a_2 (s^{\frac{1}{v}})^{n-1} + \ldots + a_n (s^{\frac{1}{v}})^1 + a_{n+1}, \tag{11}$$

for an appropriate $v \in \mathbb{N}$. Assuming $w := s^{\frac{1}{v}}$, we have

$$\widetilde{Q}(w) = a_1 w^n + a_2 w^{n-1} + \ldots + a_n w + a_{n+1}. \tag{12}$$

The fundamental theorem of algebra gives n roots for $\widetilde{Q}(w) = 0$, say $w_1, w_2, \ldots, w_n$. Consequently, $Q(s) = 0$ has n roots at $s_1 = w_1^v, s_2 = w_2^v, \ldots, s_n = w_n^v$.

Note that the zeros of a fractional-order polynomial are distributed on a Riemann surface the location of which can be easily calculated using the above change of variable. If the power of any term of (7) be an irrational number, then the equation $Q(s) = 0$ will have infinite number of roots. For example, consider the fractional-order equation

$$s^{\pi} + 1 = 0, \tag{13}$$

the roots of which are $s_k = e^{j(2k+1)}$ for every $k \in \mathbb{Z}$. All these roots are distinct because if for some $k_1, k_2, n \in \mathbb{Z}$ we have

$$\angle s_{k_2} - \angle s_{k_1} = (2k_2 + 1) - (2k_1 + 1) = 2n\pi, \tag{14}$$

(i.e., if s_{k_2} is the same as s_{k_1}) then necessarily $k_2 - k_1 = n\pi$ which is in contradiction with our first assumption that k_1, k_2 are integer numbers. Obviously, the roots of (13) are distributed on a Riemann surface with infinite number of Riemann sheets. In the rest of this paper, the symbol C_+ is used to denote the closed right-half plane (CRHP) of the first Riemann sheet.

In the following we study the notion of zero-pole cancellation for fractional systems with deeper analysis. It is a common practice to consider the transfer function of a fractional-order process as

$$G(s) = \frac{b_0 s^{\frac{m}{v}} + b_1 s^{\frac{m-1}{v}} + \ldots + b_{q-1} s^{\frac{1}{v}} + b_q}{s^{\frac{n}{v}} + a_1 s^{\frac{n-1}{v}} + \ldots + a_{p-1} s^{\frac{1}{v}} + a_p}, \tag{15}$$

where the parameter v is the smallest integer that allows interpreting G in this form. The above form is of great significance especially for identification purposes. For example, if one does an identification with the precision of two digits then v is equal to 100. Note that assuming rational numbers for the powers of the Laplace variable s is not a loss of generality because in practice all numbers are stored with a limited precision in computer and moreover, one can find a rational number in the vicinity of any real number. When one interprets a fractional-order transfer function in the form of (15) the number of poles and zeros may artificially be increased. For example, consider the transfer function

$$G(s) = \frac{s^{1/2} - 1}{s^{1/3} - 1},$$

which according to Proposition 3 has only one pole and one zero. This transfer function can be represented in the equivalent form

$$G(s) = \frac{s^{3/6} - 1}{s^{2/6} - 1},$$

which is in the form of (15). This equivalent representation has 2 poles and 3 zeros. So, the question is what happens when one represents a fractional-order polynomial like

$$Q(s) = (s^{\frac{1}{v}})^n + a_1 (s^{\frac{1}{v}})^{n-1} + \ldots + a_p, \tag{16}$$

in the equivalent form

$$Q(s) = (s^{\frac{1}{kv}})^{kn} + a_1 (s^{\frac{1}{kv}})^{k(n-1)} + \ldots + a_p. \tag{17}$$

According to Proposition 3, $Q(s)$ in (16) has n zeros which are distributed on v Riemann sheets while $Q(s)$ in (17) has kn zeros which are distributed on kv Riemann sheets. It can be shown that in the latter case, the second, the third, ..., the k'th set of v successive Riemann sheets are copies of the first v successive sheets. For example, the location of poles and zeros on the first, $(v+1)$'th, ..., $(k-1)v+1$'th sheets are the same.

Definition 4. The fractional-order polynomial

$$Q(s) = a_1 s^{\frac{n}{v}} + a_2 s^{\frac{n-1}{v}} + \ldots + a_n s^{\frac{1}{v}} + a_{n+1},$$

is *minimal* if fdeg{Q(s)} $= n$.

In the rest of this paper, it is assumed that all fractional-order polynomials are minimal unless it is mentioned explicitly. This ensures that there is no redundancy in the number of Reimann sheets. In the same manner, when it is referred to the poles (zeros) of a fractional-order transfer function it is assumed that the fractional-order polynomial in denominator (numerator) is in its minimal form. Obviously, rewriting (16) in the form of (17) will not change the basic properties of the system (such as stability) because they are determined by the poles and zeros on the first Reimann sheet. In order to proceed with resolving the problem of zero-pole cancelation in the fractional case, we need to introduce the following definition.

Definition 5. Consider two fractional-order polynomials $F(s)$ and $G(s)$. In general, we can interpret $F(s)$ and $G(s)$ as $F(s) = s^{\alpha} F_1(s)$ and $G(s) = s^{\beta} G_1(s)$ where $\alpha, \beta \in \mathbb{Q} \cup \{0\}$, and $F_1(s)$, $G_1(s)$ are two fractional order polynomials such that $F_1(0) \neq 0$ and $G_1(0) \neq 0$. We say $F(s)$ *contains* $G(s)$ if the following two conditions are satisfied:

(1) The roots of the equation $G_1(s) = 0$ be the subset of the roots of the equation $F_1(s) = 0$, i.e. $G_1(s_0) = 0$ for every s_0 such that $F_1(s_0) = 0$.
(2) $\alpha > \beta - 1$.

The following lemma attests that the ratio of two fractional order polynomials with identical zeros is analytic at those zeros.

Lemma 6. Consider the fractional-order polynomials $A(s)$ and $B(s)$ which can be represented as

$$A(s) = \prod_{k=1}^{m} (s^{1/q_1} - s_k^{1/q_1}) \prod_{k=m+1}^{n} (s^{1/q_1} - s_k^{1/q_1}),$$

and

$$B(s) = \prod_{k=1}^{m} (s^{1/q_2} - s_k^{1/q_2}),$$

where $s_1, \ldots, s_n$ are non-zero constants such that $\Re\{s_i\} \geq 0$, $i = 1, \ldots, n$ [1]. Obviously, $A(s)$ and $B(s)$ have zeros at $\{s_1, \ldots, s_n\}$ and $\{s_1, \ldots, s_m\}$, respectively. Then the function A/B is analytic at $s_1, \ldots, s_m$.

Proof. Assuming $v = \mathrm{lcm}\{q_1, q_2\}$ we can write

$$A(s) = \prod_{k=1}^{m} (s^{p_1/v} - s_k^{p_1/v}) \prod_{k=m+1}^{n} (s^{p_1/v} - s_k^{p_1/v}),$$

and

$$B(s) = \prod_{k=1}^{m} (s^{p_2/v} - s_k^{p_2/v}),$$

for appropriate chosen constants p_1 and p_2. It is now sufficient to show that

$$\frac{s^{p_1/v} - s_k^{p_1/v}}{s^{p_2/v} - s_k^{p_2/v}},$$

[1] No matter which branch of s_k^{1/q_1} or s_k^{1/q_2} is considered.

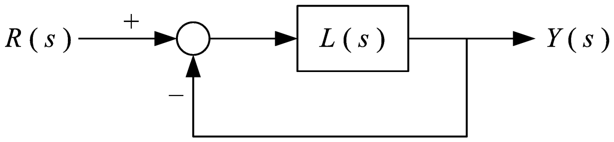

Fig. 2. The connection corresponding to Theorem 8 where L stands for the open-loop transfer function CG

is analytic at $s = s_k$ $(i = 1, \ldots, m)$. By expanding the fractional-order polynomials in numerator and denominator one finds

$$\frac{s^{\frac{p_1}{v}} - s_k^{\frac{p_1}{v}}}{s^{\frac{p_2}{v}} - s_k^{\frac{p_2}{v}}} = \frac{s^{\frac{1}{v}} - s_k^{\frac{1}{v}}}{s^{\frac{1}{v}} - s_k^{\frac{1}{v}}} \times \frac{s^{\frac{p_1-1}{v}} + s^{\frac{p_1-2}{v}} s_k^{\frac{1}{v}} + \ldots + s^{\frac{1}{v}} s_k^{\frac{p_1-2}{v}} + s_k^{\frac{p_1-1}{v}}}{s^{\frac{p_2-1}{v}} + s^{\frac{p_2-2}{v}} s_k^{\frac{1}{v}} + \ldots + s^{\frac{1}{v}} s_k^{\frac{p_2-2}{v}} + s_k^{\frac{p_2-1}{v}}},$$

or

$$\frac{s^{\frac{p_1}{v}} - s_k^{\frac{p_1}{v}}}{s^{\frac{p_2}{v}} - s_k^{\frac{p_2}{v}}} = \frac{s^{\frac{p_1-1}{v}} + s^{\frac{p_1-2}{v}} s_k^{\frac{1}{v}} + \ldots + s^{\frac{1}{v}} s_k^{\frac{p_1-2}{v}} + s_k^{\frac{p_1-1}{v}}}{s^{\frac{p_2-1}{v}} + s^{\frac{p_2-2}{v}} s_k^{\frac{1}{v}} + \ldots + s^{\frac{1}{v}} s_k^{\frac{p_2-2}{v}} + s_k^{\frac{p_2-1}{v}}},$$

which is analytic at $s_1, \ldots, s_m$.

Notice that in Lemma 6, the fractional order polynomials A and B are defined on two Riemann surfaces with different number of sheets but both have zeros at $s_1, \ldots, s_m$. It is also obvious that $\text{fdeg}\{A\} = n > \text{fdeg}\{B\} = m$.

2.2 Notion of Stability

In the fractional case, the notion of stability is different from the integer case. Interesting result is that a stable fractional system may have root(s) in the RHP of simple complex plane. For instance, a system with characteristic equation $s^2 + s^{\frac{3}{2}} + s + 11s^{\frac{1}{2}} + 10 = 0$ has roots equal to $-3 \pm j4, 4, 1$ for which the last two are in the RHP but this system is stable (Uraz, 1979). The following theorem addresses the stability problem for fractional case (Matignon, 1998, 1996).

Theorem 7. Letting $w := s^{\frac{1}{v}}$ in (15), the fractional-order transfer function $G(s) = N(s)/D(s)$ is BIBO stable if and only if the following condition is satisfied in w-plane:

$$|\arg \sigma| > \frac{\pi}{2v}, \quad \forall \sigma \in \mathbb{C}, \quad D(\sigma) = 0. \tag{18}$$

This condition is equivalent to the closed-loop system have no pole in the CRHP of the first Riemann sheet, i.e. in C_+.

3. INTERNAL MODEL PRINCIPLE FOR FRACTIONAL-ORDER SYSTEMS

The following theorem provides the necessary and sufficient conditions needed for tracking the command signal without steady-state error for fractional case. In what follows, the disturbance is assumed to be zero for simplicity.

Theorem 8. Consider the closed-loop system of Fig. 2 which is assumed to be BIBO stable. All poles of $R(s)$ are in C_+. Consider $L = P/Q$ and $R = N/D$ where (P, Q) and (N, D) are relatively prime. Then $e \to 0$ if and only if Q contains D.

Proof. $(\Rightarrow)$ With straight calculations on finds

$$E(s) = \frac{1}{1 + L(s)} R(s) = \frac{Q(s)}{Q(s) + P(s)} \frac{N(s)}{D(s)}.$$

Considering $Q(s) = s^{\alpha} Q_1(s)$ and $D(s) = s^{\beta} D_1(s)$ where $\alpha, \beta \in \mathbb{Q}^+ \cup \{0\}$, and $D_1(0)$ and $D_2(0)$ both are non-zero finite numbers, it follows that

$$E(s) = \frac{s^{\alpha} Q_1(s)}{Q(s) + P(s)} \frac{N(s)}{s^{\beta} D_1(s)}. \tag{19}$$

According to Lemma 6, $E(s)$ has no singularity in C_+ because Q contains D and $Q(s) + P(s)$ is stable. Since $\beta - \alpha < 1$ it is concluded from (Miller and Ross, 1993) that $e(t)$ is a bounded function of time. Applying the final-value theorem one finds

$$\begin{aligned} \lim_{t \to \infty} e(t) &= \lim_{s \to 0} s \frac{Q_1(s)}{Q(s) + P(s)} \frac{N(s)}{D_1(s)} s^{\alpha - \beta} \\ &= \lim_{s \to 0} \frac{Q_1(s)}{Q(s) + P(s)} \frac{N(s)}{D_1(s)} s^{\alpha - \beta + 1} \\ &= 0. \end{aligned}$$

Note however that $s = 0$ is a branch point of $E(s)$ and it is a well-known result that a multi-valued function has no limit at its branch point (LePage (1961)). See Appendix A for the possibility of applying the final-value theorem in this case.

$(\Leftarrow)$ Suppose $e \to 0$. It then follows that $E(s)$ has no pole in C_+. Then it is concluded from (19) that $Q_1(s_0)$ must be equal to zero for every s_0 in C_+ such that $D_1(s_0) = 0$. Now the final-value theorem results in

$$\lim_{s \to 0} s \frac{s^{\alpha} Q_1(s)}{Q(s) + P(s)} \frac{N(s)}{s^{\beta} D_1(s)} = 0,$$

which yields

$$1 + \alpha - \beta > 0. \tag{20}$$

Thus $Q(s)$ must contain $D(s)$. This completes the proof.

As a result of the above theorem, the closed-loop system of Fig. 1 will track the command signal

$$r(t) = t^n u(t), \tag{21}$$

without steady-state error only if the term $s^{\alpha} (\alpha > n)$ exists in the denominator of $L(s)$.

The following theorem provides the necessary and sufficient conditions needed for perfect disturbance rejection.

Theorem 9. Consider the closed-loop system of Fig. 1 which is assumed to be BIBO stable and moreover, $r = 0$. All poles of $D(s) = \mathcal{L}\{d(t)\}$ are in C_+. Let us assume $C = P_C/Q_C, G = P_G/Q_G$, and $D = N/D$ where $(P_C, Q_C), (P_G, Q_G)$, and (N, D) are relatively prime. Then $y \to 0$ if and only if $P_G Q_C$ contain D.

Proof. A proof similar to the one presented for Theorem 8 can be provided.

4. EXAMPLES

Two illustrative examples are presented in this section. Simulations are performed by approximating the fractional-order transfer function under consideration with an integer-order one. Such an approximation can be well done in MATLAB environment using the function *invfreqs*. See also (Vinagre et al., 2000) for more details and other possibilities.

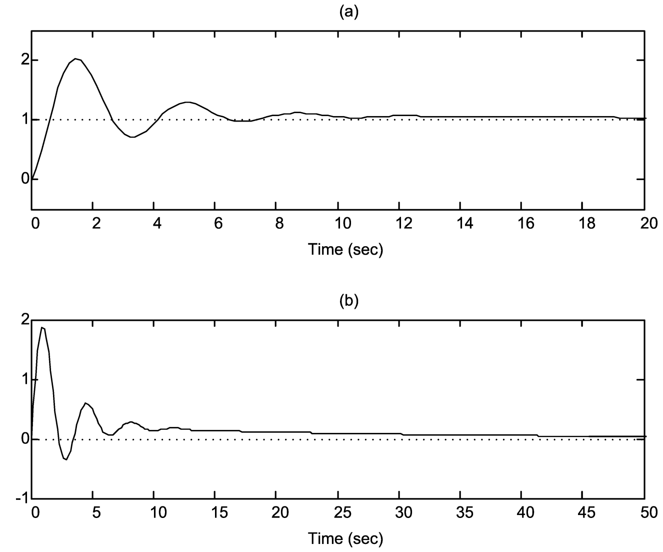

Fig. 3. The closed-loop system response to (a) step command, and (b) step disturbance corresponding to Example 4.1

4.1 Example 1

This example shows the application of a fractional element for command tracking and disturbance rejection when the plant is of integer order. Consider the closed-loop system of Fig. 1 with

$$G(s) = \frac{3}{s-1}.$$

Let the controller be a fractional integrator with transfer function

$$C(s) = \frac{1}{s^{1/2}}.$$

This transfer function is obtained by trial and error in order to reach closed-loop stability. In fact, the characteristic equation of the closed-loop system is $s^{3/2} - s^{1/2} + 3 = 0$ which satisfies the BIBO stability condition given in (18). Note that according to Theorem 8 a *bit* integration in a stable loop suffices for tracking the step command without steady-state error. It is also evident from Theorem 9 that this system can perfectly reject the step disturbance.

Figure 3(a) shows the step response of the closed-loop system when $d = 0$. As expected, error tends to zero. The closed-loop system response to the step disturbance (when $r = 0$) is shown in Fig. 3(b) which goes to zero when $t \to \infty$.

4.2 Example 2

Consider the closed-loop system of Fig. 1 with

$$G(s) = \frac{1}{s^{1/2}+1},$$

and

$$C(s) = \frac{1}{s^{4/3}}.$$

The characteristic equation of this system is

$$s^{11/6} + s^{8/6} + 1 = 0,$$

which satisfies the BIBO stability condition given in (18). Let this system be subjected to a ramp input. Using the notation of Theorem 8 one finds

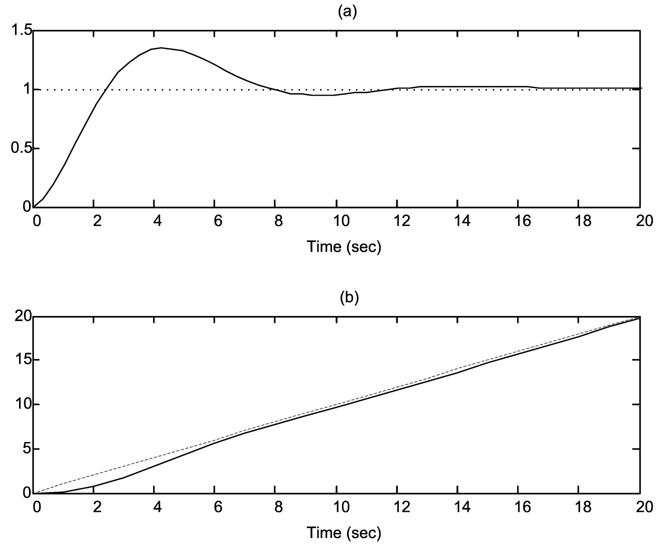

Fig. 4. The closed-loop system response to (a) step, and (b) ramp command corresponding to Example 4.2

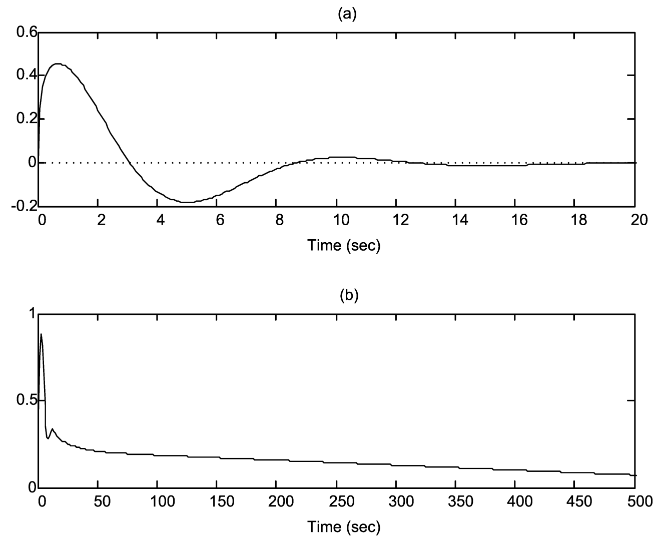

Fig. 5. The closed-loop system response to (a) step, and (b) ramp disturbance corresponding to Example 4.2

$$\alpha = \frac{4}{3}, \quad \beta = 2.$$

Since (20) is satisfied, it is concluded from Theorem 8 that this system will track the ramp (and consequently, the step) command without steady-state error. It is also concluded from Theorem 9 that the ramp disturbance will be completely rejected. Figures 4(a) and 4(b) show the responses of the closed-loop system to step and ramp commands, respectively when $d = 0$. As expected, there is no steady-state error. The system responses to step and ramp disturbances (assuming $r = 0$) are illustrated in Figs. 5(a) and 5(b), respectively which tend to zero as expected.

5. CONCLUSION

The necessary and sufficient conditions for perfect command tracking and disturbance rejection are developed for certain class of fractional-order systems. The notion of zero-pole cancellation is also studied in a well-posed mathematical framework for the systems under consideration and two theorems are presented. Two illustrative examples are studied which confirm the effectiveness of the proposed theorems.

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Appendix A. THE FINAL-VALUE THEOREM FOR FRACTIONAL CASE

Here, we show that the final-value theorem is applicable when there is a branch point at $s = 0$. Assume that $F(s) = \mathcal{L}\{f(t)\}$ is a multi-valued function of s with a branch point at $s = 0$. Then

$$\int_0^{\infty} f'(t)e^{-st}dt = sF(s) - f(0). \tag{A.1}$$

Now, let s tend to zero in the direction of positive real axis:

$$\lim_{s\to 0}\int_0^{\infty} f'(t)e^{-st}dt = \lim_{s\to 0}[sF(s) - f(0)]. \tag{A.2}$$

Since the Laplace integral is uniformly convergent we can change the order of limit and integral:

$$\int_0^{\infty} \lim_{s\to 0}[f'(t)e^{-st}]dt = \lim_{s\to 0}[sF(s)] - f(0), \tag{A.3}$$

which implies that

$$\int_0^{\infty} f'(t)dt = f(\infty) - f(0) = \lim_{s\to 0}[sF(s)] - f(0), \tag{A.4}$$

or

$$f(\infty) = \lim_{s\to 0}[sF(s)]. \tag{A.5}$$