Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Optimal Spectral Expansion for Discrete Control

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Abstract: Consideration was given to identification of discrete processes, which is reducible to the functional decomposition of discrete functions, where by the decomposition is meant the representation of a function by a formula in the basis of binary operations. A procedure of optimal formula design was based on a novel approach of spectral expansion. Both exact and asymptotic complexity estimate of the designed formulas were given.

1. INTRODUCTION

Computer science and digital technology form a complex and wide subject that extends from social implementation of technological development to deep mathematical foundations of the techniques that make this development possible. In control science digital technology is used widely for discrete control. Explicit models are required by many of modern control methods including control design (Ikonen and Najim, 2002). In the model-based approaches the controller can be seen as an algorithm operation on a model of the process (Fig. 1).

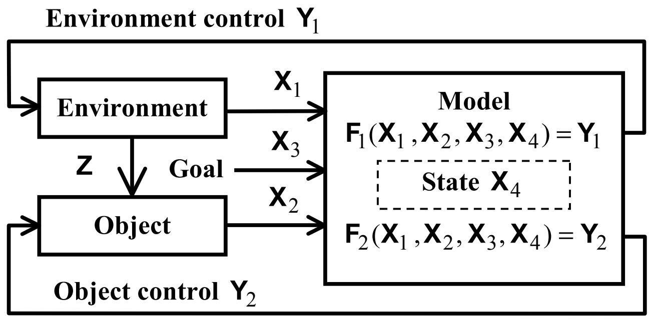

Fig. 1. Discrete control

From control design's point of view, we need to construct discrete function representation. Decomposition of discrete function is one of the universal approaches to modeling (Astola and Stankovic, 2006). The function is represented as a composition of smallest-dimensionality functions. In limit required for practical purposes, the composition is constructed of the variables and operations, which are immediately realized by the computing facility.

For discrete control it is required a compact representation of function. In spectral techniques there is an optimal method of synthesis called Karunen-Loev transformation (Achmed and Rao, 1980), where expansion is done in the eigenvectors of the function covariance matrix, thus providing its best root-mean-square approximation.

However, in distinction to Karunen-Loev transformation, in actual practice one needs to minimize not only the complexity of expansion, but also the complexity of function representation. Therefore, we pose the problem of designing the least-complexity formula representation of function. We note that such a problem does not arise with the Karunen-Loev expansion.

For the optimal Karunen-Loev expansion, the spectral functions are calculated with a certain degree of freedom, their number being equal to the number of nonzero eigenvalues. As the result, this transformation guarantees determination of the least complexity function expansion. We make use of the available degrees of freedom in the definition of the spectral functions and take their subset such that at the minimum complexity of expansion it also provided the minimum complexity of the function representation.

Our approach is also similar to approach that uses linear independence of spectral function in the orthogonal expansion (Perkowski, 1992)

This paper is devoted to the spectral discrete decomposition where the variables and functions assume values over arbitrary finite set and the choice of operations is not confined to any of their subsets. It is needed to design optimal formula representations of the discrete function. The gist of this approach lies in merging the algebraic methods and orthogonal expansions within a wide spectrum of operations. The analytical construction of the spectral expansion is used to find the upper estimate of the designed formulas.

2. NOTATIONS AND DEFINITIONS

For discrete coding of data, we take the integer sets $N_k = \{0,1,\ldots,k-1\}$, where $k > 0$ is the number of elements in N_k.

2.1 Discrete function

The function f of one variable x defined on set N_k is a map on N_k to N_{k_f} such that each element x of the definition domain N_k is related (corresponds) to a most one element $y = f(x)$ of the range space N_{k_f}. The number k is called the digit capacity of the variable, k_f, the digit capacity of the function.

Let the function f be defined on set $N_{k_1} \times N_{k_2} \times \ldots \times N_{k_n}$ and assume values on the set N_{k_f}, where $\times$ is the Cartesian product of sets. In this case, the function depends on n variables x_1, x_2, …, x_n with respective digit capacities k_1,

k_2, …, k_n, that is, $y = f(x_1, x_2, \ldots, x_n)$, or $y = f(X)$, where $X = \{x_1, x_2, \ldots, x_n\}$.

An arbitrary function is defined by the vector $F = [f_1, f_2, \ldots, f_m]$ and vector $K = [k_1, k_2, \ldots, k_n]$ of the digit capacities of the variables, $m = k_1 k_2 \ldots k_n$, where the brackets are used to denote the vectors.

2.2 Discrete operation

By the operation is meant the function, which is essentially dependent on its variables. Operation is defined by the number of operands (variables) involved in the generation of its result. Obviously, if result on a r-place operation does not depend on one of the operands, then this operation should be regarded as ($r-1$)-place operation.

The binary (two-place) operation may be defined by a matrix. The $k_1 \times k_2$ matrix defines a binary operation if it has at least one row (column) differing from the rest of the rows (columns). In this case, the first operand has the digit capacity k_1, and the second, k_2.

Let the binary operation $\circ$ be defined by the matrix $[y_{ij}]$. Application of the binary operation $\circ$ to the variables x_1 and x_2 will be denote by $x_1 \circ x_2 = [y_{ij}](x_1, x_2) = y_{x_1 x_2}$.

2. SPECTRAL DECOMPOSITION

The decomposition of a discrete function means its representation by a formula relating the variables and operations over them.

Let f be a discrete function of digit capacity k_f that depends on n variables $X = \{x_1, x_2, \ldots, x_n\}$. The variables digit capacities are $K = \{k_1, k_2, \ldots, k_n\}$. We expand f in some system of functions $G = \{g_i(X) \mid i = \overline{1,m}\}$ over finite field or integral domain $R = \langle N_k, +, \times \rangle$ defined on the discrete set N_k,

$$f(X) = \sum_{i=1}^{m} g_i(X) \times h_i \,, \qquad (1)$$

where $k_f \le k$, h_i are expansion coefficients (the elements of R).

2.1 Spectral functions

We demand an effective computation from spectral functions. For that let $g(X) = x_1 \circ_1 x_2 \circ_2 x_3 \cdots \circ_{n-1} x_n$, where $\circ_i$ are some discrete operations. In this case, the spectral function $g(X)$ may be written in matrix form,

$$x_1 \underset{\circ_1}{\overset{x_2}{\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}}} \underset{\circ_2}{\overset{x_3}{\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}}} \cdots \underset{\circ_{n-1}}{\overset{x_n}{\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}}} \qquad (2)$$

where $*$ is used to denote arbitrary elements.

Example 1. Let the formula of function $g(x_1, x_2, x_3)$ be defined in matrix form (2),

$$x_1 \underset{\circ_1}{\overset{x_2}{\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}}} \underset{\circ_2}{\overset{x_3}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}}} \to g(X).$$

In this case, $g(X) = x_1 \circ_1 x_2 \circ_2 x_3$, $K = \{2,3,2\}$ and $k = 3$. To find the vector g we calculate its truth table (Table 1). As a result of calculation, we have the vector of function **G**, $\mathbf{G} = [012010201202]$, where vector of digit capacities **K** is $[232]$.

Table 1. Truth table of function

x_1	x_2	x_3	g
0	0	0	0
1	0	0	1
0	1	0	2
1	1	0	0
0	2	0	1
1	2	0	0
0	0	1	2
1	0	1	0
0	1	1	1
1	1	1	2
0	2	1	0
1	2	1	2

Example 2. Let the spectral function g be defined by the vectors **G** and **K**, $\mathbf{G} = [011101102010201021]$, $\mathbf{K} = [323]$. To represent the function in matrix form (2) we construct of its matrix pattern,

$$x_1 \underset{\circ_1}{\overset{x_2}{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}} \underset{\circ_2}{\overset{x_3}{\begin{bmatrix} 0 & 6 & 12 \\ 1 & 7 & 13 \\ 2 & 8 & 14 \\ 3 & 9 & 15 \\ 4 & 10 & 16 \\ 5 & 11 & 17 \end{bmatrix}}} \to g(X).$$

Now we replace the numbers of the function points into matrix of operation $\circ_2$ by the values of function,

$$x_1 \underset{\circ_1}{\overset{x_2}{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}} \underset{\circ_2}{\overset{x_3}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}}} \to g(X).$$

We reduce the operation $\circ_2$ by finding the same rows into the matrix and replace numbers of these rows into the matrix of the operation $\circ_1$. After deletion of duplicate rows, we have

$$x_1 \underset{\circ_1}{\overset{x_2}{\begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}}} \underset{\circ_2}{\overset{x_3}{\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}}} \to g(X),$$

or, in symbolic form,

$$g(X) = x_1 \circ_1 x_2 \circ_2 x_3 ,$$

where

$$\circ_1 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}, \ \circ_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

2.2 Reductive spectral expansion

As $g_i(X) \times h_i = g_i'(X) \times \tau$, where τ is a neutral element of the algebra R relative to the multiplication (unit), the function $g_i'(X)$ is got from the function $g_i(X)$ by multiplication the matrix of last operation $\circ_{n-1}$ by the constant h_i. Then the spectral expansion (1) may be represented as matrix equation,

$$\left[\begin{array}{cccc|ccc} g_{11} & g_{21} & \cdots & g_{M1} & * & \cdots & * \\ g_{12} & g_{22} & \cdots & g_{M2} & * & \cdots & * \\ g_{13} & g_{23} & \cdots & g_{M3} & * & \cdots & * \\ g_{14} & g_{24} & \cdots & g_{M4} & * & \cdots & * \\ g_{15} & g_{25} & \ddots & g_{M5} & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{1m} & g_{2m} & \cdots & g_{Mm} & * & \cdots & * \end{array}\right] \times \left[\begin{array}{c} \tau \\ \tau \\ \vdots \\ \tau \\ \hline \sigma \\ \vdots \\ \sigma \end{array}\right] = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_m \end{bmatrix},$$

where some coefficients are equal to zero σ (a neutral element of the algebra R relative to the addition) and the residuary coefficients equal to unit τ.

Because of zero coefficients existence, we need only M spectral functions in expansion (1),

$$f(X) = \sum_{i=1}^{M} g_i(X) \times \tau, \quad M \le m .$$

From $g_i(X) \times \tau = g_i(X)$, we have

$$f(X) = \sum_{i=1}^{M} g_i(X) . \tag{3}$$

Hence the requirements of the algebra R can be weaken and the reductive expansion (3) can be fulfilled over a group $< N_k, + >$ defined on the set N_k by binary operation + that is called addition, such that for all a and b the equations $a + x = b$ and $x + a = b$ have unique solutions with respect to $x \in N_k$. It is meant that the addition matrix has no repeating elements in each row and column.

Example 3. A formula to compute the first 24 decimal digits of $\pi = 3.14159265358979323846265$ modulo 3 is required. In this case, $\mathbf{F} = [011120202022010020210202]$ and $\mathbf{K} = [2223]$. Let $R =< N_3, + >$ and

$$+ = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

where $\sigma = 0$ and each row and each column has no repeating elements. Then function f has the following representation,

$$f(X) = g_1(X) + g_2(X) ,$$

where

$$x_4 \underset{\circ_{11}}{\overset{x_1}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}}} \underset{\circ_{12}}{\overset{x_2}{\begin{bmatrix} 0 & 0 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}}} \underset{\circ_{13}}{\overset{x_3}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}}} \to g_1(X),$$

$$x_4 \underset{\circ_{11}}{\overset{x_1}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}}} \underset{\circ_{12}}{\overset{x_2}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}}} \underset{\circ_{13}}{\overset{x_3}{\begin{bmatrix} 0 & 0 \\ 2 & 2 \\ 0 & 2 \end{bmatrix}}} \to g_2(X).$$

2.3 Upper Bound Estimate

For the maximum number of addends in (3) required to realize an arbitrary function on n variables with the digit capacities k_1, k_2 … k_n, it was found (Vykhovanets, 2006),

$$M \le \frac{1}{k-\alpha} \frac{\prod_{i=1}^{n} k_i}{\sum_{i=1}^{n} k_i - \beta}, \quad \beta = k_1 + k_2 - \frac{k_1 k_2}{k-\alpha}, \tag{4}$$

where α is the parameter, $0 < \alpha < 1$, which called the generating ability of the analytical construction (2), and β is the correction coefficient taking into account the initial generating ability of the construction.

Estimate (4) can be interpreted in rather simple descriptive terms. The upper bound of the degrees of freedom of the analytical representation (3) is smaller then the operations total area. The lower bound is equivalent to the area, which is obtained by deleting one row of each operation but the first one.

Fig. 2 depicts the generating ability α vs. the number of variables n. Calculation was carried out for identical digit capacities of the group G, function, and their variables.

We note that with growing n the parameter α tends asymptotically to some value α_k, which is referred to as the limit generating ability.

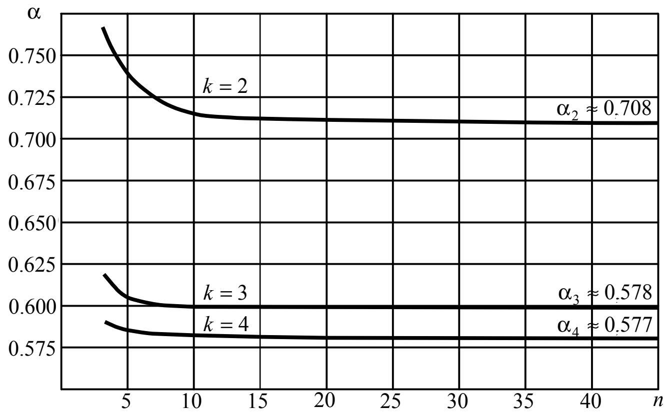

Fig. 2. Generating ability of finite region

We establish from (4) for $n \to \infty$ that

$$M \sim \frac{1}{k-\alpha_k}\,\frac{\prod_{i=1}^{n} k_i}{\sum_{i=1}^{n} k_i}, \quad \alpha_k = k - \frac{1}{k}\log_k \prod_{i=0}^{k-1}\frac{k^k - i}{i+1}. \qquad (5)$$

Analysis of (5) and Fig. 3 show that the limit generating ability α_k tends to unity with k, but its least value is attained for $k = 4$.

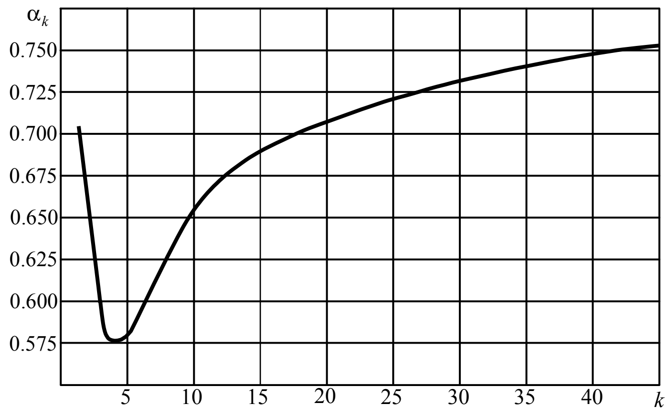

Fig. 3. Limit generating ability

Example 4. For spectral expansion of the function from Example 3, it is required $M = 2$ addends,

$$M \le \frac{1}{3-0.75}\,\frac{24}{9-(2+2-1.78)} \approx 1.57\,.$$

3. SPECTRAL SYNTHESIS

The base operation in spectral synthesis of discrete functions is the representation of vector as composition of operations, which may be defined by unknown matrices from (2).

3.1 Matrix decomposition

For a given vectors **F** and **K**, let us find $n-1$ matrices $\circ_t$, $t = \overline{1, n-1}$, such that $f(X) = g(X)$, where $g(X)$ is a spectral function. Let the matrices $\circ_i$ be defined by recurrent rule,

$$\left.\begin{aligned} K_t &= k_t K_{t-1}, \\ \circ_t &= [p_{ij}],\ p_{ij} = i + jK_t \end{aligned}\right\} \ (t = \overline{1, n-2}),$$

where $K_0 = 1$, $i = \overline{0, K_{t-1}-1}$, $j = \overline{0, k_{t+1}-1}$, and

$$\left.\begin{aligned} K_{n-1} &= k_n K_{n-2}, \\ \circ_{n-1} &= [p_{ij}],\ p_{ij} = f_q,\ \ q = i + jK_{n-1} \end{aligned}\right\},$$

where f_q is the q th element of vector **F**.

For an arbitrary function f, we obtain the matrices $\circ_t$, such as $f(X) = g(X)$, where the digit capacities of the operations are greater then k (see Example 2).

3.2 Reduction of operations

The next step in spectral synthesis consists of the reduction of $g(X)$ and corresponding modification of $\circ_t$, such that its digit capacities have to equal to k. Like approach was used for polynomial factorization of spectral bases (Vykhovanets, 2004).

For this purpose, the rows of the matrices $\circ_t$ are subdivided in equivalent classes. The matrix $\circ_t$ itself is transformed to a form containing one row from each classes.

Let $\{C_0, C_1, \ldots, C_{u-1}\}$ be the equivalent classes of the rows consisting of indexes of identical rows (or comparable rows if the row contains unspecified value). The reduced matrix is constructed as follows: the i th row of new matrix is taken to be a row from the class C_i.

To preserve equality $f(X) = g(X)$, all elements $c \in C_i$ of the matrix $\circ_{t-1}$ must be replaced by i. We obtain new matrices $\circ_{t-1}$ and $\circ_t$ such that they have no duplicate rows.

If the dimension of $\circ_t$ is not greater then k, then reduction of $\circ_t$ is assumed to be successful and the matrix of $\circ_{t-1}$ must be reduced. Otherwise, we choose only k rows of $\circ_t$ and repeat replacing the indexes of $\circ_{t-1}$ so that the indexes, which have no corresponded rows of $\circ_t$, are replaced by asterisk (unspecified value).

3.3 Residual vector

When we have $f(X) \ne g(X)$, we need to calculate residual vector $\mathbf{F}' = \mathbf{F} - \mathbf{G}$ over group R by solving the equation $g + f' = f$ with respect to f' for all elements g and f of vectors **G** and **F** correspondently.

In the next step the decomposition, the vector $\mathbf{F}'$ is used as new vector of decompose function. If the vector $\mathbf{F}'$ equals to zero, then the decomposition of f is finished.

Thus, the spectral decomposition can be described by the following recurrent rule,

$$\begin{cases} \mathbf{F}_1 = \mathbf{F}; \\ \mathbf{F}_i = \mathbf{F}_{i-1} - \mathbf{G}_{i-1}\ \ (i = \overline{2, t}); \\ \mathbf{F}_t = [\sigma\sigma\ldots\sigma], \end{cases}$$

where $\mathbf{G}_{i-1}$ is the reduction of $\mathbf{F}_{i-1}$, σ is left zero of the group R.

3.4 Swapping variables

To minimize the number of equivalent classes we use variable swapping. We establish a one-one correspondence between the value i of the variable $X \in N_m$ and the values i_j of the variables x_j by representing the number i in a n-digit positional system with bases defined by the digits capacities of variables:

$$i = (i_n i_{n-1} \dots i_1)_{k_n k_{n-1} \dots k_1},$$

where i_j is j th digit in the representation of the number i in the positional notation with vector of bases $\mathbf{K} = [k_1 k_2 \dots k_n]$.

If we change the order of variables from $\mathbf{X} = [\dots x_p \dots x_q \dots]$ to $\mathbf{X}' = [\dots x_q \dots x_p \dots]$ for calculating $f'(i)$, then we need swap the correspondents digit capacities in vector of bases from $\mathbf{K} = [\dots k_p \dots k_q \dots]$ to $\mathbf{K}' = [\dots k_q \dots k_p \dots]$.

It is very important that we need swap only the last variable of $\mathbf{X}$ as the reduction of the last operation $\circ_{n-1}$ does not depend of the order of variables expect for the last one.

Example 5. Now the formal representation of function from Example 3 will be synthesized. At first we change orders of variables and calculate new vectors of values for various last variable (Table 2).

Table 2. Swapping variables

X	**F**
$[x_1\ x_2\ x_3\ x_4]$	[011120202022010020210202]
$[x_4\ x_1\ x_2\ x_3]$	[022100122121200012200002]
$[x_3\ x_4\ x_1\ x_2]$	[022020100102122020102012]
$[x_2\ x_3\ x_4\ x_1]$	[012222002200110002100122]

The very best of orders is $\mathbf{X} = [x_4\ x_1\ x_2\ x_3]$, where the last matrix of operation has the least number of equivalent classes. In this case, we have

$$x_4 \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \underset{\circ_1}{\overset{x_1}{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \underset{\circ_2}{\overset{x_2}{\begin{bmatrix} 0 & 6 \\ 1 & 7 \\ 2 & 8 \\ 3 & 9 \\ 4 & 10 \\ 5 & 11 \end{bmatrix}}} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \underset{\circ_3}{\overset{x_3}{\begin{bmatrix} 0 & 2 \\ 2 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 1 & 2 \\ 2 & 0 \\ 2 & 0 \\ 1 & 0 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}}} \to f(X),$$

where are the following equivalent classes: $C_0 = \{0,5\}$, $C_1 = \{1,2,7,8,10\}$, $C_2 = \{3,9\}$, $C_3 = \{4\}$, and $C_4 = \{6,11\}$.

Now we must choose only k classes, where k is the digit capacity of group R, $k = 3$. Obviously the classes C_0, C_1 and C_2 are ensured the most covering of the matrix $\circ_3$.

Thus, after the reducing of $\circ_3$, we obtain

$$x_4 \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \underset{\circ_{11}}{\overset{x_1}{\begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}}} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \underset{\circ_{12}}{\overset{x_2}{\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \\ 2 & 2 \\ * & 2 \\ 1 & * \end{bmatrix}}} \begin{array}{c} 0 \\ 1 \\ 2 \end{array} \underset{\circ_{13}}{\overset{x_3}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}}} \to g_1(X),$$

where asterisk is used when there is no required row into the matrix of operation $\circ_{13}$.

After the swapping variables $[x_4\ x_1\ x_2]$ and the reducing of $\circ_{12}$ we have

$$x_4 \underset{\circ_{11}}{\overset{x_1}{\begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}}} \underset{\circ_{12}}{\overset{x_2}{\begin{bmatrix} 0 & 0 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}}} \underset{\circ_{13}}{\overset{x_3}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}}} \to g_1(X),$$

where elements denoted by asterisk are used to minimize the number of the rows classes.

Let us calculate the vector $\mathbf{G}_1$ and the residual vector $\mathbf{F}_2$,

$$\mathbf{G}_1 = [022100222122200022000000],$$

$$\mathbf{F}_2 = \mathbf{F}_1 - \mathbf{G}_1 = [022100122121200012200002].$$

In the same way we derive from $\mathbf{F}_2$ the following results:

$$x_4 \underset{\circ_{21}}{\overset{x_1}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}}} \underset{\circ_{22}}{\overset{x_2}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}}} \underset{\circ_{23}}{\overset{x_3}{\begin{bmatrix} 0 & 0 \\ 2 & 2 \\ 0 & 2 \end{bmatrix}}} \to g_2(X),$$

$$\mathbf{G}_2 = [000000200002000020200002],$$

$$\mathbf{F}_3 = [000000000000000000000000].$$

Finally we find

$$\mathbf{F} = \mathbf{G}_1 + \mathbf{G}_2,$$

$$f(X) = g_1(X) + g_2(X),$$

$$f(X) = x_4 \circ_{11} x_1 \circ_{12} x_2 \circ_{13} x_3 + x_4 \circ_{21} x_1 \circ_{22} x_2 \circ_{23} x_3,$$

where

$$\circ_{11} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}, \ \circ_{12} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}, \ \circ_{13} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix},$$

$$\circ_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \; \circ_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \; \circ_{23} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \\ 0 & 2 \end{bmatrix}.$$

4. EFFICIENCY

To study computation of discrete function, we use efficiency criteria witch taking into account both computation time and size of memory required to compute.

4.1 Criteria of efficiency

We introduce the following criteria for efficiency of the spectral expansion:

$$E = \frac{V_T}{V_S}\frac{T_T}{T_S}, \qquad (6)$$

where V_S is the size of memory required to the spectral expansion, V_T is the size of memory required to store the truth table of function, T_S is the time required to calculate a value of function by spectral expansion, and T_T is the time required to calculate a value of function by tabular method.

Thus, we calculate the efficiencies relatively tabular implementation of discrete functions, which is function-invariant. It is allowed one not only to compare different representations of the same function, but different representations of different functions as well.

4.2 Efficiency of spectral expansion

Let us estimate memory and time requirements of the tabular and of the optimal spectral method,

$$V_T = \prod_{i=1}^{n} k_i\,,\; T_T = 1;$$

$$V_S = 1\,,\; T_S \le \frac{n}{k-\alpha}\frac{\prod_{i=1}^{n} k_i}{\sum_{i=1}^{n} k_i - \beta}.$$

Then the efficiency of the reductive spectral expansion (6) can be written so,

$$E \ge (k-\alpha)\left(\bar{k} - \frac{\beta}{n}\right) > 1,$$

where $\bar{k}$ is the mean digit capacity of the variables. In asymptotic region, we have

$$E \sim (k-\alpha_k)\bar{k}\,.$$

Example 6. Let us calculate efficiency of the spectral expansion of the function from Example 5,

$$E = \frac{1}{4}(3-0.75)(2+3-\frac{4}{3-0.75}) \approx 3.81\,.$$

4.3 Efficiency of implementation

We note that to implement the tabular computation it is necessary to calculate an entry x into truth vector of function,

$$x = x_1 + k_1(x_2 + k_2(x_2 + \ldots + k_{n-1}x_n)\ldots)\,,$$

what requires $2(n-1)$ units of time. It increases the efficiency of computation in spectral form. And the spectral expansion requires some hardware to implement its operations.

In contrast to the tabular method, we may also use parallel computation of the spectral expansion (Fig. 4).

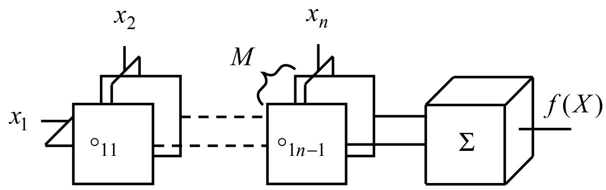

Fig. 4. Parallel computation of the spectral expansion

In this case, $M(n-1)$ operations may be fulfilled simultaneously, what requires units of time equal to $n-1$. From (6) and $T_S \le M + n - 2$, we find

$$E \ge \frac{(k-\alpha)\left(\sum_{i=1}^{n} k_i - \beta\right)\prod_{i=1}^{n} k_i}{(n-2)(k-\alpha)\left(\sum_{i=1}^{n} k_i - \beta\right) + \prod_{i=1}^{n} k_i}. \qquad (7)$$

Example 7. Let us calculate efficiency of the parallel computation of function from Example 5. Using (7), we have $E \approx 6.71$.

REFERENCES

Ahmed, N. and Rao, K.R. (1980). *Orthogonal transforms for digital signal processing.* Springer, Berlin.

Astola, J.T. and Stankovic, R.S. (2006). *Fundamentals of switching theory and logic design: A hands on approach.* Springer, Dordrecht.

Ikonen, E. and Najim, K. (2002). *Advanced processes identification and control.* Marcel Dekker, New York.

Perkovski, M.A. (1992). The generalized orthogonal expansion of function with multiple-valued inputs and some of its application. *Proc. Int. Symp. Multi-Valued Logic,* 442-450.

Vykhovanets, V.S. (2004). Polynomial factorisation of spectral bases. *Automation and Remote Control*, 65(12), 888-893.

Vykhovanets, V.S. (2006). Algebraic decomposition of discrete functions. *Automation and Remote Control*, 67(3), 361-392.