IFAC

Estimation of Attraction Domains in Wheeled Robot Control Using Absolute Stability Approach [*]

Lev B. Rapoport, [*] **Yuriy V. Morozov** [*]

[*] *Institute of Control Sciences RAS and Javad GNSS, Moscow, Russia*
(e-mail: L.Rapoport@javad.com)

Abstract: Considered is the control synthesis problem for planar motion of a wheeled robot. The mathematical model of the robot is based on kinematic relationships between the velocity of a given point of a robot platform, referred to as the target point, orientation of the platform, and control. It is supposed that all four wheels move without a lateral slippage. The front wheels are responsible for steering. The control goal is to drive the target point to the prespecified trajectory and to stabilize the motion of the target point along the prespecified trajectory. The trajectory consists of line segments and circular arcs. The current curvature of the trajectory of the target point is taken as control; it is related to the steering angle of the front wheels by a simple algebraic expression. The control is subject to two-sided constraints due to limitations on the steering angle of the front wheels. For the control law proposed, the attraction domain in the space "distance to the trajectory - orientation" is analyzed. For the initial conditions from this domain, the system is guaranteed to hit a trajectory with given exponent of stability. The numerical method based on LMIs approach is proposed to approximate the attraction domain.

Keywords: Asymptotic stabilization; Tracking; LMIs; Lyapunov methods; Mobile robots; Guidance navigation and control

1. INTRODUCTION

There are many applications like a road construction and agriculture where a vehicle must be automatically driven along a target trajectory with high level of accuracy. These and other tasks are performed by wheeled robots equipped with satellite and inertial navigation tools, see Cordesses et al. (2000), Thuilot et al. (2002), and Rapoport et al. (2006). On the other hand, automatic control does not exclude manual steering. Presence of an operator is necessary from safety considerations and on the case if unexpected disturbances in the sensors' measurements, satellites shading, or signal multipath errors throw the system out of the stability region.

In general, the control system does not posses global stability. Automatic driving to the target path is not guaranteed if the system starts from the initial state not belonging to the attraction domain in the state space. The vehicle must be equipped with a special indicator on the control panel to let the operator know whether it is safe to operate in the automatic mode. If the system estimates itself as belonging to the attraction domain, the indicator lights green, otherwise it lights red. The problem of numerical estimation of the attraction domain is addressed in the paper.

The problem, similar to the one considered in this paper is not new. In many papers (e.g., see Cordesses et al. (2000), Thuilot et al. (2002), Samson (1995), Kolmanovsky

and McClamroch (1995), Guldner and Utkin (1994) and the references therein), a control law is designed which stabilizes the motion along a line segment or stabilizes the motion towards a given point in the plane; the control may be either continuous or discontinuous (Guldner and Utkin (1994). Here, we consider the control design problem which accounts for the boundedness of controls. Under this condition, it is impossible to attain the guaranteed rate of convergence from arbitrary initial position and orientation of the platform (Rapoport (2006)). Instead, for a given norm of deviation, the domain of initial conditions is estimated such that the synthesized control provides the specified rate of its exponential decay. In the paper (Rapoport (2006)) this problem has been considered for the case of the straight line and the circular trajectory separately. In the present paper the trajectory is supposed to consist of line segments and circular arcs of the finite length. This way of parametrization corresponds to the motion of a tractor along parallel swaths or furrows across a work field, the one which is most often the case in agricultural applications. Circular arcs correspond to U-turns of the tractor at the ends of the furrows. The control law and attraction domain guaranteeing the specified rate of the exponential convergence is constructed for the composite trajectory consisting of arbitrary number of segments.

2. KINEMATIC SCHEME

The model of a wheeled robot is represented in Fig. 1. The motion is assumed to be two-dimensional, the orientation

10.3182/20080706-5-KR-1001.0373

of the robot platform is defined by a single angle. The symbol $^\mathrm{T}$ denotes the matrix transpose, vectors are supposed to be columns, $X = (x, y)^\mathrm{T}$ denotes the point of the plane. The target point is located at the middle of the rear axle of the platform and is denoted by $X_c = (x_c, y_c)^\mathrm{T}$. For the planar case the orientation is defined by an angle θ between the centre line of the platform and the x-axis. Every point X of the platform has its own instantaneous velocity vector V. Vectors, orthogonal to the instant velocities, all intersect in a single point X_0 known as instantaneous center of velocity. Let $\dot{\theta}$ be an instantaneous angular rate of the rotation of the platform. Then the following relationship holds:

$$|\dot{\theta}| = \|V\|/\|X - X_0\|. \tag{1}$$

Hereinafter, $\|\cdot\|$ denotes the Eucledian vector norm. Moreover, the condition that each of the four wheels move without lateral slippage means that the vectors of instantaneous velocities of the axles' endpoints are collinear to the planes of the wheels; the normals to each of these vectors intersect at the point X_0.
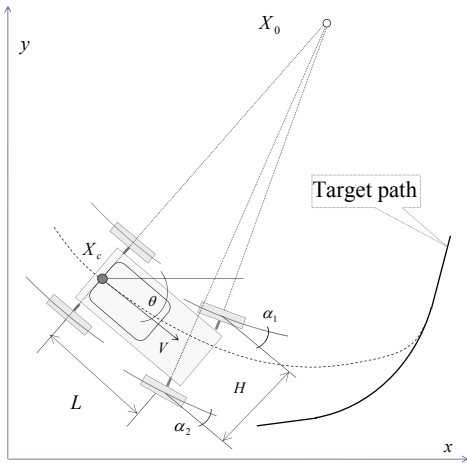

Fig. 1. The kinematics scheme of the wheeled robot.

The two rear wheels are driving and the front wheels are responsible for turning the platform. For the case of the movement along a straight line, the point X_0 is located at infinity and expression (1) yields zero angular rate. For points of the rear axle the instantaneous center of velocity coincides with the instantaneous center of curvature. Particularly, for the target point the value $\|X_c - X_0\|$ is the instantaneous radius of curvature of the trajectory (dashed line in Fig. 1) circumscribed by the target point X_c. The value $1/\|X_c - X_0\|$ reciprocal to the radius is the instantaneous curvature; denote this value by u, then $\|X_c - X_0\| = 1/u$. Let L and H be dimensions of the platform as shown in Fig. 1. Then relationships

$$\frac{uL}{1 - uH/2} = \tan\alpha_1, \quad \frac{uL}{1 + uH/2} = \tan\alpha_2 \tag{2}$$

relate the curvature u of the target point to the steering angles of the front wheels. Relationships (2) are obtained from Fig. 1 where the platform is turning left. These same relationships are valid for the case of right turn. Note that

the counterclockwise direction is taken as positive, a left turn associates with a positive value of curvature u. This value has opposite sign for right turn. The relation (2) allows to simplify the model, and the value u is further taken as control. Denoting $v_c = \|V_c\|$ in the forward motion and $v_c = -\|V_c\|$ while in the reverse motion, arrive at the following well known model:

$$\begin{aligned}
\dot{x}_c &= v_c \cos\theta, \\
\dot{y}_c &= v_c \sin\theta, \\
\dot{\theta} &= v_c u.
\end{aligned} \tag{3}$$

The limitations on the steering angle impose two-sided constraints on the value of curvature:

$$-\bar{u} \le u \le \bar{u}. \tag{4}$$

The expression for the quantity $\bar{u}$ is easily derived from the value of the maximum steering angle. Taking (4) into account, the equations (3) take the form

$$\begin{aligned}
\dot{x}_c &= v_c \cos\theta, \\
\dot{y}_c &= v_c \sin\theta, \\
\dot{\theta} &= v_c s_{\bar{u}}(u),
\end{aligned} \tag{5}$$

where $s_{\bar{u}}(u)$ is the saturation function:

$$s_{\bar{u}}(u) = \begin{cases} -\bar{u} & \text{for } u \le -\bar{u}, \\ u & \text{for } |u| < \bar{u}, \\ \bar{u} & \text{for } u \ge \bar{u}. \end{cases} \tag{6}$$

3. TARGET TRAJECTORY PARAMETRIZATION

For brevity, the line segments and circular arcs will be referred to as trajectory segments and denoted by $\mathbf{s}_i$, $i = 1, \ldots, n$. The number of segments can be as large as pleased. Every segment has its curvature c_i; the line segments have zero curvature. The following feasibility condition is supposed to be satisfied

$$\|c_i\| < \bar{u}. \tag{7}$$

Let ξ be the length parameter and l_i be the length of the segment $\mathbf{s}_i$. A sample trajectory is depicted in Fig. 2. The segments $\mathbf{s}_1$, $\mathbf{s}_3$, and $\mathbf{s}_5$ are linear. The segments $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_6$, and $\mathbf{s}_7$ are circular. In the course of motion in the neighborhood of the trajectory, one of the segments is considered current. The motion equations (5) are further rewritten in the form where the parameter ξ is taken as an independent variable. The motion along the current segment $\mathbf{s}_i$ is considered started if the parameters ξ exceeds the value b_i

$$b_1 = 0, \quad b_i = b_{i-1} + l_{i-1} \quad \text{for} \quad i = 2, \cdots, n, \tag{8}$$

and it is considered terminated as soon as ξ exceeds the value b_{i+1}. After the segment $\mathbf{s}_i$ is over, the segment $\mathbf{s}_{i+1}$ becomes current. The following assumption is supposed to be satisfied

Assumption 1. Adjacent segments have a common tangent at the connection point

The passage to the successive segment is accompanied by abrupt change of the desired curvature $\mathbf{c}_i$, which, in turn, necessitates the abrupt change of the control u or, according to formulas (2), that of the steering angle of the front wheels. In the situations where the dynamics of the front wheel drive cannot be neglected (in contrast to what was assumed further), the passage from segment to segment is accompanied by transition processes.
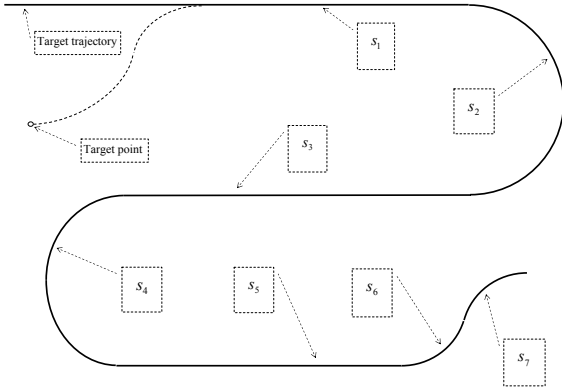
Fig. 2. A target trajectory composed of linear and circular segments.

4. CHANGE OF VARIABLES

Below, the the change of variables is considered separately for the two cases where the segment of the target trajectory is a line segment or a circular segment.

4.1 Motion along Line Segments

A current line segment $\mathbf{s}_i$ is specified by its initial point X_b and the terminal point $X_e \neq X_b$. The direction vector is given by the expression $l = (X_e - X_b)/(\|X_e - X_b\|) = (l_x, l_y)^{\mathrm{T}}$, and the normal vector pointing to the left is given by $n = (-l_y, l_x)^{\mathrm{T}}$. By changing the variables

$$X_c = X_b + (\xi - b_i)l + \eta n, \quad \theta = \psi + \arctan l_y/l_x \quad (9)$$

problem (3) reduces to the following problem:

$$\begin{aligned}
\dot{\xi} &= v_c \cos \psi, \\
\dot{\eta} &= v_c \sin \psi, \\
\dot{\psi} &= v_c u.
\end{aligned} \quad (10)$$

In the new variables, the control goal is to ensure $\eta \to 0$. The following assumptions are supposed to be satisfied.

Assumption 2. The linear velocity of the platform $v_c(t)$ is positive, separated from zero

$$v_c(t) \geq v_0 > 0 \quad (11)$$

and satisfies the existence conditions for absolutely continuous solutions of the system of differential Eqs. (10).

Assumption 3. Along the trajectories of the controlled system (10), the following relation holds:

$$\cos \psi(t) \geq \varepsilon > 0. \quad (12)$$

Assumption 3 will be further removed. As will be shown, if this assumption is satisfied at the initial state it will hold along the whole trajectory.

Further let us perform the change of variables $z_0 = \xi$, $z_1 = \eta$, $z_2 = \tan \psi$ and substitute the time derivative with the derivative in the variable ξ. The variable $z_0 = \xi$ corresponds to the projection of the target point on the linear segment and will be taken as a parameter of the segment length. In this case $c_i = 0$ and $l_i = \|X_e - X_b\|$. The variable $z_1 = \eta$ corresponds to the side deviation from target trajectory.

By the first equation in (10) and Assumptions 2 and 3, the variable ξ changes monotonically. Let $'$ denotes the derivative with respect to the variable ξ; then (10) rewrites in the form

$$\begin{aligned}
z_0' &= 1, \\
z_1' &= z_2, \\
z_2' &= u(1 + z_2^2)^{\frac{3}{2}}.
\end{aligned} \quad (13)$$

First differential equation in (13) is trivial and separates from two others. Finally arrive at the system

$$\begin{aligned}
z_1' &= z_2, \\
z_2' &= u(1 + z_2^2)^{\frac{3}{2}}.
\end{aligned} \quad (14)$$

Let $z = (z_1, z_2)^{\mathrm{T}}$. Starting from the initial value of the parameter $\xi = b_i$ and the initial state $z(b_i)$, the current linear segment is terminated when the independent variable ξ exceeds the value $b_i + l_i$.

4.2 Motion along Circular Arcs

A current circular segment $\mathbf{s}_i$ is specified by the center X_0, initial point X_b, radius R, and angular size ρ. In this case $c_i = 1/R$ and $l_i = \rho R$. We start with the case where the motion is performed in the negative direction, see Fig. 3. The case of the positive direction of motion is analyzed in a similar way.
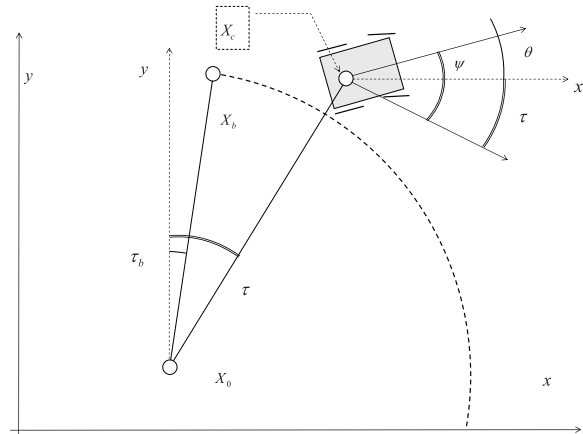

Fig. 3. Motion along a circular segment.

In the figure,

$$\tau = \arctan\left(\frac{x_c - x_0}{y_c - y_0}\right)$$

denotes the angle between the x–axis and the normal to the radius-vector $X_c - X_0$ directed towards the decrease of the angle (or the angle between the y–axis and the vector $X_c - X_0$ which is the same);

$$\tau_b = \arctan\left(\frac{x_b - x_0}{y_b - y_0}\right)$$

denotes the angle between the y–axis and the vector $X_b - X_0$; ψ denotes the angle between the centroidal axis of the platform and the normal to the radius-vector $X_c - X_0$. By the change of variables

$$\begin{aligned}
\eta &= \sqrt{(x_c - x_0)^2 + (y_c - y_0)^2}, \\
\xi &= b_i + (\tau - \tau_b)R, \\
\psi &= \theta + \tau,
\end{aligned} \quad (15)$$

re-write (3) in the form

$$
\begin{aligned}
\dot{\xi} &= v_c \frac{R}{\eta} \cos \psi, \\
\dot{\eta} &= v_c \sin \psi, \\
\dot{\psi} &= v_c u + \frac{v_c}{\eta} \cos \psi.
\end{aligned}
\tag{16}
$$

Assumptions 2 and 3 of the previous section are assumed valid. Denote $z_0 = \xi$, $z_1 = \eta - R$, $z_2 = \tan \psi$. Also, let $'$ denote the derivative with respect to the variable ξ. Then the following relations hold:

$$
\begin{aligned}
z_0' &= 1, \\
z_1' &= (\frac{z_1}{R} + 1) z_2, \\
z_2' &= u(\frac{z_1}{R} + 1)(1 + z_2^2)^{\frac{3}{2}} + \frac{1}{R}(1 + z_2^2).
\end{aligned}
\tag{17}
$$

Using the reasoning similar to that used in the previous subsection, arrive at the system

$$
\begin{aligned}
z_1' &= (\frac{z_1}{R} + 1) z_2, \\
z_2' &= u(\frac{z_1}{R} + 1)(1 + z_2^2)^{\frac{3}{2}} + \frac{1}{R}(1 + z_2^2).
\end{aligned}
\tag{18}
$$

Starting from the initial value of the parameter $\xi = b_i$ and the initial state $z(b_i)$, the current circular segment is terminated when the independent variable ξ exceeds the value $b_i + l_i$.

Summing up results of subsections *4.1* and *4.2* and taking into account Assumption 1, arrive at the following system describing motion along the target trajectory:

$$
\begin{aligned}
z_1' &= (c(\xi)z_1 + 1) z_2, \\
z_2' &= u(c(\xi)z_1 + 1)(1 + z_2^2)^{\frac{3}{2}} + c(\xi)(1 + z_2^2),
\end{aligned}
\tag{19}
$$

where the quantity $c(\xi)$ takes values c_i when the current segment becomes $\mathbf{s}_i$ according to the condition

$$
b_i \leq \xi \leq b_{i+1}.
$$

The value ξ is easily calculated for the current segment using relationships (9) or (15) and measurements of the position. Solution of the system of differential equations (19) with a discontinuous right hand side is considered in the Filippov sense, see Filippov (1988).

5. CONTROL LAW DESIGN

The choice of control u in (19) in the form

$$
u = -\frac{\sigma + c(\xi)(1 + z_2^2)}{(c(\xi)z_1 + 1)(1 + z_2^2)^{\frac{3}{2}}},
\tag{20}
$$

for some $\lambda > 0$ and

$$
\sigma = 2\lambda z_2 + \lambda^2 z_1
\tag{21}
$$

leads to differential equations of the closed loop system

$$
\begin{aligned}
z_1' &= (c(\xi)z_1 + 1) z_2, \\
z_2' &= -\sigma.
\end{aligned}
\tag{22}
$$

Control law of the form (20), based on the feedback linearization approach, was earlier considered in the papers Cordesses et al. (2000), Thuilot et al. (2002), and others. In the present paper we extend it on the case of the control with saturation. If $c(\xi) = 0$ this system is equivalent to

$z_2'' + 2\lambda z_2' + \lambda^2 z_2 = 0$ which implies the exponential decay rate of z_1 and z_2. In the small enough neighbourhood of the zero the term $c(\xi)z_1 z_2$ is negligible and the zero solution of the system (22) is also exponentially stable with the exponent of stability $-\mu$ where $0 < \mu < \lambda$. However, in general, control (20) does not satisfy the two-sided constraints (4). On the other hand, taking control in the form

$$
u = -s_{\bar{u}}\left(\frac{\sigma + c(\xi)(1 + z_2^2)}{(c(\xi)z_1 + 1)(1 + z_2^2)^{\frac{3}{2}}} \right),
\tag{23}
$$

may not guarantee that z_1 and z_2 decrease exponentially. We introduce the following definition. *Definition.* A vector function $z(\xi)$ is said to be decreasing exponentially at rate $-\mu$ for $\xi \geq 0$ if there exists a quadratic form

$$
V(z) = z^{\mathrm{T}} P z
\tag{24}
$$

with $P > 0$, $P^{\mathrm{T}} = P$ such that the following inequality holds:

$$
\frac{dV(z(\xi))}{d\xi} + 2\mu V(z(\xi)) \leq 0, \quad \xi \geq 0.
\tag{25}
$$

Rewrite the last equation in (19) taking the control u as (23)

$$
\begin{aligned}
z_2' &= -s_{\bar{u}}\left(\frac{\sigma + c(\xi)(1 + z_2^2)}{(c(\xi)z_1 + 1)(1 + z_2^2)^{\frac{3}{2}}} \right)(c(\xi)z_1 + 1)(1 + z_2^2)^{\frac{3}{2}} \\
&\quad + c(\xi)(1 + z_2^2) \doteq -\Phi(z, \sigma).
\end{aligned}
$$

Then

$$
\begin{aligned}
\Phi(z, \sigma) &= s_{\bar{u}(c(\xi)z_1 + 1)(1 + z_2^2)^{\frac{3}{2}}}\big(\sigma + c(\xi)(1 + z_2^2) \big) \\
&\quad - c(\xi)(1 + z_2^2)
\end{aligned}
\tag{26}
$$

and the system (19) takes the form

$$
\begin{aligned}
z_1' &= (c(\xi)z_1 + 1) z_2, \\
z_2' &= -\Phi(z, \sigma).
\end{aligned}
\tag{27}
$$

6. ESTIMATION OF THE ATTRACTION DOMAIN

We are aimed at characterizing the initial conditions z^0 having the property that along the trajectories of system (19) $z(\xi)$ decreases exponential at rate $-\mu$, where $0 < \mu \leq \lambda$. To estimate this domain of initial conditions, which will be denoted by $\Omega(\mu)$, the Lyapunov function (24) will be used in the framework of the absolute stability approach for time-varying systems, see Pyatnitskii (1970), Rapoport (1999). Let

$$
\Omega_0(P) = \{z : V(z) \leq 1\}.
\tag{28}
$$

Given positive values α_1 and α_2 the matrix P will be sought in the class of matrices satisfying the matrix inequalities

$$
P \geq \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & 0 \end{bmatrix}, \quad P \geq \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix},
\tag{29}
$$

meaning that the desired domain $\Omega_0(P)$ is inscribed in the rectangle

$$
\Pi(\alpha_1, \alpha_2) = \{z : -\alpha_1 \leq z_1 \leq \alpha_1, \ -\alpha_2 \leq z_2 \leq \alpha_2\}
\tag{30}
$$

We now characterize the values of μ, α_1, and α_2 that guarantee

$$\Omega_0(P) \subseteq \Omega(\mu), \quad \Omega_0(P) \subseteq \Pi(\alpha_1, \alpha_2) \tag{31}$$

for some matrix $P > 0$. Denote $d = (\lambda^2, 2\lambda)^{\mathrm{T}}$ and

$$\bar{c} = \max_{i=1,2,\cdots} |c_i|. \tag{32}$$

The following auxiliary assertion holds:

Lemma 1. Assume that for a matrix P satisfying (29) and a number α_1 satisfying the inequality

$$u_0 \doteq \bar{u}(1 - \bar{c}\alpha_1) - \bar{c} > 0, \tag{33}$$

the inclusion

$$z \in \Omega_0(P) \tag{34}$$

is valid. Then the following inequalities hold

$$-\sigma_0 \leq \sigma \leq \sigma_0, \tag{35}$$

$$\begin{array}{l} s_{u_0}(\sigma) \leq \Phi(z, \sigma) \leq \sigma \quad \text{for} \quad \sigma \geq 0, \\ \sigma \leq \Phi(z, \sigma) \leq s_{u_0}(\sigma) \text{ for} \quad \sigma \leq 0, \end{array} \tag{36}$$

where

$$\sigma_0 = \sqrt{d^T P^{-1} d}, \tag{37}$$

Proof. From conditions (34) and (29) it follows that

$$z_1^2 \leq \alpha_1^2, \quad z_2^2 \leq \alpha_2^2. \tag{38}$$

From (26) we have

$$\Phi(z, \sigma) = \begin{cases} -\sigma_1 & \text{for} \quad \sigma \leq -\sigma_1, \\ \sigma & \text{for} \quad -\sigma_1 < \sigma < \sigma_2, \\ \sigma_2 & \text{for} \quad \sigma \geq \sigma_2, \end{cases} \tag{39}$$

where

$$\sigma_1 = \bar{u}(1 + c(\xi)z_1)(1 + z_2^2)^{\frac{3}{2}} + c(\xi)(1 + z_2^2), \tag{40}$$

$$\sigma_2 = \bar{u}(1 + c(\xi)z_1)(1 + z_2^2)^{\frac{3}{2}} - c(\xi)(1 + z_2^2). \tag{41}$$

Keeping (38), (32), and (33) in mind, we obtain for $j = 1, 2$

$$\sigma_j \geq u_0. \tag{42}$$

Combination (39) and (42) gives (36). Next, consider the convex optimization problem $\sigma \rightarrow \max$ subject to constraints (34). Necessary and sufficient conditions for the extremum have the form

$$2\nu P z = d, \tag{43}$$

where $\nu > 0$ is the Lagrange multiplier. Premultiplying the last equation by z^{T} and accounting for the fact that the extremum in this case is attained at the boundary of the domain (34), we obtain

$$\nu = \frac{1}{2}\sigma^*, \tag{44}$$

where σ^* is the solution of the optimization problem. Multiplying (43) by $d^{\mathrm{T}}P^{-1}$, we arrive at

$$\nu\sigma^* = \frac{1}{2}d^T P^{-1} d. \tag{45}$$

Combining (44) with (45), we obtain $\sigma^* = \pm\sqrt{d^{\mathrm{T}}P^{-1}d}$, so that at the maximum, the equality $\sigma^* = \sigma_0 = \sqrt{d^{\mathrm{T}}P^{-1}d}$

holds. Now, formulating the minimization problem and using similar reasonings, we obtain that at the minimum point, the equality $\sigma^* = -\sigma_0$ holds, which yields (35). Proof of Lemma 1 is complete.

Along with the function $\Phi(z, \sigma)$ in the formulation of system (27), introduce the function

$$\phi(\xi, \sigma) = \beta(\xi)\sigma, \tag{46}$$

where $\beta(\xi)$ satisfies the conditions

$$k_0 \leq \beta(\xi) \leq 1, \quad k_0 = \min\{\frac{u_0}{\sigma_0}, 1\}. \tag{47}$$

The graph of the function $\Phi(z, \sigma)$, satisfying the conditions (36), is inscribed into a 'sector' on the plane σ–Φ for values σ satisfying conditions (35). Conditions (47) define the size of the sector. Further, introduce the function $\gamma(\xi)$ satisfying the conditions

$$(1 - \bar{c}\alpha_1) \leq \gamma(\xi) \leq (1 + \bar{c}\alpha_1). \tag{48}$$

We next expand the class of systems (27) by considering systems of the form

$$\begin{array}{l} z_1' = \gamma(\xi)z_2, \\ z_2' = -\beta(\xi)\sigma. \end{array} \tag{49}$$

We now require that the functions $\beta(\xi), \gamma(\xi)$ satisfy the existence conditions of absolutely continuous solution of system (49). If system (49) possesses property

$$V'(z) + 2\mu V(z) \leq 0 \tag{50}$$

for all functions $\beta(\xi)$ and $\gamma(\xi)$ satisfying conditions (47) and (48), then property (50) also holds along the trajectories of system (27) satisfying (34).

Consider the matrices

$$A_{\beta}\gamma = \begin{bmatrix} 0 & \gamma \\ -\beta\lambda^2 & -2\beta\lambda \end{bmatrix}. \tag{51}$$

Theorem 1. Assume that given numbers $\mu > 0$ and $\alpha_1 > 0$, $\alpha_2 > 0$ satisfying (33) there exists numbers $\mu > 0$ and $0 < \beta \leq 1$ such that the following linear matrix inequalities in the variable P are feasible:

$$PA_{\beta(1-\bar{c}\alpha_1)} + A_{\beta(1-\bar{c}\alpha_1)}^T P + 2\mu P \leq 0,$$

$$PA_{\beta(1+\bar{c}\alpha_1)} + A_{\beta(1+\bar{c}\alpha_1)}^T P + 2\mu P \leq 0,$$

$$PA_{1(1-\bar{c}\alpha_1)} + A_{1(1-\bar{c}\alpha_1)}^T P + 2\mu P \leq 0, \tag{52}$$

$$PA_{1(1+\bar{c}\alpha_1)} + A_{1(1+\bar{c}\alpha_1)}^T P + 2\mu P \leq 0,$$

$$\begin{bmatrix} P & \vdots & d \\ d^{\mathrm{T}} & \vdots & \frac{(\bar{u}(1 - \bar{c}\alpha_1) - \bar{c})^2}{\beta^2} \end{bmatrix} > 0, \tag{53}$$

$$P \geq \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & 0 \end{bmatrix}, \quad P \geq \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix}, \tag{54}$$

Then the domain $\Omega_0(P)$ is an attraction domain of system (19) under control (23); moreover, the condition (31) holds.

Proof. The stability of the zero solution of system (49) for all possible functions $\beta(\xi)$) and $\gamma(\xi)$ satisfying (47) and

(48) implies the stability of the zero solution of system (23) and hence, the stability of the zero solution of system (19) under control (23) and initial conditions satisfying (34). Moreover, the existence of the Lyapunov function of the form (24) satisfying condition (50) ensures the exponential decay of z with rate μ. In order that condition (50) be fulfilled for all functions $\beta(\xi), \gamma(\xi)$ satisfying (47) and (48), it is necessary and sufficient that the conditions

$$
\begin{aligned}
PA_{k_0(1-\bar{c}\alpha_1)} + A_{k_0(1-\bar{c}\alpha_1)}^T P + 2\mu P &\leq 0, \\
PA_{k_0(1+\bar{c}\alpha_1)} + A_{k_0(1+\bar{c}\alpha_1)}^T P + 2\mu P &\leq 0, \\
PA_{1(1-\bar{c}\alpha_1)} + A_{1(1-\bar{c}\alpha_1)}^T P + 2\mu P &\leq 0, \\
PA_{1(1+\bar{c}\alpha_1)} + A_{1(1+\bar{c}\alpha_1)}^T P + 2\mu P &\leq 0,
\end{aligned}
\tag{55}
$$

be satisfied, where k_0 is defined by the second condition in (47). For a given value of k_0, conditions (55) are fulfilled for a certain matrix $P > 0$, provided that for some $0 < \beta \leq k_0$, the linear matrix inequalities (52) in $P > 0$ are feasible. By (37) and (47), the condition $0 < \beta \leq k_0$ writes

$$
\beta^2 \leq \frac{u_0^2}{\sigma_0^2}
$$

or, equivalently,

$$
d^{\mathrm{T}} P^{-1} d \leq \frac{u_0^2}{\beta^2}.
\tag{56}
$$

Together with the condition $P > 0$ and (33), the last inequality means that matrix (53) is positive semi-definite. In combination with Lemma 1, this assertion leads to the following result. Under condition (34) and the conditions of Theorem 1 , the solution of system (27) is exponentially decaying with rate $-\mu$. Moreover, condition (34) holds along the whole trajectory of system (27). This completes the proof of Theorem 1.

Numerical methods (Boyd et al. (1994)) for testing the feasibility of LMI can be used. Under the conditions of Theorem 1, the set $\Omega_0(P)$ is invariant, since the function $V(z(\xi))$ is exponentially decreasing. It therefore follows that over the trajectories of system (19) subjected to control (23), the quantity $z_2^2 = \tan^2 \psi(\xi)$ is bounded by α_2^2. Hence,

$$
\cos^2 \psi(\xi) \geq \frac{1}{1 + \alpha_2^2}
$$

and the quantity $\cos\psi(\xi)$ does not change sign. If the condition

$$
\cos\psi(0) \geq \frac{1}{(1 + \alpha_2^2)^{\frac{1}{2}}} > 0,
$$

is satisfied at the initial time instant, then it remains valid for all $\xi \geq 0$. Hence, the condition of Assumption 3 holds.

7. EXAMPLE

Using LMI Lab Matlab the attraction domain has been constructed for $\bar{u} = 0.2$, $\bar{c} = 0.185$, $\lambda = 0.5$, $\alpha_1 = 0.9$, $\alpha_2 = 0.9$, $\beta = 0.23$.

In Fig. 4 the attraction domain is presented for $\mu = 0.01$. Also trajectories of the system (27) are presented for initial conditions from the attraction domains for the case of the target trajectory, consisting of circular arcs: $l_i = \pi/5, i = 1, ..., 10$, $c(\xi) = \bar{c}\,\mathrm{sign}(2(l_i - \xi))$. As easily seen, the ellipsoid is invariant.
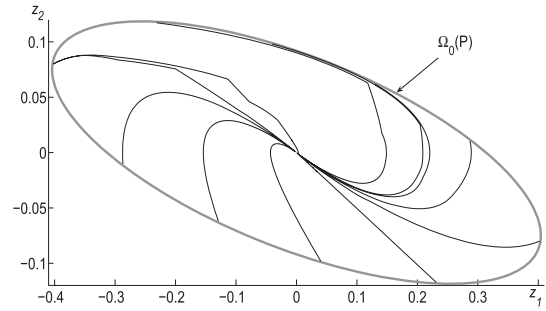

Fig. 4. Attraction domain for $\mu = 0.01$

REFERENCES

Cordesses L., Cariou C., and Berducat M. Combine Harvester Control Using Real Time Kinematic GPS. *Precision Agriculture* 2:147-161, 2000.

Thuilot B., Cariou C., Martinet P., and Berducat M. Automatic Guidance of a Farm Tractor Relying on a Single CP-DGPS. *Autonomous Robots* 15:53-61, 2002.

Rapoport L.B. et al. OCTOPUS: Multi antennae GPS/GLONASS RTK System. In *Proc. ION GPS 1999, The 12th International Technical Meeting*, pages 797–804. 1999.

Rapoport L.B. et al. Control of Wheeled Robots Using GNSS and Inertial Navigation: Control Law Synthesis and Experimental Results. In *Proc. ION GNSS 2006, The 19th International Technical Meeting*, pages 2214–2221. 2006.

Samson C. Control of Chained Systems Application to Path Following and Time-Varying Point-Stabilization of Mobile Robots. *IEEE Trans. Automat. Control*, 40:64–77, 1995.

Kolmanovsky I., McClamroch N.H. Developments in Nonholonomic Control Problems. *IEEE Control Systems Magazine*, 15:20–36, 1995.

Guldner J., Utkin V.I. Stabilization of nonholonomic mobile robots using Lyapunov functions for navigation and sliding mode control. In *Proc. 33rd IEEE Conf. Decision Control*, pages 2967–2972. 1994.

Rapoport L.B. Estimation of Attraction Domains in Wheeled Robot Control. *Automation and Remote Control* 67:1416–1435. 2006.

Pyatnitskii E.S. Absolute Stability of Time-Varying Nonlinear Systems. *Automation and Remote Control* 31: 1–9. 1970.

Rapoport L.B. Estimation of an Attraction Domain for Multivariable Lur'e Systems Using Looseless Extention of the S-Procedure. In *Proc. Amer. Control Conference*, pages 2395–2396. 1999.

Boyd S., Ghaoui L.E., Feron E. and Balakrishnan V. *Linear matrix inequalities in system and control theory.* SIAM, Philadelphia, 1994.

Filippov A.F. *Differential Equations with Discontinuous Right-Hand Sides.* Kluwer, New York, 1988.

Soetanto D., Lapierre L., Pascoal A. Adaptive, Non-Singular Path Following, Control of Dynamic Wheeled Robots. In *Proc. 42nd IEEE Conference on Decision and Control*, Vol. 2, pages 1765 – 1770. 2003.