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Algebraic Criteria for Consensus Problems of Discrete-Time Networked Systems ⋆

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Abstract: This paper is mainly devoted to the algebraic criteria for consensus problems of discrete-time networked systems with the fixed and switching topology. A special eigenvector ω of the Laplacian matrix is first correlated with the connectivity of a digraph, and then the relations between a class of Laplacian-type matrix and the stochastic matrix are established. Based on these tools, some necessary and/or sufficient algebraic conditions are proposed, which can directly determine whether the consensus problem can be solved or not. Furthermore, it is proved that only the agents corresponding to the positive elements of ω contribute to the group decision value and decide the collective behavior of the system. Particularly for the fixed topology case, it is shown that not only the role of each agent is exactly proportional to the value of the corresponding element of ω but also the group decision value can be calculated by such a vector and the initial states of all agents.

1. INTRODUCTION

In the last decade, due to the broad applications of networked systems in the fields of mobile robots, unmanned air vehicles(UAVs), autonomous underwater vehicles(AUVs), etc., where the coordination control of all agents is in the central position, the importance of the consensus problem in these fields has been well recognized and many results obtained(see Jadbabaie et al. (2003)∼Lv and Jia (2007), and references therein). In these literatures, both the continuous-time and discrete-time update schemes are extensively studied. For example, Jadbabaie et al. (2003) study a simplified Vicsek's model with discrete dynamics and show the alignment of the heads of all agents under the conditions that the interaction undirected graphs are jointly connected. Olfati and Murray (2004) propose and solve average consensus problem with continuous-time dynamics, where the balanced digraph plays a central role. Further, Ren and Beard (2005) extend the results in Jadbabaie et al. (2003) to digraphs and point out that the consensus can be reached if the union of interaction digraphs contain a spanning tree across each bounded time interval, in which both the continuous and discrete update schemes are discussed. Another important work on consensus problem is Moreau (2005) where the author shows that for the discrete-time update scheme, the conditions in Ren and Beard (2005) are also necessary. Recently, Liu et al. (2007) extend the results about the average consensus problem to the switching topology cases. Obviously, all the above results depend on the structure property of interaction digraphs, for convenience, we uniformly call them as ***geometrical criteria***. Different from the above mentioned literatures, Xiao et al. (2006) propose a sufficient and necessary condition for discrete-time networked systems with fixed topology to reach consensus by employing the eigenvalues of stochastic matrix.

Although the above literatures have provided the conditions for the networked system to solve the consensus problem, the details about the consensus procedure are not clear. Meanwhile, the relations between the general consensus problem and the average consensus problem are not clearly revealed. The answers to these questions will give a deep insight into the consensus problem and provide more information to control such a networked system. With this in mind, our main objective in this paper is to develop the ***algebraic criteria*** for the consensus problem with discrete-time dynamics, which preliminary answers the above questions and can be directly used to determine whether the consensus problem can be solved or not. To this end, we first construct a special nonnegative left eigenvector ω of Laplacian matrix and then correlate it with the connectivity of digraph. It is shown that the digraph is strongly connected(weakly connected with spanning tree) if and only if the vector ω is positive(nonnegative). Further, if a connected digraph is balanced, then it must be strongly connected, meanwhile, all elements of ω are equal. Because of its properties, we use vector ω to study the consensus problem. The proposed results provide a set of necessary/sufficient algebraic conditions for all agents to achieve consensus. It is worth noting that more information about consensus procedure is revealed. On the one hand, it is proved that only the agents corresponding to the positive elements of ω decide the collective behavior of the system and contribute to the group decision value, i.e., they are leaders of the system. On the other hand, for the fixed topology case, the vector ω not only can be used to calculate the group decision value but also exactly measures the role of each agent.

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All these new facts indicate that if the elements of ω are not equal, then different agent plays different role in consensus procedure.

2. PRELIMINARIES AND BACKGROUND

In this section, we introduce some notations in graph theory and matrix theory which are used throughout this paper.

Let $I = \{1, 2, \cdots, n\}$ be an index set and $G = (V, E, A)$ be a weighted digraph of order n with the set of nodes $V = \{v_1, v_2, \cdots, v_n\}$, set of edges $E \subseteq V \times V$, and a weighted adjacency matrix $A = [a_{ij}]$ with nonnegative adjacency elements $a_{ij} \geq 0$ and $a_{ii} = 0$ for all $i, j \in I$. As $a_{ij} > 0$, it means that there exists an edge from the node v_j to the node v_i (i.e., the node v_i receives the information sent by the node v_j), and further, the node v_j is called the *parent* of the node v_i and the node v_i is the *child* of the node v_j. If a node has no parent, it is called a *root*. The set of neighbors of the node v_i is denoted by N_i and defined as $N_i = \{v_j : a_{ij} > 0\}$. The in-degree and out-degree of the node v_i are defined as $deg_{in}(v_i)$ and $deg_{out}(v_i)$, respectively, just as follows:

$$deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji}, \;\; deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij}. \tag{1}$$

The *degree matrix* of G is a diagonal matrix denoted by $\triangle = [\triangle_{ij}]$ where $\triangle_{ij} = 0$ for all $i \neq j$ and $\triangle_{ii} = deg_{out}(v_i)$. The Laplacian matrix of a weighted digraph G is defined as

$$L(G) = \triangle - A. \tag{2}$$

A *walk* in a directed graph(digraph) G is a sequence of edges such that the terminal node of one edge is the initial node of the next. A *path* is a walk that does not include any node twice, except that its first node might be the same as its last. A digraph G is said to be *strongly connected* if and only if for every pair of distinct nodes v_i and v_j in V, there is a directed path from v_i to v_j. A digraph G is called *weakly connected* if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. A digraph G is called *disconnected* if it is not even weak. In addition, a weighted digraph G is called *balanced* if its out-degree equal to its in-degree, i.e., $\mathbf{1}^\top L = 0$ with $1 = [1, \cdots, 1]^\top \in R^{n \times 1}$. In addition, a *subgraph* S of a digraph G is a digraph whose set of nodes and set of edges are all subsets of G. A *spanning subgraph* of G is a digraph whose nodes is the same as the digraph G. A *directed tree* of a digraph G is such a subgraph in which every node, except the root, has exactly one parent. A *spanning tree* of a spanning subgraph of G.

A vector $x = [x_1, \cdots, x_n]^\top$ is called positive if each element of x is positive(i.e., $x_i > 0, \forall i \in I$). A vector x is called nonnegative if each element of x is nonnegative(i.e., $x_i \geq 0$, $\forall i \in I$) and there exists at least one nonzero element. As all elements in x are zero, the vector x is called zero vector. Following the same line, a matrix $A \in R^{m \times n}$ is called positive if all its elements are positive and matrix A is called nonnegative if all its elements are nonnegative. Furthermore, A *stochastic matrix* is a square nonnegative matrix whose rows sums to 1. And a *doubly stochastic matrix* is a square nonnegative matrix, each of whose rows and columns sums to 1. Now, some lemmas are introduced because they will be used below.

Lemma 1. Given a matrix $S = [a_{ij}] \in R^n$, where $a_{ii} \geq 0$, $a_{ij} \leq 0$, $\forall i \neq j$, and $\sum_{j=1}^{n} a_{ij} = 0$ for each j, then S has at least one zero eigenvalue and all of the nonzero eigenvalues are in the open right half plane. Furthermore, S has exactly one zero eigenvalue if and only if the digraph with S has a spanning tree.

Proof. See the Lemma 3.3 in Ren and Beard (2005). □

Lemma 2. If a nonnegative matrix $A = [a_{ij}] \in R^n$ has the same positive constant row sums given by $\mu > 0$, then μ is an eigenvalue of A with an associated eigenvector $\mathbf{1}$ and $\rho(A) = \mu$, where $\rho(\cdot)$ denotes the spectral radius. In addition, the eigenvalue μ of A has algebraic multiplicity equal to one, if and only if the digraph associated with A has a spanning tree.

Proof. See Lemma 3.4 in Ren and Beard (2005). □

Lemma 3. Let $A \in R^n$ and suppose that A is irreducible and nonnegative, then
(a). $\rho(A) > 0$;
(b). $\rho(A)$ is an eigenvalue of A;
(c). there is a positive vector x such that $Ax = \rho(A)x$;
(d). $\rho(A)$ is an algebraically(and hence geometrically) simple eigenvalue of A.

Proof. See the Horn and Johnson (1985), page 508, Theorem 8.4.4. □

3. PROBLEM STATEMENT

In this paper, we consider a networked system with n autonomous agents, which are labeled through 1 to n. All these agents share a common state space R and each of them updates its state based upon the information received from the neighbor agents. In what follows, we give the model of the considered system:

$$x_i[k+1] = \frac{1}{\sum_{v_j \in N_i} \beta_{ij}[k]} \sum_{v_j \in N_i} \beta_{ij}[k] x_i[k], \;\; i \in I, \tag{3}$$

where $i, j \in I$, $k \in \{0, 1 \cdots, n\}$ is the discrete time index, $x_i \in R$ denotes the information state of the agent v_i for all i. $\beta_{ij}[k]$ denotes the confidence of the agent v_i to the information sent by the agent v_j at time k. As $v_j \in N_i$, $\beta_{ij}[k] > 0$; otherwise, $\beta_{ij}[k] = 0$. If $\beta_{ii}[k] > 0$ for some $i \in I$, it means that the agent v_i can receive the information of itself at time k, i.e., the agent v_i has a loop. On the contrary, if $\beta_{ii}[k] = 0$, it means that the agent v_i has no loop.

System (3) can solve consensus problem means that as $k \longrightarrow \infty$, the information value of all agents are equal, i.e., $\lim_{k\to\infty} x_i(k) = \lim_{k\to\infty} x_j(k) = x^*$ for all $i, j \in I$. x^* is called *group decision value*. The special cases with $x^* = max_{\|x_i\|} x_i$, $x^* = min_{\|x_i\|} x_i, x^* = \frac{1}{N}\sum_{i=1}^{N} x_i$ are called *max-consensus problem*, *min-consensus problem* and *average consensus problem*, respectively. Let $x = [x_1, x_2, \cdots, x_n]^\top$, $h_{ij} = \frac{\beta_{ij}[k]}{\sum_{v_j \in N_i} \beta_{ij}[k]}$, then equation (3) can be rewritten in a compact form

$$x[k+1] = H[k]x[k]. \tag{4}$$

Clearly, $H[k]$ is a row stochastic matrix. In what follows, we define a Laplacian-type matrix as follows.

$$L_h[k] = I_n - H[k], \tag{5}$$

where $I_n \in R^n$ denotes identity matrix. Let

$$\begin{aligned} a_{ii}[k] &= 1 - h_{ii}[k], \qquad && i \in I; \\ a_{ij}[k] &= h_{ij}[k], && i, j \in I,\ i \neq j. \end{aligned} \tag{6}$$

Then the elements of matrix $L_h[k]$ is $a_{ii}[k] \geq 0$ for all $i \in I$ and $-a_{ij}[k] \leq 0$ for all $i \neq j$, $i, j \in I$. Moreover, for a given row i in matrix L_h, the following equation is satisfied.

$$a_{ii}[k] = \sum_{j=1, j\neq i}^{n} a_{ij}[k] \quad i, j \in I. \tag{7}$$

Thus, L_h is a Laplacian-type matrix. Therefore, we can say that the digraph G_1 represented by $H[k]$ is the same as the digraph G_2 represented by L_h by introducing some notation as follows.

Firstly, because the operation in (5) can not change the neighbor relations among the agents, so the edges in G_1 are the same as the edges in G_2 except for the loops. Secondly, if there exists a loop in G_1, the corresponding principal diagonal element in L_h is less than 1. Thus, if we take $a_{ii} < 1$ in L_h as a loop in G_2 with weight $h^*_{ii} = 1 - a_{ii} = h_{ii}$, then the loops in G_2 is also equal to loops in G_1. From the above, we can say that G_1 and G_2 are the same digraph. Now, the above analysis can be summarized as the following lemma.

Lemma 4. Let the matrix $H[k]$ defined in (4) represent the digraph G_1, the corresponding Laplacian-type matrix $L_h[k]$ defined in (5) represent the digraph G_2, let $-a_{ij}[k]$ in $L_h[k]$ denote the edge from the agent v_i to the agent v_j with weight $a_{ij}[k]$, and $a_{ii}[k] < 1$ in $L_h[k]$ denote the **loop** of the agent v_i with weight $1 - a_{ii}[k]$ in G_2, then the digraph G_1 is the same as the digraph G_2.

Note 1. Although the matrix $L_h[k]$ has the form of the Laplacian matrix, it is not a true Laplacian matrix because the digraph represented by a Laplacian matrix has no loops. However, this difference does not affect the use of the properties of the Laplacian matrix to study the considered problem.

In what follows, we first give the algebraic conditions for connectivity of digraph. These results are the basis of our method.

4. ALGEBRAIC CRITERIA FOR CONNECTIVITY OF DIGRAPH

Obviously, all the information about a digraph is reflected by its Laplacian matrix, L. For a given node $v_i (i \in I)$, the row i of L denotes how much the other nodes directly affect the node v_i. By contrast, the column i of L reflects how much the other nodes are affected by the node v_i. In this sense, the study of Laplacian matrix is helpful for us to learn something about the digraph, especially to understand the structure information of digraph. Here, we first construct a vector ω from the Laplacian matrix L as follows.

$$\omega = [\omega_1, \omega_2, \cdots, \omega_n]^\top = [det(L_{11}), det(L_{22}), \cdots, det(L_{nn})]^\top, \tag{8}$$

where $det(L_{ii})$ denotes the determinant of matrix L_{ii}, and $L_{ii} \in R^{(n-1)\times(n-1)}$ $(i \in I)$ is obtained from L by deleting the row i and the column i. Now, we are in a position to establish the relations between such a vector ω and the connectivity of digraph.

Theorem 1. Suppose that digraph G contains a spanning tree, L is its Laplacian matrix, let ω be defined in (8), then

$$\omega^\top L = 0. \tag{9}$$

and $\omega \geq 0$.

Proof. See Appendix A. □

Corollary 1. Suppose that G is a strongly connected digraph, L is its Laplacian matrix, let ω be defined in (8), then

$$\omega^\top L = 0. \tag{10}$$

and $\omega > 0$.

Proof. See Appendix B. □

Corollary 2. Suppose that a weakly connected digraph G contains a spanning tree and L is its Laplacian matrix, let vector ω be defined in (8), then

$$\omega^\top L = 0.$$

Furthermore, $\omega \geq 0$ and has at least one zero element.

Proof. See Appendix C. □

Theorem 2. Suppose that G represents a digraph and L is its Laplacian matrix, then G is strongly connected if and only if the vector ω defined in (8) is positive.

Proof. By Corollary 1 and Lemma 1, the conclusion obviously holds. We omit the detail due to the limitation of space. □

Theorem 3. Suppose that G represents a digraph and L is its Laplacian matrix, then G is weakly connected but contains a spanning tree if and only if the vector ω defined in (8) is nonnegative and has at least one zero element.

Proof. By Corollary 1, Corollary 2 and Lemma 1, the conclusion obviously holds. We omit the detail due to the limitation of space. □

Remark 1. It is noted that all determinants of the principal square submatrix of order $n-1$ must be zero if it contains the submatrix D which is induced by the nodes corresponding to nonzero elements of ω.

Remark 2. Compared with Theorem 2 and Theorem 3, the structure difference between a strongly connected digraph and a weakly connected digraph with spanning tree is clearly reflected by vector ω, i.e., these two type of digraph can be easily distinguished from each other by employing ω.

Remark 3. It is noted that for a disconnected digraph, it is possible that its Laplacian matrix may have a positive left eigenvector associated with its one of zero eigenvalues. It does not contradict with the results in Theorem 2 and Theorem 3.

Corollary 3. Suppose that G is a weakly connected digraph of order n and contains a spanning tree, L is its laplacian matrix, let vector ω be defined in (8), then the subgraph induced by the nodes corresponding to positive elements of ω is strongly connected, meanwhile, L has the following decomposition,

$$L = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}, \tag{11}$$

where A_{22} is the submatrix corresponding to the positive elements of ω.

Proof. See Appendix D. □

Corollary 4. Suppose that G is a connected digraph and L is its Laplacian matrix, then G is a balanced digraph if and only if the vector ω of L is positive. Furthermore, $\omega_i = \omega_j$ for all $i, j \in I$.

Proof. Due to the limitation of space, we omit the detail. □

Remark 4. Corollary 4 is important because it implies that a connected balanced digraph must be strongly connected, i.e., it is impossible that a weakly connected digraph is balanced. In other words, either a strongly connected digraph or a disconnected digraph with strongly connected components has the possibility to be a balanced digraph.

5. ALGEBRAIC CRITERIA FOR CONSENSUS PROBLEM

In this section, we study the algebraic conditions for system (3) to solve consensus problem. For this purpose, we first introduce some definitions and results in digraph theory and matrix theory which will be used in the below.

Definition 1. (Horn and Johnson (1985), P: 516, Definition 8.5.0)Primitive: A nonnegative matrix $A \in R^n$ is said to be *primitive* if it is irreducible and has only one eigenvalue of maximum modulus.

Lemma 5. Let $A \in R^n$ be nonnegative and irreducible, if there exists at least one main diagonal element positive, then A is a primitive.

Proof. See Horn and Johnson (1985), P522, Problem 5. □

Lemma 6. Let $A = [a_{ij}] \in R^n$ be a stochastic matrix, if A has an eigenvalue $\lambda = 1$ with algebraic multiplicity equal to one, and all the other eigenvalues satisfy $|\lambda| < 1$, then A is SIA, that is, $\lim_{m\to\infty} A^m \to \mathbf{1}v^\top$, where v satisfies $A^\top v = v$ and $1^\top v = 1$. Furthermore, each element of v is nonnegative.

Proof. See Lemma 3.7 in Ren and Beard (2005). □

Lemma 7. With time invariant topology, system (4) solves a consensus problem if and only if 1 is an algebraically (and hence geometrically) simple eigenvalue of H, and is the unique eigenvalue of maximum modulus.

Proof. See Theorem 1 in Xiao et al. (2006). □

In what follows, we give our main results.

Theorem 4. Suppose that system (4) has time-invariant topology H, L_h is its Laplacian-type matrix defined in (5) and the corresponding left eigenvector ω of L_h is defined in (8), let submatrix H_{22} of H denote the subgraph induced by the nodes corresponding to the positive elements in ω, then system (4) solves consensus problem if and only if (i). ω is nonnegative; (ii). H_{22} is primitive.

Proof. To begin with, let G_1 and G_2 be digraph represented by H and L_h, respectively.

Necessity: We first show vector ω is nonnegative. Because system (4) can solve consensus problem, 1 is an algebraically (and hence geometrically) simple eigenvalue of H by Lemma 7. So digraph G_1 contains a spanning tree by Lemma 2, i.e., digraph G_2 also contains a spanning tree by Lemma 4. Thus, the vector ω is nonnegative by Theorem 3.

Next, we show the second part. Let submatrix A_{22} of L_h denote the subgraph of G_2 which is induced by the agents corresponding to the positive elements of ω, then A_{22} is irreducible by Corollary 3. Thus the submatrix H_{22} of H is also irreducible by Lemma 4. In what follows, let the number of positive elements of ω be $m < n$($m = n$ will be discussed at the end), we can renumber the agents in G_1 and G_2 such that L_h and H have the following form by Corollary 3 and equation (5).

$$L_h = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix} \Longleftrightarrow H = \begin{bmatrix} H_{11} & H_{12} \\ \mathbf{0} & H_{22} \end{bmatrix} \tag{12}$$

where $A_{11} \in R^{(n-m)\times(n-m)}$ and $H_{11} \in R^{(n-m)\times(n-m)}$ represent the agents corresponding to zero elements of ω, $A_{22} \in R^{m\times m}$ and $H_{22} \in R^{m\times m}$ represent the agents corresponding to positive elements of ω just as in the above analysis. By the decomposition in (12), 1 is an eigenvalue of H_{22}. Because 1 is also the unique eigenvalue of maximum modulus of H by Lemma 7, thus 1 is the unique eigenvalue of maximum modulus of H_{22}. Combining the fact that H_{22} is nonnegative and irreducible, H_{22} is primitive by definition 1.

Sufficiency: Because of $\omega \geq 0$, then digraph G_2 contains a spanning tree by Theorem 3, i.e., G_1 also contains a spanning tree by Lemma 4. Then 1 is an algebraically (and hence geometrically) simple eigenvalue of H by Lemma 2. At the same time, 1 is unique eigenvalue of maximum modulus of H_{22} because it is primitive. Thus, 1 is the unique eigenvalue of maximum modulus of H, then system (4) solves consensus problem by Lemma 7.

As $m = n$, $H_{22} = H$, the proof is similar as the above. □

Theorem 4 is important because it gives another necessary and sufficient condition for the considered problem. Since the primitive of a nonnegative irreducible matrix is also not easy to verify in practice, an improved corollary is given as follows.

Corollary 5. Suppose that system (4) has fixed topology H, L_h defined in (5) is its Laplacian-type matrix and a left eigenvector ω of L_h is defined in (8), let A_{22} and H_{22} respectively denote the subgraph of L_h and H induced by the agents which correspond to the positive elements of ω, then system (4) solves consensus problem if (i). ω is nonnegative; (ii). A_{22} has at least one main diagonal element less than 1(or: H_{22} has at least one main diagonal element positive).

Proof. Let G_1 and G_2 be digraph represented by H and L_h, respectively. It is further noted that L_h and H has the form as (12).

Because ω is nonnegative, digraph G_2 contains a spanning tree by Theorem 3, i.e., digraph G_1 is also contains a spanning tree by Lemma 4. Thus, 1 is an algebraically (and hence geometrically) simple eigenvalue of H by Lemma 2. In addition, because that A_{22} has at least one main diagonal element less than 1 is clearly equivalent to that H_{22} has at least one main diagonal element positive by Lemma 4, we only consider H_{22} with positive main diagonal elements. Because H_{22} is a irreducible nonnegative matrix with positive main diagonal elements, we have H_{22} is primitive by Lemma 5. Thus, system (4) can solve consensus problem by Theorem 4. □

Remark 5. The condition (ii) in Corollary 5 is equivalent to the requirement that there exists at least one loop in H_{22}(or: A_{22}). Such a condition is suitable and easy to be satisfied.

Remark 6. Clearly, the corresponding results reported in Ren and Beard (2005) is a special case of Corollary 5.

As system (4) can solve the consensus problem, another important and attractive question is what value is reached by the group and how much each agent contributes to final group decision value. Before answering such a problem, suppose that ω is nonnegative and let

$$\begin{aligned}\omega_l =&[\omega_1^l, \cdots, \omega_n^l]^\top = \frac{1}{\sum_i \omega_i}\omega, \\ \omega_r =&[1, 1, \cdots, 1]^\top,\end{aligned} \tag{13}$$

where ω is defined in (8), $\omega_l^\top L = 0$, $L\omega_r = 0$ and $\omega_l^\top \omega_r = 1$. Then, we have the following theorem.

Theorem 5. Suppose that system (4) can solve consensus problem, let its information topology be represented by stochastic matrix H, and L_h be the corresponding Laplacian-type matrix. Let the left/right eigenvector ω_l/ω_r of L_h be defined in (13), then the group decision value is

$$x^* = \omega_l{}^\top \mathbf{x}(0) = \sum_i \omega_i^l x_i(0) \tag{14}$$

where $\mathbf{x}(0) = [x_1(0),\ x_2(0),\ \cdots,\ x_n(0)]^\top$, and $x_i(0)$ denotes the initial state of the agent v_i.

Proof. Because the system (4) can solve consensus problem, 1 is an algebraically (and hence geometrically) simple eigenvalue of H, meanwhile, 1 also is the unique eigenvalue of maximum modulus by Theorem 4. On the other hand, from equation (5) and (13), ω_l/ω_r also are left/right eigenvector of H associated with eigenvalue 1 by simple calculation. Thus, by Lemma 6, we have $\lim_{k\to\infty} H^k = \omega_r \omega_l^\top$, i.e.,

$$\mathbf{x}^* = \lim_{k\to\infty} \mathbf{x}[k] = \lim_{k\to\infty} H^k \mathbf{x}(0) = \omega_r \omega_l{}^\top \mathbf{x}(0) \tag{15}$$

where $\mathbf{x}^* = [x^*,\ x^*, \cdots,\ x^*]^\top \in R^{n\times 1}$. So $x^* = \sum_i \omega_i^l x_i(0)$ by (15). □

Remark 7. Theorem 5 clearly reflects the role of each agent in consensus procedure, i.e., the agent v_i with large ω_i^l contributes more to the final group decision value than the other agent v_j with small ω_j, where $i, j \in I$.

In contrast to the above results about the fixed topology case, a result about the switching topology case is discussed in the below. To begin with, suppose that there exists constants $k \geq 0$, $T > 0$, and let the interaction topology at each time $l \in [k, k+T)$ be represented by a stochastic matrix $H(l)$. Then the *union* H_u of these interaction stochastic matrixes $H(k), \cdots, H(k+T-1)$ is defined as,

$$H_u = \frac{1}{T}\sum_{l=k}^{k+T-1} H(l). \tag{16}$$

Clearly, H_u is also a stochastic matrix and all historical edges during interval $[k,\ k+T)$ are reflected by it. Now, a result for the switching topology case can be given, which reproduces in algebraic form a special case of Theorem 2 in Moreau (2005).

Theorem 6. Let $G[k] \in \overline{G}$ be a switching interaction graph at time $t = kT$ and the diagonal elements of $G[k]$ be positive, suppose that there exists an infinite sequence of uniformly bounded,non-overlapping time intervals $[k_jT, k_{j+1}T),\ j = 1, 2\cdots,$ starting at $k_1 = 0$, let $H_u(j)$ be the union of interaction digraphs during each interval $[k_jT, k_{j+1}T)$ and $L_u(j)$ be the corresponding Laplacian matrix of $H_u(j)$. Then, system (4) achieves consensus asymptotically if and only if the vector $\omega(j)$ of $L_u(j)$ is nonnegative, where $L_u(j)$ and $\omega(j)$ are defined in (5) and (8), respectively.

Proof. The conclusion obviously holds by Theorem 3, Lemma 4 and Theorem 2 in Moreau (2005). □

Remark 8. From Theorem 6, we can not say more about the contribution of each agent to the group decision value. However, if the element $\omega_i(j)$ is zero for all time interval j, then agent i must give no contribution to the group decision value. This is a new fact that is not reflected by Theorem 2 in Moreau (2005).

6. ALGEBRAIC CRITERIA FOR AVERAGE CONSENSUS PROBLEM

In this section, we study the average consensus problem of networked system with discrete-time dynamics. As reported in Olfati and Murray (2004) , the balanced digraph plays a crucial role in solving such a problem. Here motivated by Theorem 5 and Remark 7, we first give the algebraic conditions for the system (4) with fixed topology to solve average consensus problem as follows.

Theorem 7. Suppose that system (4) has fixed information topology H, L_h defined in (5) is its laplacian-type matrix and a left eigenvector ω of L_h is defined in (8), then system (4) solves average consensus problem if and only if (i). ω is positive and $\omega_i = \omega_j$ for all $i, j \in I$; (ii). H is primitive.

Proof. The conclusion obviously holds by Theorem 4 and Theorem 5. □

Remark 9. For the fixed topology case, it is easy to verify that if system (4) can solve average-consensus problem, then H is a double stochastic matrix, i.e., the digraph G represented by H(or L_h) is a balanced digraph.

Similar to the Theorem 4, primitive of H is not easy to be tested in practice, therefore the following corollary is given to overcome this difficulty.

Corollary 6. Suppose that system (4) has fixed information topology H, L_h defined in (5) is its laplacian-type matrix and a left eigenvector ω of L_h is defined in (8), then system (4) solves average consensus problem if and only if (i). ω is positive and all elements in ω are equal; (ii). L_h has at least one main diagonal element less than 1(or: H has at least one main diagonal element positive).

Proof. Sufficiency of the conclusion obviously holds by Theorem 4, Theorem 5 and Corollary 5. The Necessity of the proof is omitted due to the limitation of the space. □

Remark 10. In Corollary 6, if further assume that each agent can receive its own information, i.e., $h_{ii} > 0$ for all $i \in I$, then the necessary and sufficient algebraic condition for system (4) to solve average consensus problem is $\omega > 0$ and $\omega_i = \omega_j$ for all $i, j \in I$.

As the discussion in section 5, let he union of the interaction digraphs be defined as (16), then a result about the switching topology case can be given as follows.

Theorem 8. Let $G[k] \in \overline{G}$ be the interaction digraph with positive diagonal elements at time $t = kT$ and $L_h(k)$ be its Laplacian-type matrix, suppose that there exists an infinite sequence of uniformly bounded, non-overlapping time intervals $[k_jT,\ k_{j+1}T),\ j = 1, 2, \cdots$, starting at $k_1 = 0$, let $G_u(j)$ be the union of the interaction digraphs during interval $[k_jT,\ k_{j+1}T)$ and $L_h^u(j)$ be the laplacian-type matrix of $G_u(j)$, then system (4) asymptotically achieves average consensus if i). $1^\top L_h(k) = 0$, $\forall k \in [k_jT,\ k_{j+1}T)$; ii). The left eigenvector $\omega(j)$ of $L_h^u(j)$ is positive, where $L_h^u(j)$ and $\omega(j)$ are defined by (5) and (8).

Proof. The conclusion obviously holds by Theorem 2, Theorem 4 in Liu et al. (2007). □

Remark 11. Combining with Corollary 4, Theorem 8 and Remark 4, we know that if system (4) can solve average consensus problem, then each connected component of its interaction digraph at time k is strongly connected and balanced.

7. CONCLUSION

This paper has developed the algebraic criteria for the consensus problem of discrete-time networked systems with fixed and switching topology. The proposed results not only can algebraically determine whether the consensus problem can be solved or not but also clearly reveal that the average consensus problem is a special case of the general consensus problem. More importantly, for the fixed topology case, a necessary and sufficient algebraic condition has been derived, which extends the information topology to non-self loop topology. Furthermore, it is shown that only the agents corresponding to the positive elements of ω contribute to the group decision value but the other agents corresponding to zero elements of ω give no contribution to the group decision value except for converging to it. In addition, for the fixed topology case, the consensus procedure has been distinctly clarified. All these new facts give us a deep insight into the consensus procedure.

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APPENDIX

A. Proof of theorem 1

Proof. Because digraph G may be reducible, we prove it directly. The proof is divided into four steps as follows.

Step 1: Because the digraph G has a spanning tree, $rank(L) = n-1$ by Lemma 1. Therefore, square matrix L contains at least one nonsingular submatrix of order $n-1$. Without loss of generality, assume that L_{ij} is such a submatrix which is obtained by deleting the row i and the column j of L. So the row vectors of L except for the row i are linearly independent. Now, let L^* denote the submatrix of L obtained by deleting the row i of L as follows.

$$L^* = \begin{bmatrix} a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{i-1,1} & -a_{i-1,2} & \cdots & -a_{i-1,n} \\ -a_{i+1,1} & -a_{i+1,2} & \cdots & -a_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

It is clear that $L^* \in R^{(n-1)\times n}$ is a row full rank matrix(i.e., $rank(L^*) = n-1$). At the same time, the sum of each row of L^* is zero because L is a Laplacian matrix. Therefore, any $n-1$ column vectors of L^* must be linearly independent. Thus, the first $(n-1)\times(n-1)$ leading principal submatrix A of L is nonsingular if we renumber the node v_i as the node v_n and the node v_n as the node v_i. To start with, let $B = [-a_{1n}, -a_{2n}, \cdots, -a_{n-1,n}]^\top$ and $C = [-a_{n1}, -a_{n2}, \cdots, -a_{n,n-1}]$, then there must exist an inverse column permutation matrix $P \in R^{n\times n}$ such that

$$LP = L\begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & a_{nn} \end{bmatrix} \times \begin{bmatrix} A^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{n-1} & B \\ CA^{-1} & a_{nn} \end{bmatrix} = L^{**},$$

where I_{n-1} is identity matrix of order $n-1$.

Step 2: Let

$$\begin{aligned} D =& CA^{-1} = [-a_{n1}, -a_{n2}, \cdots, -a_{n,n-1}]A^{-1} \\ =& [-b_1, -b_2, \cdots, -b_{n-1}], \end{aligned}$$

substitute it into L^{**}, and then perform some elementary column transformations on matrix L^{**}, we have

$$L^{**} \sim \begin{bmatrix} I_{n-1} & 0 \\ D & a_{nn} - DB \end{bmatrix} = L_1^{**}.$$

Since the above operations on L are invertible, $rank(L_1^{**}) = rank(L) = n-1$. Thus, we directly have $a_{nn} - DB = 0$ and get the following two equivalent equations

$$\omega^\top L = 0 \Longleftrightarrow \omega^\top L_1^{**} = 0 \tag{17}$$

In what follows, we calculate the value of b_i for all $i = 1, \cdots, n-1$. Due to $A^{-1} = \frac{1}{det(A)} A^*$, where A^* is the adjoint matrix of A, we have

$$
\begin{aligned}
D =& \frac{1}{det(A)} CA^* \\
=& \frac{1}{det(A)} C \begin{bmatrix} A_{11}^* & A_{21}^* & \cdots & A_{n-1,1}^* \\ A_{12}^* & A_{22}^* & \cdots & A_{n-1,2}^* \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,n-1}^* & A_{2,n-1}^* & \cdots & A_{n-1,n-1}^* \end{bmatrix},
\end{aligned}
$$

where $A_{ij}^* = (-1)^{i+j} det(A_{ij})$, A_{ij} is the submatrix of A by deleting the row i and the column j. Then combining with the vector C defined in the above, we have

$$
\begin{aligned}
-b_i =& \frac{1}{det(A)} (-a_{n1}A_{i1}^* - a_{n2}A_{i2}^* - \cdots - a_{n,n-1}A_{i,n-1}^*) \\
=& \frac{(-1)^i}{det(A)} [(-1)^2 a_{n1} det(A_{i,1}) + (-1)^3 a_{n2} det(A_{i,2}) \\
& + \cdots + (-1)^n a_{n,n-1} det(A_{i,n-1})].
\end{aligned} \tag{18}
$$

Let $\triangle_i$ denote the $(n-1)\times(n-1)$ submatrix of L as follows

$$
\triangle_i = BL \begin{Bmatrix} 1,2,\cdots,i-1,i+1,\cdots,n \\ 1,2,\cdots,i-1,i,i+1,\cdots,n-1 \end{Bmatrix}, \tag{19}
$$

i.e., $\triangle_i$ is obtained by deleting the row i and the column n of L . Noting that $\triangle_i$ is a square matrix of order $n-1$, we have

$$
\begin{aligned}
det(\triangle_i) =& (-1)^{n+1} a_{n1} det(\triangle_{n1}) + (-1)^{n+2} a_{n2} det(\triangle_{n2}) + \\
& \cdots + (-1)^{2n-1} a_{n,n-1} det(\triangle_{n,n-1}),
\end{aligned} \tag{20}
$$

where $\triangle_{ij}$ is submatrix of $\triangle_i$ by deleting the row i and the column j of $\triangle_i$. From equations (18) and (20), we have $det(\triangle_{n1}) = det(A_{i1})$, $det(\triangle_{n2}) = det(A_{i2}), \cdots$, $det(\triangle_{n,n-1}) = det(A_{i,n-1})$. Thus, equation (20) becomes

$$
\begin{aligned}
det(\triangle_i) =& (-1)^{n+1} a_{n1} det(A_{i1}) + (-1)^{n+2} a_{n2} det(A_{i2}) \\
& + \cdots + (-1)^{2n-1} a_{n,n-1} det(A_{i,n-1}) \\
=& (-1)^{n-1} [(-1)^2 a_{n1} det(A_{i1}) + (-1)^3 a_{n2} det(A_{i2}) \\
& + \cdots + (-1)^n a_{n,n-1} det(A_{i,n-1})].
\end{aligned} \tag{21}
$$

Then by (18) and (21), we have

$$
b_i = \begin{cases} \dfrac{(-1)^i}{det(A)} det(\triangle_i), \text{if } n = 2k, k \in N^+; \\ \dfrac{(-1)^{i-1}}{det(A)} det(\triangle_i) \text{if } n = 2k+1, k \in N^+. \end{cases} \tag{22}
$$

Step 3: In this step, we show that $b_i \geq 0$ for all $i = 1,2,\cdots,n-1$ and give a formula to calculate ω defined in (8) by b_i. In what follows, we only consider the case in which n is even and then show $b_2 \geq 0$ for $i = 2$. The others can be proved by the similar way. Under these assumptions, we have

$$
\begin{aligned}
det(A)b_2 =& det(\triangle_2) \\
=& det \begin{bmatrix} a_{11} & -a_{12} & \dots & -a_{1,n-1} \\ -a_{31} & -a_{32} & \dots & -a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \dots & -a_{n,n-1} \end{bmatrix} \\
=& (-1)^{n-3} det \begin{bmatrix} a_{11} & -a_{13} & \dots & -a_{12} \\ -a_{31} & a_{33} & \dots & -a_{32} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \dots & -a_{n2} \end{bmatrix} \\
=& (-1)^{n-3} det \begin{bmatrix} a_{11} & -a_{13} & \dots & a_{11} - \sum_{j=2}^{n-1} a_{1,j} \\ -a_{31} & a_{33} & \dots & a_{33} - \sum_{j=1,j\neq 3}^{n-1} a_{3,j} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \dots & -\sum_{j=1}^{n-1} a_{n-1,j}] ^\top \end{bmatrix} \\
=& (-1)^{n-4} det \begin{bmatrix} a_{11} & -a_{13} & \dots & -a_{1n} \\ -a_{31} & a_{33} & \dots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n3} & \dots & a_{nn} \end{bmatrix} = det(L_{22}),
\end{aligned} \tag{23}
$$

where the third step is obtained by some elementary column transformations; the fourth step is obtained by adding the first $n-2$ columns to the last column; the fifth step used the properties of Laplacian matrix L, $\sum_{j=1}^n a_{ij} = 0$ for each i; and the last step is obtained by using the fact that n is even. It is clear that L_{22} is the principal submatrix of L . By *step 2*, we know $det(A) > 0$ and $det(L_{22}) \geq 0$, which directly implies $b_2 \geq 0$. Just as the same analysis, we can show $b_i = det(L_{ii})/det(A) \geq 0$ for $i \in \{1,3,\cdots,n-1\}$. Thus, we have

$$
b_1 = \frac{det(L_{11})}{det(A)}; b_2 = \frac{det(L_{22})}{det(A)}; \cdots; b_{n-1} = \frac{det(L_{n-1,n-1})}{det(A)} \tag{24}
$$

Substitute (24) into the right equation of (17) and let the free variable $\omega_n = det(A) = det(L_{nn})$, we have

$$
\omega_1 = det(L_{11}); \omega_2 = det(L_{22}); \cdots; \omega_n = det(L_{nn}). \tag{25}
$$

such that $\omega^\top L = 0$.

Step 4: In this step, we show ω is nonnegative. For convenience, we denote eigenvalues of L_{ii} as λ_i^k, where $i \in I$ and $k \in \{1,2,\cdots,n-1\}$. Because each principal minor L_{ii} of Laplacian matrix L is diagonally dominant and main diagonal element of it is nonnegative, and then by Gerŝgorin disc Theorem, each eigenvalue λ_i^k of L_{ii} is in the right open plane, i.e., $\lambda_i^k \geq 0$ for all k. Therefore, $det(L_{ii}) = \lambda_i^1 \lambda_i^2 \cdots \lambda_i^{n-1} \geq 0$ for all $i \in I$. Combining with *Step 1* and *Step 3* ω is nonnegative.

Combining all of the above, the conclusion holds. □

B: Proof of corollary 1

Proof. Because the strongly connected digraph must have a spanning tree, the nonnegative vector ω defined by (8) satisfies $\omega^\top L = 0$ by Theorem 1. To this end, we only need to show that each element of ω is positive. Because L is irreducible, the nonnegative matrix $P = \rho(L)I_n - L$ also is irreducible, where $\rho(L)$ is spectral radius of L and I_n is the identity matrix of order n. Let $v = [1,1,\cdots,1]^\top \in R^{n\times 1}$ and note that $Lv = 0$, we have

$$
Pv = (\rho(L)I_n - L)v = \rho(L)v,
$$

which implies that the sum of all rows in P are identical, then $\rho(L)$ also is spectral radius of P by Lemma 2, i.e., $\rho(P) = \rho(L)$. Meanwhile, the following equation also holds.

$$
P^\top \omega = (\rho(L)I_n - L)^\top \omega = \rho(P)\omega,
$$

where the equation $L^\top \omega = 0$ is used. Because $P^\top$ also is irreducible and nonnegative, thus $\omega > 0$ by Lemma 3. □

C: Proof of corollary 2

Proof. Because the weakly connected digraph G contains a spanning tree, we have $\omega^\top L = 0$ and ω is nonnegative by Theorem 1. In what follows, we prove that there exists at least one zero element in ω. Since the digraph G contains a spanning tree, then $rank(L) = n-1$ by Lemma 1. In addition, because G is weakly connected, there exist at least two nodes v_i and v_j such that there has no path from v_i to v_j, i.e., $a_{ji} = 0$. Let the set S denote the nodes such that for each node $v_s \in S$ there has a path from v_s to v_j, and let the set T denote the other nodes which are not in the set S, which means that there has no path from the node $v_t \in T$ to the node $v_s \in S$. Suppose that the number of the nodes in the set S is m, then we can renumber the nodes and write the Laplacian matrix of digraph as the following form

$$L = \begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix}, \tag{26}$$

in which the node v_i is renumbered as node v_1 and the node v_j is renumbered as node v_n, and $A \in R^{(n-m)\times(n-m)}$,$B \in R^{(n-m)\times m}$,$\mathbf{0} \in R^{m\times(n-m)}$, $D \in R^{m\times m}$. From (26), the first $n-m$ rows contain all nodes in the set T and the last m rows contain all nodes in the set S. It is worth noting that B is a nonzero submatrix due to the weakly connectedness of the digraph G.

Clearly, submatrix D of L is also a Laplacian matrix. Meanwhile, the subgraph of G represented by D also has a spanning tree because the possible root node of G must be in the set S. We prove it by contradiction. Suppose the node $v_k \in T$ is a root node, then there must exist a path from v_k to any other node in the set S, which is a contradiction. Then we have $v_k \in S$. Therefore, $rank(D) = m-1$ by Lemma 1. It implies that the row vectors of D are linear dependent. Without loss of generality, we assume that there exist a series of scalars $\alpha_1, \alpha_2, \alpha_{m-1}$, not all them are zeros, such that

$$d_m = \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_{m-1} d_{m-1}, \tag{27}$$

where $d_1, d_2, \cdots, d_m$ denote the row vector of D. Then, by equation (26), the Laplacian matrix L and its first leading principal minor L_{11} of order $n-1$ can be respectively written as follows.

$$L = \begin{bmatrix} a_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \mathbf{0} & \mathbf{0} & D \end{bmatrix}; \qquad L_{11} = \begin{bmatrix} A_{22} & B_2 \\ \mathbf{0} & D \end{bmatrix}, \tag{28}$$

with

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

where D is as same as (26) and all the other blocks have the suitable dimension. Then we have

$$\begin{aligned} \omega_1 =& det(L_{11}) = det\left(\begin{bmatrix} A_{22} & B_2 \\ \mathbf{0} & D \end{bmatrix}\right) \\ =& det\left(\begin{bmatrix} A_{22} & B_2 \\ \mathbf{0} & D^* \\ \mathbf{0} & d_m - \sum_{i=1}^{m-1} \alpha_i d_i \end{bmatrix}\right) = 0, \end{aligned} \tag{29}$$

where D^* is a submatrix of D obtained by deleting the last row of D. Thus, there exists at least one zero element in ω, which completes the proof. □

Remark 12. From the above proof, it is clear that all determinants of the $(n-1)\times(n-1)$ leading principal minor of L must be zero if it contains the submatrix D. Thus, the first $n-m$ elements in ω are zero.

D: Proof of corollary 3

Proof. Because digraph G contains a spanning tree, then just as the proof of Corollary 2 does, we can renumber the nodes and rewrite the Laplacian matrix of digraph G as equation (26) and the corresponding submatrix is also denoted by A, B, $\mathbf{0}$ and D. It is noted that the first $n-m$ elements in ω are zeros by Corollary 2 and Remark 12. Without loss of generality, suppose that the numbers of zero elements in ω are k, then $k \geq n-m$. Thus, the vector ω can be rewritten as

$$\omega = [0, \cdots, 0, \omega_{k+1}, \cdots, \omega_n]^\top \in R^{n\times 1}.$$

In what follows, we discuss it by two cases.

Case 1: If $k = n-m$, we show that the last m nodes induce a strongly connected digraph. From (26), D is also a Laplacian matrix. Due to $\omega^\top L = 0^{n\times 1}$, we have

$$\omega^{*\top} D = 0^{m\times 1},$$

where $\omega^* = [\omega_{k+1}, \cdots, \omega_n]^\top \in R^{m\times 1}$. Due to $\omega^* > 0$, by Theorem 2, the subgraph induced by the nodes in D is strongly connected.

Case 2: If $k > n-m$, we show that the last $n-k$ nodes induce a strongly connected subgraph. Due to $k > n-m$, it means that there are $l = m+k-n$ nodes in D corresponding to the zero elements in ω. Without loss of generality, suppose that the first l nodes in D correspond to the zero elements. Thus, (26) can be rewritten as

$$L = \begin{bmatrix} A & B_{11} & B_{12} \\ \mathbf{0} & D_{11} & D_{12} \\ \mathbf{0} & D_{21} & D_{22} \end{bmatrix}, \tag{30}$$

where A is same as the corresponding submatrix in (26), $D_{11} \in R^{l\times l}$, $D_{22} \in R^{(n-k)\times(n-k)}$, the other matrixes have the compatible dimension. Just as the *case 1*, we also have

$$\omega^{*\top} D = 0^{(n-k)\times 1}, \tag{31}$$

where $\omega^* = [0, \cdots, 0, \omega_{k+1}, \cdots, \omega_n]^\top \in R^{m\times 1}$. Let d_{ij} denote the entries in D for all $i, j \in \{1, 2, \cdots, m\}$ and $d_t = [d_{1,t}, \cdots, d_{m,t}]^\top$ denote the first l columns of D, $t \in \{1, 2, \cdots, l\}$. Then, combining with equation (31), we have

$$\omega_{k+1} d_{l+1,t} + \omega_{k+2} d_{l+2,t} + \cdots + \omega_n d_{m,t} = 0. \tag{32}$$

Because the elements in submatrix D_{21} are nonpositive and the vector $\omega^{**} = [\omega_{k+1}, \cdots, \omega_n]^\top$ are positive, so the elements in D_{21} must be zero, i.e., we have

$$L = \begin{bmatrix} A & B_{11} & B_{12} \\ \mathbf{0} & D_{11} & D_{12} \\ \mathbf{0} & \mathbf{0} & D_{22} \end{bmatrix}. \tag{33}$$

Thus, D_{22} is a Laplacian matrix and $\omega^{**\top} D_{22} = 0^{(n-k)\times 1}$. Due to ω^{**} positive, then the subgraph induced by the nodes in D_{22} is strongly connected by Theorem 2. Combining all the above analysis, the conclusion is true. □