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Second Order Sliding Mode (SOSM) Approach to Orbital Stabilization of Friction Pendulum via Position Feedback

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Abstract: A second order sliding mode (SOSM) approach is proposed for orbital stabilization of a friction pendulum, operating under uncertain conditions. Only position measurements are assumed to be available. A SOSM velocity observer is developed and included into the closed-loop system, driven by the quasihomogeneous controller that provides orbital stabilization of the pendulum. Performance issues of the observer-based position feedback synthesis are illustrated in a simulation study.

1. INTRODUCTION

Primary concern of the paper is to develop a SOSM-based approach to output feedback stabilization of friction mechanical systems. To facilitate exposition a test problem, chosen for treatment, is confined to orbital stabilization of an inverted pendulum over position measurements. Although the classical friction model, consisting of the viscous friction and the Coulomb friction, is only under study, the dynamic LuGre friction modeling is involved into the simulation runs made to account for most of the experimentally observed friction behavior and demonstrate the robustness of the proposed SOSM-based position feedback synthesis against different friction forces.

The quasihomogeneous state feedback synthesis, recently developed in Orlov *et. al.* (2008), is utilized and accompanied with a SOSM-based velocity observer, inspired from Davila *et. al.* (2005), to present the position feedback, solving the problem in question. A nonsmooth Lyapunov function is constructed for the over-all plant-observer system. In contrast to Davila *et. al.* (2005), where the developed geometrical method is only capable of proving the observer stability in the open-loop, our method handles the closed-loop stability.

The resulting closed-loop system is shown to track the model orbit in a sliding mode of the second order, even in the presence of external disturbances with an a priori known magnitude bound (see the original work of Levant (1993) for second order sliding modes and Bartolini *et. al.* (1998); Fridman and Levant (1996, 2002) for advanced results in the area). The proposed synthesis is thus expected to yield desired robustness properties against the discrepancy between the real friction and that described in the model. In particular, advantage of the controller constructed may be outstanding if Coulomb friction, typically ignored in the existing controllers design, is relatively strong for the actuator's power.

As in Orlov *et. al.* (2008), a modification of the Van der Pol oscillator is introduced into the synthesis as a reference model. This modification is made to shape the oscillator limit cycle to a harmonic one. Moreover, the limit cycle production of the modified Van der Pol oscillator possesses a single harmonic (as opposed to multiharmonics of a standard harmonic oscillator) and the oscillator parameters specify amplitude, frequency, and damping of oscillation. Similar to Orlov *et. al.* (2008), amplitude, frequency, and damping can thus be modified dynamically by simply changing the oscillator parameters.

The control law, enforcing the system to slide along a periodic orbit of the phase space, and a SOSM velocity observer, being coupled together, yield a novel unified framework for orbital stabilization of a friction pendulum using position measurements only. The resulting closed-loop system possesses its own limit cycle, producing a prescribed harmonic whose frequency and amplitude can be modified dynamically at our will.

Capabilities of the proposed orbitally stabilizing synthesis are illustrated in a simulation study of a laboratory pendulum. Good controller performance is concluded from this study in spite of the dynamic nature of friction forces, involved into simulations, and the presence of actuator dynamics. Thus, the model description of the pendulum presents a simple underactuated system and the extension to orbital stabilization of underactuated systems seems

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possible. However, this is not trivial and remains beyond the present investigation.

The rest of the paper is organized as follows. Section 2 is focused on the observer to be used in Section 3 as a velocity estimation in the quasihomogeneous orbital stabilization of a friction pendulum. Section 4 presents numerical results and Section 5 finalizes the paper with some conclusions.

2. ORBITAL STABILIZATION OF FRICTION PENDULUM

2.1 Problem Statement

The state equation of the controlled one-link pendulum, depicted in Fig. 2, is given by

$$(ml^2 + J)\ddot{q} = mgl\sin(q) - F(\dot{q}) + \tau \tag{1}$$

where q is the angle made by the pendulum with the vertical, m is the mass of the pendulum, l is the distance to the center of mass, J is the moment of inertia of the pendulum about the center of mass, g is the gravity acceleration, $F(\dot{q})$ is the friction force, τ is the control torque.

In order to describe the friction force $F(\dot{q})$ the classical model is utilized

$$F(\dot{q}) = \alpha_v\dot{q} + \alpha_c sign(\dot{q}). \tag{2}$$

The above model comes with the viscous friction coefficient $\alpha_v > 0$, the Coulomb friction level $\alpha_c > 0$, and the standard notation $sign(\dot{q})$ for the signum function of the angular velocity. Subject to (2), the right-hand side of the dynamic system (1) is piece-wise continuous. Throughout, solutions of such a system are defined in the sense of Filippov (1988) as that of a certain differential inclusion with a multi-valued right-hand side.

An interesting problem for mechanical systems, affected with friction, is the way we have access to the data. Usually we have access to position data, but not to velocity data. With this in mind, *our objective* is to track the output $x(t)$ of the asymptotic harmonic generator

$$\ddot{x} + \varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} + \mu^2 x = 0, \tag{3}$$

using position measurements only, while also attenuating the effect of friction, i.e., the limiting relation

$$\lim_{t\to\infty}[q(t) + x(t)] = 0 \tag{4}$$

is to be satisfied for trajectories of the closed-loop system.

A modification of the Van der Pol oscillator, resulting in the asymptotic harmonic generator (3), was proposed in Roup and Bernstein (2002). Motivation behind the use of the harmonic generator (3) as a reference model and quasihomogeneous state feedback synthesis, solving the above problem, were given in Orlov *et. al.* (2008). To attain our objective, using position measurements only, we accompany the afore-mentioned quasihomogeneous synthesis with SOSM-based observer design.

3. OBSERVER DESIGN AND CONTROL SYNTHESIS

3.1 Observer design

To begin with, we present a SOSM observer

$$\dot{\widehat{q_1}} = \widehat{q_2} + w sign(q_1 - \widehat{q_1})$$

$$\dot{\widehat{q_2}} = \frac{1}{ml^2 + J}(mgl\sin(q_1) - F(\widehat{q_2}) + w_1 sign(q_1 - \widehat{q_1}) + \tau)$$
$$+ w_1 sign(q_1 - \widehat{q_1}) \tag{5}$$

that copies the pendulum structure with the positive parameters $w, w_1 > 0$, used to improve the observer performance, and $F(\widehat{q_2}) = \alpha_v\widehat{q_2} + \alpha_c sign(\widehat{q_2})$.

3.2 Control Synthesis

Let us now utilize the quasihomogeneous state feedback, developed in Orlov *et. al.* (2008). This control law is readily modified to the position feedback by running in parallel the SOSM observer (5) and substituting the velocity estimate $\widehat{q_2}$ from the SOSM observer into the state feedback for the velocity.

Due to (1), (2), (3), the error tracking dynamics in terms of the tracking error

$$y(t) = q(t) + x(t) \tag{6}$$

is given by

$$(ml^2 + J)\ddot{y} = mgl\sin(q) - \alpha_v q_2 - \alpha_c sign(q_2) + \tau$$
$$-(ml^2 + J)\{\varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} + \mu^2 x\}. \tag{7}$$

The quasihomogeneous control low, extracted from Orlov *et. al.* (2008) and modified to use the velocity estimation $\widehat{q_2}$ rather than the velocity itself, is as follows

$$\tau = (ml^2 + J)\{\varepsilon[(x^2 + \frac{\dot{x}^2}{\mu^2}) - \rho^2]\dot{x} + \mu^2 x\} - mgl\sin(q)$$
$$-\alpha_v\dot{x} - \alpha sign(y) - \beta sign(\dot{\widehat{y}}) - hy - p\dot{\widehat{y}} \tag{8}$$

where $\dot{\widehat{y}} = \widehat{q_2} + \dot{x}$, and the parameters are governed

$$h, p \geq 0,\ \beta > M + \alpha_c,\ \alpha > \beta + M + \alpha_c. \tag{9}$$

The error dynamics (7), driven by the dynamic position feedback (8), take the form

$$(ml^2 + J)\ddot{y} = -\alpha_c sign(q_2) - \alpha sign(y) - \beta sign(\dot{\widehat{y}})$$
$$-hy - p\dot{\widehat{y}} - \alpha_v\dot{y}. \tag{10}$$

Relating the quasihomogeneous synthesis from Orlov (2005a), the above controller has been composed of the nonlinear compensator

$$u_c = (ml^2+J)\{\varepsilon[(x^2+\frac{\dot{x}^2}{\mu^2})-\rho^2]\dot{x}+\mu^2 x\}-mgl\sin(q), \quad (11)$$

the relay part (the so-called twisting controller from Fridman and Levant (1996, 2002))

$$u_h = -\alpha sign(y) - \beta sign(\dot{\hat{y}}),$$

and the linear remainder

$$u_l = -hy - p\dot{\hat{y}} - \alpha_v \dot{y}.$$

For later use, the tracking error dynamics (10) is brought into the canonical form

$$\begin{aligned} \dot{y_1} &= y_2 \\ \dot{y_2} &= k[-\alpha_c sign(q_2) - \alpha sign(y_1) - \beta sign(\widehat{y_2}) \\ &\quad -hy_1 - p\widehat{y_2} - \alpha_v y_2] \end{aligned} \quad (12)$$

whereas the observation error dynamics

$$\begin{aligned} \dot{\eta_1} &= \dot{q_1} - \dot{\widehat{q_1}} = q_2 - (\widehat{q_2} - wsign(q_1 - \widehat{q_1})) \\ &= \eta_2 - wsign(\eta_1) \\ \dot{\eta_2} &= \dot{q_2} - \dot{\widehat{q_2}} = k\left(-\alpha_v \eta_2 - \alpha_c(sign(q_2) + sign(q_2 - \eta_2))\right) \\ &\quad -w_1 sign(\eta_1) \end{aligned} \quad (13)$$

are given in terms of the estimation error

$$\begin{aligned} \eta_1 &= q_1 - \widehat{q_1} \\ \eta_2 &= q_2 - \widehat{q_2}. \end{aligned} \quad (14)$$

We are now in a position to state our main result.

Theorem 1. The over-all error system (12), (13), driven by the dynamic position feedback (8) subject to the parameter subordination (9), is globally asymptotically stable.

Proof: The proof follows the line of reasoning used in the proof of (Orlov *et. al.*, 2008, Theorem 3) and it is based on the nonsmooth Lyapunov function

$$V(y,y_2,\eta_1,\eta_2) = \frac{1}{2}(kh\nu y_1^2 + \nu y_2^2 + \eta_2^2) + k\nu\alpha|y_1| + \gamma|\eta_1| \quad (15)$$

where $k = \frac{1}{ml^2+J} > 0$ and

$$\begin{aligned} \gamma w &> 2(p+\alpha_v)\left[\nu(\alpha_c+\beta) + \frac{w_1+\gamma}{k}\right]^2, \\ \nu &\le \frac{(p+\alpha_v)\alpha_v}{p^2}. \end{aligned} \quad (16)$$

The time derivative of $V(y_1(t), y_2(t), \eta_1(t), \eta_2(t))$, computed along the trajectories of (12), (13), is given by

$$\begin{aligned} \dot{V} &= kh\nu y_1 \dot{y_1} + \nu y_2 \dot{y_2} + \eta_2 \dot{\eta_2} \\ &\quad + k\nu\alpha sign(y_1)\dot{y_1} + \gamma sign(\eta_1)\dot{\eta_1} \\ &= kh\nu y_1 y_2 + k\nu y_2[-\alpha_c sign(q_2) - \alpha sign(y_1) \\ &\quad -\beta sign(\widehat{y_2}) - hy_1 - p\widehat{y_2} - \alpha_v y_2] \\ &\quad + k\eta_2\left[-\alpha_v\eta_2 - \alpha_c\left(sign(q_2) + sign(q_2-\eta_2)\right)\right] \\ &\quad -\eta_2 w_1 sign(\eta_1) + k\nu\alpha sign(y_1) y_2 \\ &\quad +\gamma sign(\eta_1)(\eta_2 - wsign(\eta_1)). \end{aligned} \quad (17)$$

Setting $\widehat{y_2} = y_2 - \eta_2$ and making rather lengthy manipulations yield

$$\begin{aligned} \dot{V} &= -k\nu\alpha_c y_2 sign(q_2) - k\beta\nu y_2 sign(\widehat{y_2}) - kp\nu\widehat{y_2}y_2 \\ &\quad -k\nu\alpha_v y_2^2 - k\alpha_v\eta_2^2 \\ &\quad -k\alpha_c\eta_2 sign(q_2) - k\alpha_c\eta_2 sign(\widehat{q_2}) \\ &\quad +\gamma\eta_2 sign(\eta_1) - w\gamma - \eta_2 w_1 sign(\eta_1) \\ &= -k\nu\alpha_c sign(q_2)y_2 - k\nu\beta sign(\widehat{y_2})(\widehat{y_2}+\eta_2) \\ &\quad -k\nu\alpha_v y_2^2 - kp\nu y_2(y_2-\eta_2) - k\alpha_v\eta_2^2 \\ &\quad -k\alpha_c\eta_2\left(sign(q_2) - sign(\widehat{q_2})\right) + \gamma\eta_2 sign(\eta_1) \\ &\quad -w\gamma - w_1 sign(\eta_1)\eta_2 \\ &\quad -k\nu\alpha_c sign(q_2)y_2 - k\nu\beta|\widehat{y_2}| - k\nu\beta\eta_2 \\ &\quad -k\nu\alpha_v y_2^2 - kp\nu y_2^2 + kp\nu y_2\eta_2 \\ &\quad -k\alpha_v\eta_2^2 - k\alpha_c|q_2|\left(1 - sign(q_2)sign(\widehat{q_2})\right) \\ &\quad -k\alpha_c|\widehat{q_2}|\left(1 - sign(q_2)sign(\widehat{q_2})\right) \\ &\quad +\gamma sign(\eta_1)\eta_2 - w\gamma - w_1 sign(\eta_1)\eta_2 \\ &\le -k\nu\alpha_c sign(q_2)(\eta_2+\widehat{y_2}) - k\nu\beta|\widehat{y_2}| \\ &\quad -k\nu\beta sign(\widehat{y_2})\eta_2 - k\nu\alpha_v y_2^2 - kp\nu y_2^2 + kp\nu y_2\eta_2 \\ &\quad -k\alpha_v\eta_2^2 + \gamma sign(\eta_1)\eta_2 \\ &\quad -w\gamma - w_1 sign(\eta_1)\eta_2 \\ &\le -k\nu\alpha_c sign(q_2)\eta_2 - k\nu\alpha_c sign(q_2)\widehat{y_2} \\ &\quad -k\nu\beta|\widehat{y_2}| - k\nu\beta sign(\widehat{y_2})\eta_2 - k\nu\alpha_v y_2^2 \\ &\quad -kp\nu y_2^2 + kp\nu y_2\eta_2 - k\alpha_v\eta_2^2 \\ &\quad +\gamma sign(\eta_1)\eta_2 - w\gamma - w_1 sign(\eta_1)\eta_2 \\ &= \eta_2[-k\nu\alpha_c sign(q_2) - k\nu\beta sign(\widehat{y_2}) - w_1 sign(\eta_1) \\ &\quad +\gamma sign(\eta_1)] - k\nu|\widehat{y_2}|[\beta \\ &\quad -\alpha_c sign(q_2)sign(\widehat{y_2})] - k(p+\alpha_v)\nu y_2^2 \\ &\quad -kp\nu y_2\eta_2 + k\alpha_v\eta_2^2 - w\gamma. \end{aligned} \quad (18)$$

Now employing the well-known inequality

$$ab < \frac{a^2}{2\epsilon_1} + \frac{\epsilon_1 b^2}{2}, \quad (19)$$

the resulting inequality (18) is rewritten as follows

$$\begin{aligned}\dot{V} &\leq \frac{\eta_2^2}{2\epsilon_1} + \frac{\epsilon_1}{2}\left(k\nu\alpha_c + k\nu\beta + w_1 + \gamma\right)^2 - w\gamma \\ &\quad -k\nu|\widehat{y}_2|\left[\beta - \alpha_c sign(q_2) sign(\widehat{y}_2)\right] \\ &\quad -k(p+\alpha_v)\nu y_2^2 + \frac{kp\nu}{2\epsilon_2}y_2^2 + \frac{kp\epsilon_2\nu}{2}\eta_2^2 - k\alpha_v\eta_2^2 \\ &\leq -w\gamma + \frac{\epsilon_1}{2}\left[k\nu\alpha_c + k\nu\beta + w_1 + \gamma\right]^2 \\ &\quad -k\nu|\widehat{y}_2|\left[\beta - \alpha_c\right] - \left[p + \alpha_v - \frac{p}{2\epsilon_2}\right]k\nu y_2^2 \\ &\quad -\left[k\alpha_v - \frac{kp\epsilon_2\nu}{2} - \frac{1}{2\epsilon_1}\right]\eta_2^2.\end{aligned} \tag{20}$$

Letting

$$\epsilon_1 > \frac{1}{k\alpha_v}\ ,\ \epsilon_2 \geq \frac{p}{2(p+\alpha_v)} \tag{21}$$

and taking the parameter subordinations (9), (16) into account it follows that $\dot{V}(t)$ is negative definite along the trajectories of (12), (13). Thus, the global asymptotic stability of the error system (12), (13) is established, and the proof of Theorem 1 is completed.

4. NUMERICAL RESULTS

Performance issues of the quasihomogeneous synthesis using position feedback were tested on numerical experiments using the software MATLAB. The pendulum of mass $m = 0.5234\ kg$, centered at $l = 0.108\ m$, and the inertia $J = 0.006\ kg \cdot m^2$ about the center of mass. This parameters are taken from a real pendulum from CICESE laboratory. The friction in the motor brushes and bearings was identified with the parameters $\alpha_v = 0.00053\ N \cdot m \cdot s/rad$ and $\alpha_c = 0.05492\ N \cdot m$. The initial position of the pendulum and that of the modified Van der Pol oscillator, selected for the experiments, were $q(0) = 0.5\ rad$ and $x(0) = 0.01\ rad$, whereas all the velocity initial conditions were set to $\dot{q}(0) = 0.1$.

In our numerical study the controller gains in (8) were set to $h = 0$, $p = 0$, $\alpha = 5\ N \cdot m$, $\beta = 1\ N \cdot m$ whereas the reference parameters were tuned to $\varepsilon = 20\ [rad]^{-2}s^{-1}$, $\rho = 2\ rad$, $\mu = 2\ s^{-1}$.

We simulated two cases of the quasihomogeneous observer-based controller. First we ran simulations with no disturbances. In order to test the robustness of the nonlinear observer (5) we then introduced a parametric disturbance, changing the Coulomb friction level to $\alpha_c = 0.05$. The resulting trajectories are depicted in Figure 1. This figure demonstrates that the controller stabilizes the disturbance-free system motion around the desired trajectory and rejects the parametric uncertainties.

In order to additionally illustrate capabilities of the nonlinear observer we involved into simulations the Lugre model of friction phenomena. This model is well-known Canudas-de-Wit *et. al.* (1995) to provide an adequate description of friction phenomena even at low velocities and especially while crossing the zero velocity. The Lugre model is given by

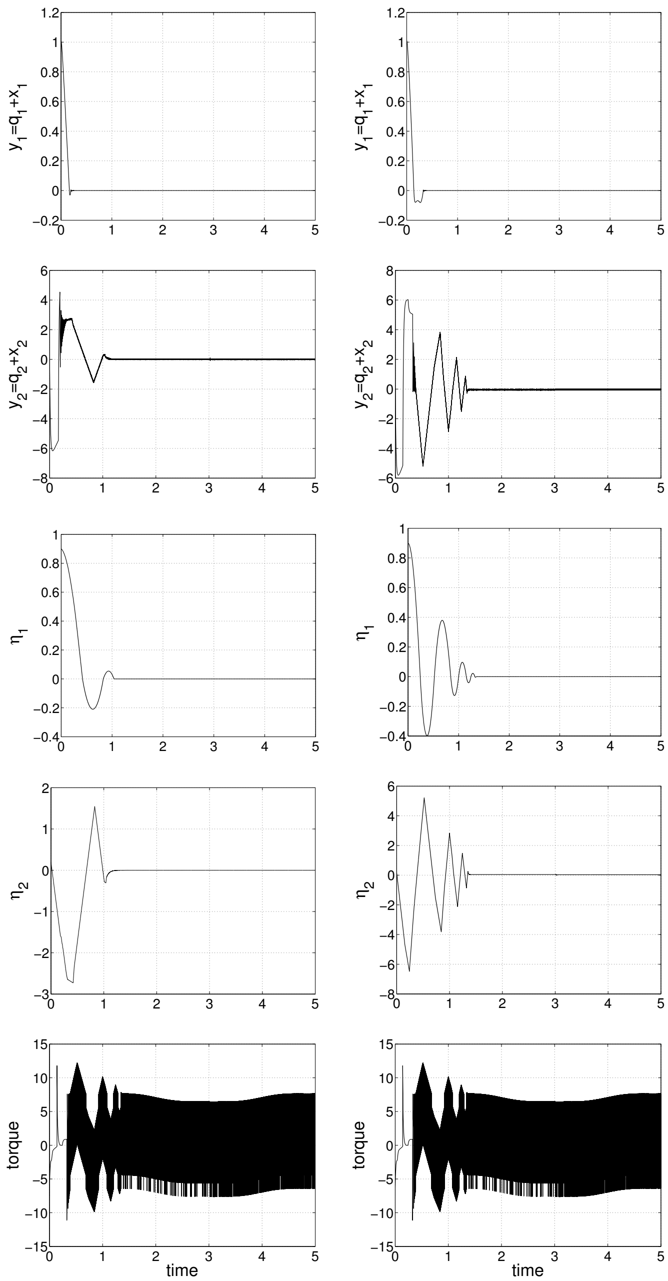

Fig. 1. Orbital stabilization of the pendulum: left column for the no disturbance case; right column for the permanent parameter variation.

$$\begin{aligned}F_l &= \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 v \\ \dot{z} &= v - \frac{|v|}{g(v)}z\end{aligned} \tag{22}$$

where $g(v) = \alpha_c + (F_s - \alpha_c)e^{-v/{v_0}^2}$, σ_0 is the stiffness, σ_1 a damping coefficient, α_c is the Coulomb friction level, F_s is the level of the stiction force, v_s is the Stribeck velocity and v is the viscous friction.

To carry out capabilities of the nonlinear observer against friction model discrepancies, we substituted the dynamic Lugre friction F_l, given by (22), for the friction force F into the plant equation (1). The numerical values of the friction parameters, used in the simulations, are presented in Table 1, whereas the plant parameters as well as the

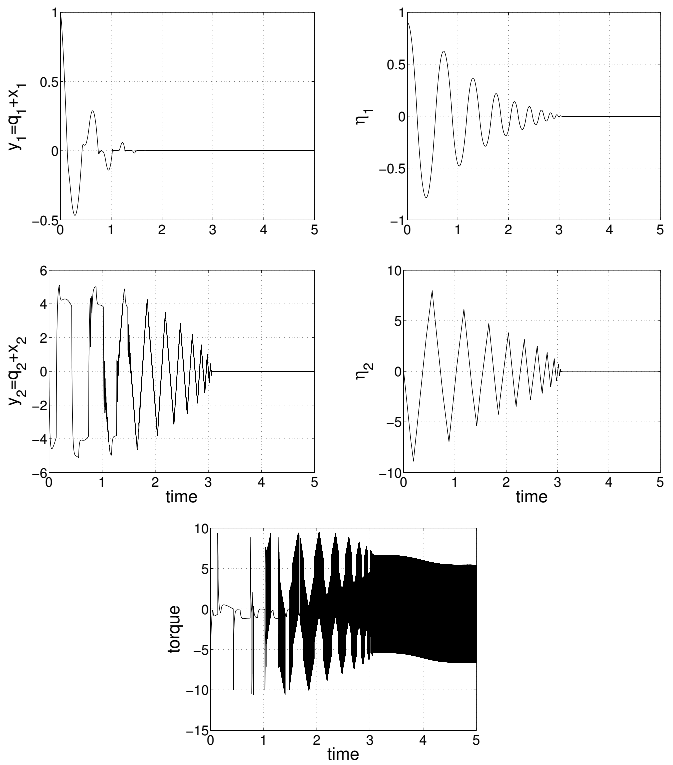

Fig. 2. Orbital stabilization of the pendulum: robustness against unmodeled dynamics resulting from the LuGre friction substitution.

initial conditions were the same as before. The resulting trajectories are depicted in Figure 2. It is concluded from Figure 2 that good performance of the controller is achieved in spite of friction model discrepancies.

Table 1. Lugre model parameters.

Notation	**Value**	**Unit**
σ_0	0.1	N/rad
σ_1	0.1	$N \cdot s/rad$
σ_2	0.0053	$N \cdot m \cdot s/rad$
F_s	0.004	N
v_0	0.01	rad/s

5. CONCLUSIONS

Orbital stabilization of a friction pendulum, operating under uncertain conditions, is under study. The quasihomogeneity-based control synthesis is utilized to design a variable structure controller that drives the state of the pendulum to a model orbit in finite time. The resulting closed-loop system is capable of tracking the model orbit in a sliding mode of the second order, even in the presence of external disturbances with an a priori known magnitude bound.

A well-known Van der Pol oscillator is modified to possess a stable limit cycle, governed by a standard linear oscillator equation. The proposed modification is introduced into the synthesis as an asymptotic harmonic generator of the periodic motion. The resulting closed-loop system is capable of moving from one orbit to another by simply changing the parameters of the modified Van der Pol oscillator.

Capabilities of the quasihomogeneous synthesis and its robustness against parametric disturbances and unmodeled friction dynamics are illustrated in an experimental study. The developed approach is hoped to suggest a practical framework for orbital stabilization of mechanical manipulators and it has additionally been supported by real-time experiments made for an underactuated two-link pendulum robot (Pendubot), required to swing up from its downward position to the upright position. This work is in progress and it will be reported elsewhere.

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REFERENCES

Alvarez J., Orlov Y. and Acho L. (2000). An invariance principle for discontinous dynamic systems with application to a coulomb friction oscillator. *Int. J. of Dynamic Systems, Measurement and Control*, **122**, 687–690.

Bartolini G., Ferrara A. and Usai E. (1998). Chattering avoidance by second-order sliding mode control. *IEEE Trans. Autom. Contr.*, **43**, 241–246.

Canudas-de-Wit C., Espiau B. and Urrea C. (2002). Orbital Stabilization of Under-actuated Mechanical Systems. *15th Triennial World Congress*, IFAC, Spain.

Canudas de Wit C., Olsson H., Astrom K.J., and Lischinsky P. (1995). A new model for control of systems with friction. *IEEE Transactions on Automatic Control*, **40**, 419–425.

Chevallereau C., Abba G., Aoustin Y., Plestan F., Westervelt E., Canudas-de-Wit C. and Grizzle J.W. (2003). Rabbit: A testbed for advance control theory. *IEEE Control Systems Magazine*, October, 57–79.

Davila J., Fridman L., and Levant A. (2005). Second-order sliding-mode observer for mechanical systems. *IEEE Trans. Autom. Contr.*, **50**, 1785–1789.

Filippov A. F. (1988). *Differential equations with discontinuous right-hand sides*. Dordrecht: Kluwer Academic Publisher.

Fridman L. and Levant A. (1996). Higher order sliding modes as a natural phenomenon in control theory, in *Robust Control via variable structure and Lyapunov techniques*, Garafalo and Glielmo (eds.), Lecture notes in control and information science, Berlin, Springer, 217, 107–133.

Fridman L. and Levant A. (2002). Higher order sliding modes, in *Sliding mode control in engineering*, W. Perruquetti and J.-P. Barbout (eds.), New York: Marcel Dekker, 53–102.

Hill D. and Moylan P. (1976). The stability of nonlinear dissipative systems. *IEEE Trans. Auto. Ctrl.*, **21**, 708–711.

Khalil H. (2002). *Nonlinear systems, third edition*, New Jersey: Prentice Hall.

Levant A. (1993). Sliding order and sliding accuracy in sliding mode control. *Int. J. Contr.*, **58**, 1247–1263.

Orlov Y. (2005a). Finite-time stability and robust control synthesis of uncertain switched systems. *SIAM Journal on Control and Optimization*, **43**, 1253–1271.

Orlov Y. (2005b). Finite time stability and quasihomogeneous control synthesis of uncertain switched systems with application to underactuated manipulators. *Proc. of the 44th Conference on Decision and Control*, Seville, Spain.

Orlov Y., AguilarL., and Cadiou J. (2003a). Switched chattering control vs. backlash/friction phenomena in electrical servo-motors. *Int. J. Contr.*, **76**, 959–967.

Orlov Y., Alvarez J., Acho L., and Aguilar L. (2003b). Global position regulation of friction manipulators via switched chattering control. *Int. J. Contr.*, **76**, 1446–1452.

Orlov Y., Acho L., and Aguilar L. (2004). Quasihomogeneity approach to the pendubot stabilization around periodic orbits. *Proc. 2nd IFAC Symposium on Systems, Structure and Control*, CD-ROM, Oaxaca-Mexico.

Orlov Y., Aguilar L., and Acho L. (2005). Zeno mode control of underactuated mechanical systems with application to the Pendubot stabilization around the upright position . *Proc. 16th IFAC World Congress*, CD-ROM, Prague-Czech.

Orlov Y., Aguilar L., Acho L., and Ortiz A. (2008). Asymptotic harmonic generator and its application to finite time orbital stabilization of a friction pendulum with experimental verification. *Int. J. Contr.*, **81**, 227–234.

Roup A.V. and Bernstein D.S. (2002). Adaptive stabilization of a class nonlinear systems with nonparametric uncertainty. *IEEE Trans. Auto. Control.*, **46**, no. 11, 1821–1825.

Shiriaev A., Perram J.W., and Canudas-de-Wit C. (2005). Constructive tool for orbital stabilization of underactuated nonlinear systems: virtual constraints approach. *IEEE Trans. Autom. Contr.*, **50**, 1164–1176.

Sira-Ramirez H. (1987). Harmonic response of variable-structure-controlled Van der Pol oscillators. *IEEE Trans. Circuits and Systems*, **34**, 103–106.

Slotine J.-J. and Li W. (1991). *Applied Nonlinear Control*. New Jersey: Prentice Hall.

Utkin V.I., Guldner J. and Shi J. (1999). *Sliding modes in Electromechanical Systems*. London: Taylor and Francis.

Wang H.-H. and Krstic M. (2000). Extremun Seeking for Limit Cycle Minimization. *IEEE Trans. Aut. Contrl.*, **45**, 12, 2432–2437.