Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

# Robust $\mathcal{H}_2$ filtering for continuous time systems with linear fractional representation ⋆

R. H. Korogui and J. C. Geromel *

* *DSCE / School of Electrical and Computer Engineering, UNICAMP, CP 6101, 13083-970, Campinas, SP, Brazil*
*email: korogui,geromel@dsce.fee.unicamp.br*

**Abstract:** This paper introduces a new approach to $\mathcal{H}_2$ robust filtering design for continuous-time LTI systems subject to linear fractional parameter uncertainty representation. The novelty consists on the determination of a performance certificate, in terms of the gap between lower and upper bounds of a minimax programming problem which defines the optimal robust equilibrium cost. The calculations are performed through convex programming methods, applying slack variables, known as multipliers, to handle the fractional dependence of the plant transfer function with respect to the parameter uncertainty. The theory is illustrated by means of a practical application involving an induction motor with uncertain leakage inductance.

Keywords: Robust estimation; linear systems; LMIs; convex optimization.

## 1. INTRODUCTION

Over the past years great attention has been devoted to the problem of robust filter design for systems subject to parameters uncertainty. The main difficulty stems from the necessity to design an unique linear filter able to cope with different models generated by a set of uncertain parameters, keeping the estimation error norm below some guaranteed level. For more details on this subject, see Jain [1975], Martin and Mintz [1983], Xie and Soh [1994], Geromel [1999], Souza and Trofino [1999], Li et al. [2002], Barbosa et al. [2005], Geromel and Regis [2006] and Scherer and Köse [2006] among others.

For systems with known parameters, the minimization of the $\mathcal{H}_2$ norm yields the celebrated Kalman filter, which is linear and has the same order of the plant, see Anderson and Moore [1979]. To deal with parameters uncertainty, the optimal filter is characterized by the equilibrium solution of a minimax optimization problem, which can be interpreted as a Man-Nature game (see Martin and Mintz [1983]), and its equilibrium solution (if any) provides the best filter for the worst parameter uncertainty. Unfortunately, in the general case, the equilibrium solution is very hard to calculate and only recently, for a particular class of polytopic parameter uncertainty, its existence has been proven in Geromel and Regis [2006]. Due to this fact, in the general case, it is not yet known the order of the optimal filter and it is not even known if it is finite; but the results of Geromel and Regis [2006] suggest that the order of the optimal filter is, in general, greater than the order of the plant.

In this paper we deal with continuous-time LTI systems in linear fractional representation with parameter uncertainty of polytopic type which enables us to take into account nonlinear dependence of the state space matrices with respect to the parameter uncertainty, a situation that often occurs in practice. We do not calculate the equilibrium solution of the already mentioned Man-Nature game. Instead, we determine lower and upper bounds to the equilibrium $\mathcal{H}_2$ cost as a way to certify the optimality gap and, by consequence, the distance from a particular filter to the optimal robust filter. The lower bound of the cost is minimized and provides a filter of order, prior of eventual poles and zeros cancellations, equal to the order of the plant times the number of vertices of the convex polytopic domain. Based on the result of this first step, we determine a robust filter with order equal to the order of the filter associated to the lower bound of the equilibrium cost. The greater order of the filter compared to the order of the plant appears to be essential to reduce conservatism, yielding more accurate results, when compared to the previous robust filter design procedures. See Geromel and Korogui [2007] for a quite complete comparison with other methods available in the literature for the case of polytopic systems. The present paper extends the recent results of Geromel and Korogui [2007] to cope with parameter uncertainty of linear fractional type, see Iwasaki and Hara [1998].

The paper is organized as follows. In the next section we state the $\mathcal{H}_2$ robust filtering problem and the model for the uncertain system to be dealt in the sequel. In Section 3 we proceed by the calculation of a lower bound to the equilibrium solution of the Man-Nature game, as well its determination by means of LMIs is discussed. Section 4 is devoted to determine the robust filter and an associated upper bound to the equilibrium cost and some implications of the results are remarked. In Section 5 we

⋆ This work was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and by Conselho Nacional de Pesquisa e Desenvolvimento (CNPq)

analyze the application of the results to the estimation of rotor flux in an induction motor with uncertain leakage inductance. Finally, Section 6 contains the conclusions and final remarks.

The notation used throughout is standard. Capital letters denote matrices and small letters denote vectors. For scalars, small Greek letters are used and $\mathbb{N} = \{1, \cdots, N\}$. For real matrices or vectors $(')$ indicates transpose. For square matrices $\mathrm{Tr}(X)$ denotes the trace function of $X$ being equal to the sum of its eigenvalues and, for the sake of easing the notation of partitioned symmetric matrices, the symbol $(\bullet)$ denotes generically each of its symmetric blocks. The operator $\mathrm{diag}(X, Y)$ generates a block diagonal matrix in whose main diagonal are the matrices $X$ and $Y$. For matrices or transfer functions $X_\lambda$ denotes the linear parameter dependence $X_\lambda := \sum_{i=1}^{N} \lambda_i X_i$, where $\lambda$ belongs to the unitary simplex

$$\Lambda = \left\{ \lambda \in \mathbb{R}^N \ : \ \sum_{i=1}^{N} \lambda_i = 1 \ , \quad \lambda_i \geq 0 \right\} \tag{1}$$

Finally, the notation

$$G(\zeta) = C(\zeta I - A)^{-1}B + D = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \tag{2}$$

is used for transfer functions of continuous time systems, where the real matrices $A$, $B$, $C$ and $D$ of compatible dimensions define a possible state space realization and $G(\omega)$ denotes $G(\zeta)$ calculated at $\zeta = j\omega$, where $\omega \in \mathbb{R}$. For any real signal $\xi$, defined in the continuous time domain, $\hat{\xi}$ denotes its Laplace transform.

## 2. PROBLEM FORMULATION

Figure 1 shows the basic filtering structure design in terms of transfer functions, where $F(\omega)$ denotes the filter to be designed and $H(\omega)$ denotes a LTI system subject to structured uncertainties characterized by the following state space representation

$$\begin{aligned} \dot{x} &= Ax + Eq + Bw \\ p &= C \begin{bmatrix} x \\ w \end{bmatrix} + Dq \\ q &= \Delta p \ \ , \quad \Delta \in \Xi \\ y &= C_y x + D_y w \\ z &= C_z x + D_z w \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state, $q \in \mathbb{R}^m$ and $p \in \mathbb{R}^r$ are internal variables of the model, $w \in \mathbb{R}^{m_w}$ is an external disturbance, $y \in \mathbb{R}^{r_y}$ is the measured output, $z \in \mathbb{R}^{r_z}$ is the output to be estimated and $\Xi$ is the set of all feasible parameters uncertainty, defined by

$$\Xi = \mathrm{co}\{\Delta_i \ : \ i \in \mathbb{N}\} \tag{4}$$

where $\mathrm{co}\{\cdot\}$ denotes the convex hull generated by $N$ known matrices $\Delta_i$ for all $i \in \mathbb{N}$. Hence, any element of the set $\Xi$ can be written in the form $\Delta_\lambda$ for some $\lambda \in \Lambda$. All matrices are supposed to be of compatible dimensions, yielding the following definition of the transfer function $H(\omega)$ as being

$$H(\lambda, \omega) = \begin{bmatrix} T(\lambda, \omega) \\ S(\lambda, \omega) \end{bmatrix} = \left[ \begin{array}{c|c} A_\Delta(\lambda) & B_\Delta(\lambda) \\ \hline C_y & D_y \\ C_z & D_z \end{array} \right] \tag{5}$$

where

$$[\, A_\Delta(\lambda) \ \ B_\Delta(\lambda) \,] = [\, A \ \ B \,] + E(I - \Delta_\lambda D)^{-1} \Delta_\lambda C \tag{6}$$

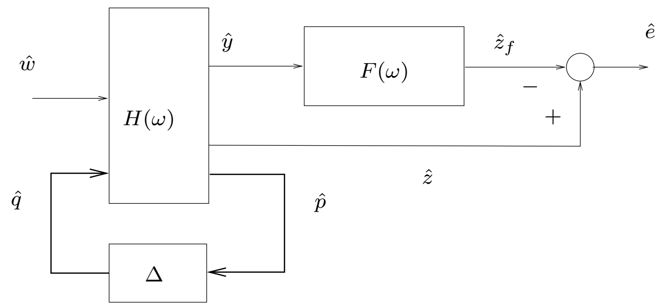


Fig. 1. Filtering Structure

This relationship makes clear the nonlinear dependence of the state space representation of the plant, with respect to $\lambda \in \Lambda$, whenever $D \neq 0$. It is assumed that $\det(I - \Delta_\lambda D) \neq 0$ for all $\lambda \in \Lambda$. Notice that this model is quite general and reduces to the structured LFT description considered in Tuan et al. [2003] from a particular choice of matrices $C$, $D$, $E$ and the structure of $\Delta \in \Xi$. For this system, the filter $F(\omega)$ has to be designed in such a way that its output is the best estimate of $\hat{z}$ that can be obtained from the data contained in $\hat{y}$. Formally, the problem is expressed as

$$\min_{F \in \mathcal{F}} \max_{\lambda \in \Lambda} J(F(\omega), H(\lambda, \omega)) \tag{7}$$

where $J(F(\omega), H(\lambda, \omega)) = \|E_F(\lambda, \omega)\|_2^2$ is the $\mathcal{H}_2$ squared norm of the transfer function from the exogenous input $\hat{w}$ to the estimation error $\hat{e}$, that is $E_F(\lambda, \omega) = S(\lambda, \omega) - F(\omega)T(\lambda, \omega)$, and the set $\mathcal{F}$ is used to impose some desired characteristics to the optimal filter as, for instance, asymptotical stability and causability.

The equilibrium solution of (7) is very difficult to calculate (see Rockafellar [1970]). The main reason is the highly nonlinear dependence of the transfer function $H(\lambda, \omega)$ with respect to $\lambda \in \Lambda$, which makes the max problem in (7) hard to solve. In numerous works, problem (7) is addressed by defining the so called guaranteed cost $J_u(F(\omega))$ satisfying $J(F(\omega), H(\lambda, \omega)) \leq J_u(F(\omega))$ for all $\lambda \in \Lambda$ and a feasible set $\mathcal{F}_u \subset \mathcal{F}$. The main motivation to this approach is that when $\mathcal{F}_u$ is constrained to contain only the full order filters of $\mathcal{F}$, the filtering design problem $\min_{F \in \mathcal{F}_u} J_u(F(\omega))$ is convex and, thus, solvable by means of any LMI solver, see Geromel [1999], Geromel et al. [1998] and Boyd et al. [1994].

In this paper we follow the same lines adopted in Geromel and Korogui [2007] and we extend those results to cope with the linear fractional representation of the plant. First we determine a lower bound to (7), by solving a problem that can be written in terms of LMIs. The optimal optimistic filter obtained in this way has order equal to the order of the plant times the number of vertices of the unitary simplex $\Lambda$ (see Geromel and Regis [2006], Geromel and Korogui [2007]), putting aside eventual poles and zeros cancellations. Afterwards, the filter associated to the lower bound defines a parametrization which enables us to determine a robust filter with a certification of the distance to the optimal robust filter provided by the equilibrium solution of problem (7).

## 3. OPTIMISTIC PERFORMANCE

In this section, our purpose is to calculate a lower bound to the equilibrium cost (7), since in the general case of uncertain polytopic systems its global solution is virtually impossible to be exactly calculated. A lower bound of (7) is determined from

$$\begin{aligned}
&\min_{F\in\mathcal{F}}\max_{\lambda\in\Lambda} J(F(\omega), H(\lambda,\omega)) \geq \\
&\geq \min_{F\in\mathcal{F}}\max_{i\in\mathbb{N}} J(F(\omega), H(e_i,\omega)) \\
&\geq \min_{F\in\mathcal{F}}\max_{\lambda\in\Lambda} \left\| \sum_{i=1}^{N} \lambda_i \left( S(e_i,\omega) - F(\omega)T(e_i,\omega)\right)\right\|_2^2 \qquad (8)
\end{aligned}$$

where $e_i$ is the $i$-th row of the identity matrix and it defines one of the $N$ vertices of the parameter polytope $\Lambda$. The first inequality follows from the fact that the set of all vertices of $\Lambda$ is a subset of $\Lambda$ and the last one comes from the convexity of the functional $\|\cdot\|_2^2$ and, consequently, the indicated maximum is attained at one vertex of the convex polytope $\Lambda$.

Using the results of Geromel and Korogui [2007], the minimax problem on the right hand side of (8) can be exactly solved. Thus, a lower bound to the equilibrium solution of (7) can be stated as

$$J_L = \min_{F\in\mathcal{F}}\max_{\lambda\in\Lambda} \|E_F(\lambda,\omega)\|_2^2 \qquad (9)$$

where the error transfer function $E_F(\lambda,\omega) = S_\lambda(\omega) - F(\omega)T_\lambda(\omega)$ depends linearly on $\lambda \in \Lambda$. Considering the filter state space realization

$$F_L(\omega) = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] \qquad (10)$$

and defining the matrices of compatible dimensions $A_E = \text{diag}(A_\Delta(e_1), \cdots, A_\Delta(e_N))$, $C_Y = [C_y, \cdots, C_y]$, $C_Z = [C_z, \cdots, C_z]$ and

$$B(\lambda) = \begin{bmatrix} \lambda_1 B_\Delta(e_1) \\ \vdots \\ \lambda_N B_\Delta(e_N) \end{bmatrix} \qquad (11)$$

the error transfer function $E_F(\lambda,\omega)$ produced by the filter (10) is given by

$$E_F(\lambda,\omega) = \left[\begin{array}{cc|c} A_E & 0 & B(\lambda) \\ B_L C_Y & A_L & B_L D_y \\ \hline C_Z - D_L C_Y & -C_L & D_z - D_L D_y \end{array}\right] \qquad (12)$$

where it is noticed that only the input matrix $B(\lambda)$ is affected by the parameter uncertainty $\lambda \in \Lambda$. A point to be emphasized is that matrix $A_E$ is of dimension $nN \times nN$, in accordance to the fact that the transfer functions $S_\lambda(\omega)$ and $T_\lambda(\omega)$ are of order $nN$ (prior to possible poles and zeros cancellations). Then the following theorem gives the solution of problem (9).

*Theorem 1.* The filtering design problem (9) is equivalent to the convex programming problem

$$J_L = \inf_{\sigma, W_i, X, L, K} \{\sigma \ : \ \text{Tr}(W_i) < \sigma \ , \ i \in \mathbb{N}\} \qquad (13)$$

where $W_i$ and $X$ are symmetric matrices and $K$, $L$ are matrices of compatible dimensions satisfying the linear equality constraint $D_z - KD_y = 0$,

$$\begin{bmatrix} W_i & \bullet \\ XB(e_i) + LD_y & X \end{bmatrix} > 0 \qquad (14)$$

for all $i \in \mathbb{N}$ and

$$\begin{bmatrix} A_E' X + XA_E + LC_Y + C_Y' L' & \bullet \\ C_Z - KC_Y & -I \end{bmatrix} < 0 \qquad (15)$$

Moreover, the optimal filter is given by

$$F_L(\omega) = \left[\begin{array}{c|c} A_E + X^{-1}LC_Y & -X^{-1}L \\ \hline C_Z - KC_Y & K \end{array}\right] \qquad (16)$$

**Proof.** The proof of this theorem, based on the result of Geromel and Regis [2006], can be found in Geromel and Korogui [2007]. For this reason it is omitted here. □

Although the obtained filter in (16) has order $nN$, it has been verified in Geromel and Korogui [2007], by means of several examples, that due to cancelations of poles and zeros its order is, in general, sensibly smaller than $nN$. However, the order of $F_L(\omega)$ remains greater than $n$, a fact that decisively contributes to improve performance.

## 4. ROBUST PERFORMANCE

In the previous section we have determined a filter $F_L(\omega)$ associated to the minimum lower bound of the filter design problem (7). However, $F_L(\omega)$ is not a robust filter, since its performance level $J_L$ can not be guaranteed for all $\lambda \in \Lambda$ or, in other words, for all $\Delta \in \Xi$. Our goal in this section is to design a robust filter $F_H(\omega) \in \mathcal{F}_H \subset \mathcal{F}$ associated to a robust performance level $J_H$ guaranteed for all $\lambda \in \Lambda$.

Adopting the same reasoning presented in Geromel and Korogui [2007], we propose to choose the set $\mathcal{F}_H$ as the set of all LTI causal filters of the form

$$F_H(\omega) = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_H & D_H \end{array}\right] \qquad (17)$$

where $A_L$ and $B_L$ are the matrices from the already determined state space realization of the optimistic filter $F_L(\omega)$, in (16), and $C_H$ and $D_H$, of compatible dimensions, are to be determined. The rationale behind this approach is that $F_L(\omega) \in \mathcal{F}_H$ for an appropriate choice of matrices $C_H$ and $D_H$. So, we can define an upper bound to problem (7) as

$$\min_{F\in\mathcal{F}}\max_{\lambda\in\Lambda} \|E_F(\lambda,\omega)\|_2^2 \leq \min_{F\in\mathcal{F}_H}\max_{\lambda\in\Lambda} \|E_F(\lambda,\omega)\|_2^2 \qquad (18)$$

where $E_F(\lambda,\omega) = S(\lambda,\omega) - F(\lambda,\omega)T(\lambda,\omega)$ is the estimation error transfer function produced by a filter $F(\omega) \in \mathcal{F}_H$. The main difficulty we have to face in order to solve the problem stated in the right hand side of (18) stems from the nonlinear dependence of transfer functions $S(\lambda,\omega)$ and $T(\lambda,\omega)$ with respect to the uncertain parameter $\lambda \in \Lambda$. Hence, from the state space realization of any feasible filter $F_H(\omega) \in \mathcal{F}_H$ and taking into account (3), the state space realization of the estimation error is

$$\begin{aligned}
\dot{\tilde{x}} &= \mathcal{A}\tilde{x} + \mathcal{E}q + \mathcal{B}w \\
p &= \mathcal{C}\begin{bmatrix} \tilde{x} \\ w \end{bmatrix} + \mathcal{D}q \\
q &= \Delta p \ , \ \Delta \in \Xi \\
e &= \mathcal{C}_e\tilde{x} + \mathcal{D}_e w
\end{aligned} \qquad (19)$$

where $\tilde{x}' = [x_F' \ x']$ is the state vector composed by the state vectors of the filter $x_F$ and the plant $x$. The indicated matrices are given by $\mathcal{C} = [\,0 \ C\,]$, $\mathcal{D} = D$,

$$\mathcal{A} = \begin{bmatrix} A_L & B_L C_y \\ 0 & A \end{bmatrix}, \ \mathcal{E} = \begin{bmatrix} 0 \\ E \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B_L D_y \\ B \end{bmatrix} \qquad (20)$$

and

$$\mathcal{C}_e' = \begin{bmatrix} -C_H' \\ C_z' - C_y' D_H' \end{bmatrix}, \ \mathcal{D}_e = D_z - D_H D_y \qquad (21)$$

From the above relations the transfer function from the external disturbance $\hat{w}$ to the estimation error $\hat{e}$ can be readily calculated as being

$$E_F(\lambda, \omega) = \left[ \begin{array}{c|c} \mathcal{A}_\Delta(\lambda) & \mathcal{B}_\Delta(\lambda) \\ \hline \mathcal{C}_e & \mathcal{D}_e \end{array} \right] \qquad (22)$$

where

$$[\, \mathcal{A}_\Delta(\lambda) \ \mathcal{B}_\Delta(\lambda) \,] = [\, \mathcal{A} \ \mathcal{B} \,] + \mathcal{E}(I - \Delta_\lambda \mathcal{D})^{-1} \Delta_\lambda \mathcal{C} \qquad (23)$$

Taking into account this state space representation for the estimation error $E_F(\lambda, \omega)$, we can state the following theorem.

*Theorem 2.* Assume the filter $F_H(\omega)$ given in (17) is such that $\mathcal{D}_e = 0$ and there exist a symmetric multiplier $\Pi$ and a positive definite matrices $\mathcal{P}$ and $W$ of appropriate dimensions satisfying the LMIs

$$[\, I \ \Delta_\lambda' \,] \, \Pi \begin{bmatrix} I \\ \Delta_\lambda \end{bmatrix} > 0 \ , \ \forall \lambda \in \Lambda \qquad (24)$$

$$\begin{bmatrix} \mathcal{A}'\mathcal{P} + \mathcal{P}\mathcal{A} & \bullet & \bullet \\ \mathcal{B}'\mathcal{P} & -I & \bullet \\ \mathcal{E}'\mathcal{P} & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{C}' & 0 \\ \mathcal{D}' & I \end{bmatrix} \Pi \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{bmatrix} < 0 \qquad (25)$$

$$\begin{bmatrix} W & \bullet \\ \mathcal{C}_e' & \mathcal{P} \end{bmatrix} > 0 \qquad (26)$$

The $\mathcal{H}_2$ squared norm of the estimation error satisfies $\|E_F(\lambda, \omega)\|_2^2 < \mathrm{Tr}\,(W)$ for all $\lambda \in \Lambda$.

**Proof.** Consider $\lambda \in \Lambda$ arbitrary but fixed. Multiplying (24) by $p$ to the right and by its transpose to the left, taking into account that $q = \Delta_\lambda p$ we obtain

$$\begin{bmatrix} p \\ q \end{bmatrix}' \Pi \begin{bmatrix} p \\ q \end{bmatrix} > 0 \qquad (27)$$

On the other hand, from (19) it is seen that

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \\ \hline q \end{bmatrix} \qquad (28)$$

which, together with (27) and (25) implies that

$$\begin{bmatrix} \tilde{x} \\ w \\ q \end{bmatrix}' \begin{bmatrix} \mathcal{A}'\mathcal{P} + \mathcal{P}\mathcal{A} & \bullet & \bullet \\ \mathcal{B}'\mathcal{P} & -I & \bullet \\ \mathcal{E}'\mathcal{P} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \\ q \end{bmatrix} < 0 \qquad (29)$$

This inequality can be further factorized using again the model (19) since the components of the vector $[\tilde{x}' \ w' \ q']$ are not indepedent. Indeed, from the relation

$$\begin{bmatrix} \tilde{x} \\ w \\ \hline q \end{bmatrix} = \begin{bmatrix} I \\ (I - \Delta_\lambda \mathcal{D})^{-1} \Delta_\lambda \mathcal{C} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ w \end{bmatrix} \qquad (30)$$

we conclude that

$$\begin{bmatrix} \mathcal{A}'\mathcal{P} + \mathcal{P}\mathcal{A} & \bullet \\ \mathcal{B}'\mathcal{P} & -I \end{bmatrix} + \begin{bmatrix} \mathcal{P}\mathcal{E} \\ 0 \end{bmatrix} (I - \Delta_\lambda \mathcal{D})^{-1} \Delta_\lambda \mathcal{C} + \\ + \mathcal{C}'\Delta_\lambda'(I - \mathcal{D}'\Delta_\lambda')^{-1} \begin{bmatrix} \mathcal{P}\mathcal{E} \\ 0 \end{bmatrix}' < 0 \qquad (31)$$

which using (23) can be written in the final form

$$\begin{bmatrix} \mathcal{A}_\Delta(\lambda)'\mathcal{P} + \mathcal{P}\mathcal{A}_\Delta(\lambda) & \bullet \\ \mathcal{B}_\Delta(\lambda)'\mathcal{P} & -I \end{bmatrix} < 0 \qquad (32)$$

As a consequence, using the fact that $\mathcal{P} > 0$, $\mathcal{D}_e = 0$ and the state space representation of the estimation error transfer function (22), we have

$$\mathcal{P}^{-1} > \int_0^\infty e^{\mathcal{A}_\Delta(\lambda) t} \mathcal{B}_\Delta(\lambda) \mathcal{B}_\Delta(\lambda)' e^{\mathcal{A}_\Delta(\lambda)' t} dt \qquad (33)$$

implying that

$$\|E_F(\lambda, \omega)\|_2^2 < \mathrm{Tr}\left(\mathcal{C}_e \mathcal{P}^{-1} \mathcal{C}_e'\right) \\ < \mathrm{Tr}\,(W) \qquad (34)$$

where the last inequality follows from the Schur Complement of inequality (26). This concludes the proof of the proposed theorem since $\lambda \in \Lambda$ is arbitrary. □

At this point some remarks are appropriate. First the constraint (24) represents a set of infinity linear matrix inequalities, one for each $\lambda \in \Lambda$. In the sequel it will be shown how to convert it into a set of $N$ LMIs, each one corresponding to a vertex of the unitary simplex $\Lambda$. Second, the result of this theorem provides a slight generalization of the previous results on multiplier theory by Iwasaki and Hara [1998] since non-independent parameter uncertainties acting on both matrices $A$ and $B$ can be handled with no additional difficulty. The third one, more important, is a consequence of the particular structure of the LMIs (24), (25) and (26) that enable us to search for a parameter dependent solution. Actually, considering the set of multipliers of the form

$$\Pi_i = \begin{bmatrix} R_i & -G \\ -G' & -Q \end{bmatrix}, \ i \in \mathbb{N} \qquad (35)$$

where the indicated matrices are of compatible dimensions and $Q > 0$, it is simply verified that the LMIs

$$[\, I \ \Delta_i' \,] \, \Pi_i \begin{bmatrix} I \\ \Delta_i \end{bmatrix} > 0, \ i \in \mathbb{N} \qquad (36)$$

assure that (24) holds. The next theorem takes these aspects under consideration in order to give a new version of Theorem 2 where the matrix variables are parameter dependent, contributing to reduce conservatism on the evaluation of the robust filter guaranteed performance.

*Theorem 3.* Assume the filter $F_H(\omega)$ given in (17) is such that $\mathcal{D}_e = 0$ and there exist symmetric multipliers $\Pi_i$ of the form (35) and positive definite matrices $\mathcal{P}_i$ and $W_i$ of appropriate dimensions satisfying the LMIs

$$[\, I \ \Delta_i' \,] \, \Pi_i \begin{bmatrix} I \\ \Delta_i \end{bmatrix} > 0 \qquad (37)$$

$$\begin{bmatrix} \mathcal{A}'\mathcal{P}_i + \mathcal{P}_i\mathcal{A} & \bullet & \bullet \\ \mathcal{B}'\mathcal{P}_i & -I & \bullet \\ \mathcal{E}'\mathcal{P}_i & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{C}' & 0 \\ \mathcal{D}' & I \end{bmatrix} \Pi_i \begin{bmatrix} \mathcal{C} & \mathcal{D} \\ 0 & I \end{bmatrix} < 0 \qquad (38)$$

$$\begin{bmatrix} W_i & \bullet \\ \mathcal{C}_e' & \mathcal{P}_i \end{bmatrix} > 0 \qquad (39)$$

for all $i \in \mathbb{N}$. The $\mathcal{H}_2$ squared norm of the estimation error satisfies $\|E_F(\lambda, \omega)\|_2^2 < \sum_{i=1}^N \lambda_i \mathrm{Tr}\,(W_i)$ for all $\lambda \in \Lambda$.

**Proof.** The proof follows from the multiplication of inequalities (37), (38) and (39) by $\lambda_i \geq 0$ and summing up the results for all $i = 1, \cdots, N$. Doing this, it is verified that the conditions of Theorem 2 are fulfilled for the parameter dependent matrix variables $\Pi_\lambda$, $\mathcal{P}_\lambda$ and $W_\lambda$. Hence, we conclude that the inequality

$$\begin{aligned} \|E_F(\lambda,\omega)\|_2^2 &< \mathrm{Tr}\left(\mathcal{C}_e \mathcal{P}_\lambda^{-1} \mathcal{C}_e\right) \\ &< \mathrm{Tr}\left(W_\lambda\right) \\ &< \sum_{i=1}^{N} \lambda_i \mathrm{Tr}\left(W_i\right) \end{aligned} \tag{40}$$

holds for all $\lambda \in \Lambda$ from which the claim follows. □

The result of Theorem 3 is important in the sense that it provides a way to determine a filter of the form (17) associated to the upper bound $J_H$ of the optimal equilibrium cost (7). Since the constraint (39) depends linearly on the filter matrices $C_H$ and $D_H$ they can be included, with no additional difficulty, in the set of variables of the robust filter design problem

$$J_H = \inf_{\sigma, W_i, \mathcal{P}_i, \Pi_i, C_H, D_H} \{\sigma \ : \ \mathrm{Tr}(W_i) < \sigma \ , \ i \in \mathbb{N}\} \tag{41}$$

where the matrix variables $W_i$, $\mathcal{P}_i$, $\Pi_i$ for all $i \in \mathbb{N}$ and $C_H$, $D_H$ satisfy the LMIs (37), (38) and (39).

In the next section the theory presented so far is applied to the design of a robust filter for an induction motor with uncertain leakage inductance.

## 5. PRACTICAL APPLICATION

Following Krishnan [2001] and Duval et al. [2006] an induction motor is governed by the following nonlinear differential equation

$$\dot{\eta} = f(\eta, \upsilon) \tag{42}$$

where the state vector $\eta = [i_{ds} \ i_{qs} \ \phi_{dr} \ \phi_{qr} \ \nu]'$ contains stator currents, the rotor flux linkages and the mechanical rotor speed, respectively. The input is defined by the vector $\upsilon = [\upsilon_{ds} \ \upsilon_{qs} \ T_l]'$ containing the stator voltages and load torque. For a constant input $\upsilon_0$ the equilibrium point $\eta_0$ is determined from $f(\eta_0, \upsilon_0) = 0$ yielding the linearized model

$$\dot{x} = A_\Delta x + B_\Delta w \tag{43}$$

where $x = \eta - \eta_0$, $w = \upsilon - \upsilon_0$ and matrices $A_\Delta$, $B_\Delta$ depend on the operation point $(\eta_0, \upsilon_0)$ and on the leakage inductance $L_f$, assumed to be an uncertain parameter with nominal value $L_{f0} = 3.7$ mH and variation $\delta L_f = \pm 15\%$ as indicated in Duval et al. [2006].

Hence, the linearized model of the induction motor can be recast as (3) where the state space nominal matrices are

$$A = \begin{bmatrix} -589.19 & 377.00 & 2749.30 & 96797.30 & -2.86 \\ -377.00 & -589.19 & -96797.30 & 2749.30 & -84.65 \\ 1.18 & 0 & -10.17 & 18.85 & 0.01 \\ 0 & 1.18 & -18.85 & -10.17 & 0.31 \\ 0.21 & 6.26 & 98.25 & -57.38 & -0.70 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 270.27 & 0 & 0 & 0 & 0 & 0 \\ 0 & 270.27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -20 & 0 & 0 & 0 \end{bmatrix}$$

Notice that the last three columns of $B$ correspond to the measurement noise, included in the external perturbation vector $w$. On the other hand, making explicit the dependence of (43) with respect to the leakage inductance $L_f$, defining $\Delta = \mathrm{diag}\left(\delta L_f, \delta L_f\right)$, the uncertainty model state space are given by matrices

Table 1. $\mathcal{H}_2$ performance - Induction motor

| Order | $J_H$ | $\max_{\lambda \in \Lambda} J(F_H(\omega), H(\lambda,\omega))$ | $J_L$ |
|---|---|---|---|
| 6 | 0.2763 | 0.2661 | 0.2660 |
| 5 | 0.2448 | 0.2448 | 0.2448 |

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ E = \begin{bmatrix} 270.27 & 0 \\ 0 & 270.27 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 2.18 & 0 & -10.17 & -358.15 & 0.01 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.18 & 358.15 & -10.17 & 0.31 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $D \neq 0$ is necessary for the construction of the nonlinear parameter dependence model. Assuming that the stator currents $i_{ds}$ and $i_{qs}$ and the mechanical rotor speed $\nu$ are measured variables for all $t \geq 0$ and that the measurement device is corrupted by some external noise with unitary intensity, we set

$$C_y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ D_y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, our goal is to estimate the currents $L_m^{-1}\phi_{dr}$ and $L_m^{-1}\phi_{qr}$ where $L_m = 0.116$ H is the magnetizing inductance supposed to be constant. This yields to the output matrices

$$C_z = \begin{bmatrix} 0 & 0 & 8.62 & 0 & 0 \\ 0 & 0 & 0 & 8.62 & 0 \end{bmatrix}, \ D_z = 0$$

Using this data and applying Theorem 1 we obtained an optimistic filter $F_L(\omega)$ of order 10 (twice the order of the plant) and a lower bound to the equilibrium solution of problem (7) given in the first row of Table 1. Applying the Matlab function `minreal` with tolerance $tol = 10^{-5}$, it has been verified that 4 pairs of poles and zeros could be cancelled with no impact on the filter performance. From this operation a 6-th order minimal realization filter has been obtained. To design a robust filter $F_H(\omega)$ parameterized by the minimal realization of $F_L(\omega)$ just calculated, we have solved problem (41). The upper bound to the equilibrium solution of problem (7) is also given in the first row of Table 1. For completeness the second row of Table 1 gives the nominal filter of 5-th order corresponding to the nominal plant with $\delta L_f = 0$. It is interesting to notice that the optimality gap of the robust filter is very small, less that 4% of the lower bound $J_L$. Hence, in practice, we can say that the robust filter we have just proposed is *almost optimal* as far as the filter design problem (7) is under consideration.

In this particular case, we have verified that the filter design problems (13) and (41) were ill-conditioned. This is due to the fact that the eigenvalues of the nominal matrix $A$ are spread in the complex plane. Indeed, they present time constants varying from 2 [msec] to 500 [msec] approximately. Particular attention must be devoted to the numerical solution of the above mentioned problems mainly because, in the present case, they involve a large number of variables.

Figure 2 shows the time simulation of the actual and the estimate of currents $L_m^{-1}\phi_{dr}$ and $L_m^{-1}\phi_{qr}$ for $\delta L_f = -15\%$

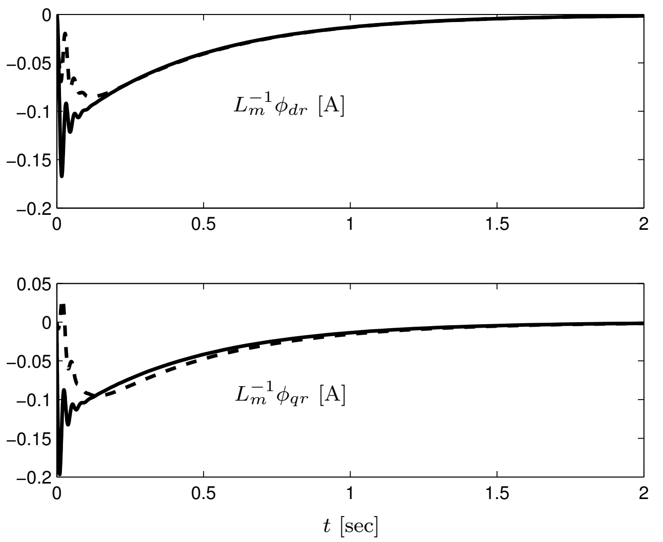


Fig. 2. Time simulation

corresponding to the worst uncertainty. In both frames the solid line shows the actual values of the currents and, in dashed lines, the estimate of the currents provided by the 6-th order robust filter $F_H(\omega)$ previously determined and considering an impulsive perturbation in the load torque. It is notice that the robust filter is effective since after 0.2 [sec] approximately the estimation error is virtually zero on both variables.

## 6. CONCLUSION

In this paper a new approach to $\mathcal{H}_2$ robust filter design for continuous-time LTI systems subject to linear fractional parameter uncertainty representation has been proposed. It is based on the determination of lower and upper bounds of the equilibrium solution of a minimax problem. A robust filter is constructed from the optimal solution of the problems defined by both bounds. The most interesting characteristic of the design method proposed is that these problems are expressed in terms of linear matrix inequalities without the limitation that the order of the filter must be equal the order of the plant. Moreover, it was possible to certify the performance of the robust filter from the estimation of the optimality gap. A practical application involving an induction motor with uncertain leakage inductance has been considered for illustration. A 6-th order robust filter has been designed for a 5-th order plant. The optimality gap was verified to be sufficient small in such a way that the robust filter can be classified as near optimal within a precision of less than 4%. The time simulation of the robust filter and the plant have shown the quality of the estimation strategy under parameter uncertainty. Some points deserve more attention in the future. First, the determination of the robustness properties of the filter $F_L(\omega)$ associated to the lower bound since this could avoid the computational effort needed to determine the filter associated to the minimum upper bound. Second, the generalization of the present results to cope with $\mathcal{H}_\infty$ norm.

## REFERENCES

B. D. O. Anderson and J. B. Moore. *Optimal Filtering*, Englewood Cliffs, NJ, Prentice Hall, 1979.

K. A. Barbosa, C. E. de Souza and A. Trofino. Robust $H_2$ filtering for uncertain linear systems: LMI based methods with parametric Lyapunov functions. *Syst. Contr. Lett.*, Vol. 54, No 3, pp. 251–262, 2005.

S. P. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.

C. Duval, G. Clerc and Y. Le Gorrec. Induction machine control using robust eigenstructure assignment. *Control Engineering Practice*, Vol. 14, pp. 29–43, 2006.

J. C. Geromel and L. A. V. Regis. $H_2$ Optimal robust filtering. *European Journal of Control*, Vol. 12, No 1, pp. 30–39, 2006.

J. C. Geromel. Optimal linear filtering under parameter uncertainty. *IEEE Trans. Signal Proces.*, Vol. 47, No. 1, pp. 168–175, 1999.

J. C. Geromel, M. C. de Oliveira and L. Hsu. LMI Characterization of Structural and Robust Stability. *Linear Algebra and Its Applications*, Vol. 285, pp. 69–80, 1998.

J. C. Geromel and R. H. Korogui. $\mathcal{H}_2$ Robust filter design with performance certificate via convex programming. *Automatica (to appear)*, 2007.

T. Iwasaki and S. Hara. Well-posedness of feedback systems: Insights into exact robustness analysis and approximate computations. *IEEE Trans. Automat. Cont.*, vol. 43, No 5, pp. 619–630, 1998.

N. B. Jain. Guaranteed error estimation in uncertain systems. *IEEE Trans. on Automat. Contr.*, Vol. 20, pp. 230–232, 1975.

R. Krishnan. *Electric Motor Devices : Modelling, Analysis and Control*, Upper Saddle River, NJ, Prentice Hall, 2001.

L. Li, Z. Q. Luo, T. N. Davidson, K. M. Wong and E. Bossé. Robust filtering via semidefinite programming with applications to target tracking. *SIAM J. on Optim.*, Vol. 12, No. 3, pp.740–755, 2002.

C. J. Martin and M. Mintz. Robust Filtering and Prediction for Linear Systems with Uncertain Dynamics : A Game - Theoretic Approach. *IEEE Trans. Automat. Contr.*, Vol. 28, No. 9, pp. 888–896, 1983.

R. Rockafellar. *Convex Analysis.* Princeton Press, 1970.

C. W. Scherer and I. E. Köse. Robust $\mathcal{H}_2$ Estimation with Dynamic IQCs: A Convex Solution. *IEEE Conf. on Decision and Control*, USA, 2006.

C. E. de Souza and A. Trofino. An LMI approach to the design of robust $H_2$ filters. *Recent Advances on Linear Matrix Inequalities Methods in Control*, Laurent El Ghaoui and Silviu-Iulian Niculescu Editors, SIAM, Englewood Cliffs, NJ, 1999.

H. D. Tuan, P. Apkarian and T. Q. Nguyen. Robust filtering for uncertain nonlinearly parametrized plants. *IEEE Trans. Signal Proces.*, Vol. 51, No. 7, pp. 1806–1815, 2003.

L. H. Xie and Y. C. Soh. Robust Kalman filtering for uncertain systems. *Syst. Contr. Lett.*, Vol. 22, pp. 123–129, 1994.