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Parameter Estimation of Two-Dimensional Linear Differential Systems via Fourier Based Modulation Function

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Abstract: In this paper, parameter identification of two-dimensional (2-D) linear differential systems via two-dimensional modulating functions is proposed. In this method, a partial differential equation on the finite time intervals converts into an algebraic equation linear in parameters. Then the parameters of the system can be estimated using the least squares algorithm. The underlying computations utilize a 2-D fast Fourier transform algorithm on polynomials of the data without the need for estimating unknown initial or/and boundary conditions at the beginning of each finite time interval. Numerical simulations are presented to confirm the theoretical results.

1. INTRODUCTION

The identification of continuous-time systems is a problem of considerable importance that has applications in various areas, such as astrophysics, economics, control, and signal processing (Garnier *et al*., 2004, Larsson *et al*., 2002, 2004, 2006). The most obvious reason for working with continuous-time models is that most physical systems are inherently continuous in time. Therefore, the parameters in the models often have a physical interpretation (Larsson *et al*., 2004).

There exist a number of alternative approaches for identification of continuous-time dynamic systems. Some methods avoid differentiation by identifying a discrete model and converting to continuous-time models using the bilinear transformation (Garnier *et al*., 2004). In these cases, sampling times play an important role. Direct continuous-time identification can be done either in time domain and frequency domain (Garnier *et al*., 2004).

Two-dimensional system identification is a difficult task. During the last three decades, although several new methods and algorithms have been proposed for one dimensional (1-D) system identification (Garnier *et al*., 2004, Larsson *et al*., 2002, 2004, 2006, Unbenhauen *et al*., 1988, 1998), but 2-D identification has not received so much attention.

In this paper, the system identification method proposed by Pearson *et al.* (Pearson *et al*., 1985) is evaluated and extended to 2-D continuous-time systems that governed by partial differential equations (PDE).

The main motivation for the development of these techniques and our study is that a large number of two-dimensional control system synthesis tasks are the most natural and the easiest to perform by using continuous-time models; therefore, it is advantageous to develop identification techniques that directly give the continuous-time representation.

In the proposed method, trigonometric functions are used as modulating functions. In this case, two-dimensional (2-D) fast Fourier transform in evaluating numerical integrations can be used. Thus, this method is provided a fast algorithm for the identification of two-dimensional continuous-time systems.

This correspondence is organized as follows: The problem formulation, basic algorithm, and computational considerations are presented in section 2. Section 3 provides numerical simulations in order to illustrate the effectiveness of the proposed method. Finally, section 4 contains conclusions.

2. TWO-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEM IDENTIFICATION

2.1 Problem Formulation

Consider a 2-D linear continuous-time system defined by partial differential equations as follows:

$$\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} a_{n_1-i_1,n_2-i_2}\, p_1^{i_1} p_2^{i_2}\, y(t_1,t_2) = \sum_{i_1=0}^{n'_1}\sum_{i_2=0}^{n'_2} b_{n'_1-i_1,n'_2-i_2}\, p_1^{i_1} p_2^{i_2}\, u(t_1,t_2) \quad ; \quad a_{0,0}=1 \tag{1}$$

where $(y(t_1,t_2), u(t_1,t_2))$ is an input-output pair of two-dimensional system. (n_1,n_2) and (n'_1,n'_2) are order of system; $n_1 \ge n'_1, n'_2$, $n_2 \ge n'_1, n'_2$. p_1, p_2 are denoted the differential operators $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$, respectively.

The objective is to estimate the unknown parameter coefficient $(a_{i_1,i_2}, b_{j_1,j_2})$ using finite time-series of input and output data ranging from $t_1 = 0$ to $t_1 = T_1$ and $t_2 = 0$ to

$t_2 = T_2$.

In this paper, the order of the linear differential equation is assumed to be known; however, in general the model order is not known and must be estimated.

2.2 Modulating Functions

$\phi(t_1,t_2)$ is a 2-D modulating function of order (n_1,n_2) relative to a fixed time interval $[0,T_1]\times[0,T_2]$ if it sufficiently smooth and possesses the property that

$$\begin{aligned} &\phi^{(i_1,i_2)}(t_1,t_2)\Big|_{t_2=0} = 0 \quad , \quad \phi^{(i_1,i_2)}(t_1,t_2)\Big|_{t_2=T_2} = 0 \\ &\phi^{(i_1,i_2)}(t_1,t_2)\Big|_{t_1=0} = 0 \quad , \quad \phi^{(i_1,i_2)}(t_1,t_2)\Big|_{t_1=T_1} = 0 \\ &i_1 = 0,1,\ldots,n_1 \;,\; i_2 = 0,1,\ldots,n_2 \;;\; (i_1,i_2) \neq (n_1,n_2) \end{aligned} \tag{2}$$

where $\phi^{(i_1,i_2)}(t_1,t_2) = \dfrac{\partial^{(i_1+i_2)}\phi}{\partial t_1^{i_1}\,\partial t_2^{i_2}}$. The multiplication or modulation of both sides of (1) with $\phi(t_1,t_2)$, integration both sides of the assumed system model equation (with unknown coefficients) over the time windows $[0,T_1]$ and $[0,T_2]$, and utilizing integration-by-parts, while noting (2), leads to the relation,

$$\begin{aligned} &\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}(-1)^{(i_1+i_2)}a_{n_1-i_1,n_2-i_2}\int_0^{T_1}\int_0^{T_2}\phi^{(i_1,i_2)}(t_1,t_2)\,y(t_1,t_2)\,dt_1\,dt_2 = \\ &\sum_{i_1=0}^{n_1'}\sum_{i_2=0}^{n_2'}(-1)^{(i_1+i_2)}b_{n_1'-i_1,n_2'-i_2}\int_0^{T_1}\int_0^{T_2}\phi^{(i_1,i_2)}(t_1,t_2)\;u(t_1,t_2)\,dt_1\,dt_2 \end{aligned} \tag{3}$$

With the conditions given in (2), the differential equation (1) has been transformed into an integral equation (3). The roles of y and ϕ within the integrals are interchanged. It is noted that the prime reasons for using such modulating functions are to avoid differentiating the data and to avoid estimating unknown initial conditions for time limited data; in the other words, modulating function methods allow for arbitrary initial conditions (Pearson *et al.*, 1985, Co *et al.*, 1990).

The simplest approach in building two-dimensional modulating functions is Kronecker product of two modulating functions in 1-D, one for the t_1 direction, one for the t_2 direction. In mathematics, the Kronecker product, denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a block matrix. So if $\phi_1^{(i_1)}(t_1)$ and $\phi_2^{(i_2)}(t_2)$ are one-dimensional modulating functions for t_1 and t_2 directions, then $\phi^{(i_1,i_2)}(t_1,t_2) = (\phi_1^{(i_1)}(t_1))\otimes(\phi_2^{(i_2)}(t_2))^T$ is a two-dimensional modulating function.

The one-dimensional modulating functions can be chosen to satisfy the following properties (Pearson *et al.*, 1985, Co *et al.*, 1990):

$$\phi_1^{(i_1)}(0) = \phi_1^{(i_1)}(T_1) = 0 \quad ; \quad i_1 = 0,1,\ldots,n_1 \tag{4a}$$

$$\phi_2^{(i_2)}(0) = \phi_2^{(i_2)}(T_2) = 0 \quad ; \quad i_2 = 0,1,\ldots,n_2 \tag{4b}$$

There are many functions to satisfy these properties. In this work, trigonometric functions are used as modulating functions, because these functions are sufficiently smooth and the use of fast Fourier transform in the evaluating numerical integrations can be allowed (Pearson *et al.*, 1985).

2.3 Parameter Identification of Two-Dimensional Linear Differential Systems via Trigonometric Functions

Consider the set of 1-D commensurable sinusoids as follows

$$\begin{aligned} f(t) = \{&1, \cos(-m_1 w_o t), \sin(-m_1 w_0 t), \cos(-m_2 w_o t), \\ &\sin(-m_2 w_0 t),\ldots,\cos(-m_M w_o t), \sin(-m_M w_0 t)\} \end{aligned} \tag{5}$$

where $w_0 = \dfrac{2\pi}{T}$ and $(m_1, m_2, \ldots, m_M)$ are selected positive integers satisfying $m_1 < m_2 < \ldots < m_M$ (Pearson *et al.*, 1985). Within the $(2M+1)$-dimensional function space spanned by the set in (5), there exist a $(2M+1-n)$-dimensional subspace of modulating functions of order n represented by the vector function $\phi(t)$ as follows (Pearson *et al.*, 1985).

$$\phi(t) = C\,f(t) \tag{6}$$

The matrix C in the above equation has rank $(2M+1-n)$ and it is determined by the solution to Vandermonde type matrix equations (Pearson *et al.*, 1985). Now, consider the $f_1(t_1)$ and $f_2(t_2)$ as follows:

$$\begin{aligned} f_1(t_1) = [&1, \cos(-m_1 w_o t_1), \sin(-m_1 w_0 t_1), \ldots, \\ &\cos(-m_M w_o t_1), \sin(-m_M w_0 t_1)]^T \\ &m_1 < m_2 < \ldots < m_M \quad , \quad w_0 = \frac{2\pi}{T_1} \end{aligned} \tag{7a}$$

$$\begin{aligned} f_2(t_2) = [&1, \cos(-m_1' w_o' t_2), \sin(-m_1' w_0' t_2), \ldots, \\ &\cos(-m_{M'}' w_o' t_2), \sin(-m_{M'}' w_0' t_2)]^T \\ &m_1' < m_2' < \ldots < m_{M'}' \quad , \quad w_0' = \frac{2\pi}{T_2} \end{aligned} \tag{7b}$$

The one-dimensional modulating functions represented by

$$\phi_1(t_1) = C_1\,f_1(t_1) \tag{8a}$$

$$\phi_2(t_2) = C_2\,f_2(t_2) \tag{8b}$$

Then, the 2-D modulating function is given by

$$\begin{aligned} \phi(t_1,t_2) &= (\phi_1(t_1))\otimes(\phi_2(t_2))^T \\ &= C_1\,f_1(t_1)\,f_2^T(t_2)\,C_2^T \end{aligned} \tag{9}$$

It can be seen from (7) and (9) that the time derivatives of $\phi_1(t_1)$, $\phi_2(t_2)$ have the representation as follow (Pearson *et al.*, 1985):

$$(-1)^{(i_1)}\,\phi_1^{(i_1)}(t_1) = C_1\, D_1^{i_1}\, f_1(t_1), \quad i_1 = 0,1,2,\ldots \tag{10a}$$

$$(-1)^{(i_2)}\,\phi_2^{(i_2)}(t_2) = C_2\, D_2^{i_2}\, f_2(t_2), \quad i_2 = 0,1,2,\ldots \tag{10b}$$

where D_1, D_2 are operational matrixes defined by the block diagonal structure (Pearson *et al.*, 1985):

$$D_1 = -w_0\, diag\left[0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & M \\ -M & 0 \end{bmatrix}\right] \tag{11a}$$

$$D_2 = -w_0'\, diag\left[0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & M' \\ -M' & 0 \end{bmatrix}\right] \tag{11b}$$

Then

$$(-1)^{(i_1+i_2)}\,\phi^{(i_1,i_2)}(t_1,t_2) = C_1\, D_1^{i_1}\, f_1(t_1)\, f_2^T(t_2)\,(D_2^{i_2})^T\, C_2^T \tag{12}$$

Using the above equation, the equation (3) can be rewritten as

$$\begin{aligned}&\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} a_{n_1-i_1,n_2-i_2}\, C_1\, D_1^{i_1}\left(\int_0^{T_1}\!\!\int_0^{T_2} f_1(t_1)\, f_2^T(t_2)\, y(t_1,t_2)\, dt_1\, dt_2\right)(D_2^{i_2})^T\, C_2^T = \\ &\sum_{i_1=0}^{n_1'}\sum_{i_2=0}^{n_2'} b_{n_1'-i_1,n_2'-i_2}\, C_1\, D_1^{i_1}\left(\int_0^{T_1}\!\!\int_0^{T_2} f_1(t_1)\, f_2^T(t_2)\; u(t_1,t_2)\, dt_1\, dt_2\right)(D_2^{i_2})^T\, C_2^T\end{aligned} \tag{13}$$

(U,Y) are defined as

$$Y = \int_0^{T_1}\!\!\int_0^{T_2} f_1(t_1)\, f_2^T(t_2)\, y(t_1,t_2)\, dt_1\, dt_2 \tag{14a}$$

$$U = \int_0^{T_1}\!\!\int_0^{T_2} f_1(t_1)\, f_2^T(t_2)\, u(t_1,t_2)\, dt_1\, dt_2 \tag{14b}$$

Then

$$\begin{aligned}&\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} a_{n_1-i_1,n_2-i_2}\; C_1\, D_1^{i_1}\; Y\,(D_2^{i_2})^T\, C_2^T = \\ &\sum_{i_1=0}^{n_1'}\sum_{i_2=0}^{n_2'} b_{n_1'-i_1,n_2'-i_2}\, C_1\, D_1^{i_1}\, U\,(D_2^{i_2})^T\, C_2^T\end{aligned} \tag{15}$$

The above equation can be converted into vector format as follows:

$$\begin{aligned}&\sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2} a_{n_1-i_1,n_2-i_2}\; vec(C_1\, D_1^{i_1}\; Y\,(D_2^{i_2})^T\, C_2^T) = \\ &\sum_{i_1=0}^{n_1'}\sum_{i_2=0}^{n_2'} b_{n_1'-i_1,n_2'-i_2}\; vec(C_1\, D_1^{i_1}\, U\,(D_2^{i_2})^T\, C_2^T)\end{aligned} \tag{16}$$

where vec is an operator that converts a matrix into a vector. The above equation can be rewritten as linear regression equation in the standard form

$$\Gamma\,\theta = vec(C_1\, D_1^{n_1}\; Y\,(D_2^{n_2})^T\, C_2^T) \tag{17}$$

where, the parameter vector θ and the partitioned matrix Γ defined by

$$\theta = [-a_{0,1}\ldots - a_{0,n_2}\ldots - a_{n_1,0}\ldots - a_{n_1,n_2}\; b_{0,0}\ldots b_{n_1',n_2'}]^T \tag{18}$$

$$\begin{aligned}\Gamma = [&vec(C_1 D_1^{n_1}\, Y (D_2^{n_2-1})^T\, C_2^T)\ldots vec(C_1 D_1^{n_1}\, Y\, C_2^T) \\ &\ldots vec(C_1 Y (D_2^{n_2})^T\, C_2^T)\ldots vec(C_1\, Y\, C_2^T) \\ &\ldots vec(C_1 D_1^{n_1'}\, U (D_2^{n_2'})^T\, C_2^T)\ldots vec(C_1\, U\, C_2^T)]\end{aligned} \tag{19}$$

Now consider the linear regression equation (17) and assume the matrix Γ has full column rank; then the one-shot least squares estimation is given by (Ljung 1999)

$$\hat{\theta} = [(\Gamma)^T\,\Gamma]^{-1}(\Gamma)^T\; vec(C_1\, D_1^{n_1}\, Y\,(D_2^{n_2})^T\, C_2^T) \tag{20}$$

Hence, it is assumed that a sufficient number of one-dimensional modulating functions have been chosen so that

$$\begin{aligned}&(2M+1-n_1)(2M'+1-n_2) \geq \\ &\qquad (n_1+1)(n_2+1)+(n_1'+1)(n_2'+1)-1\end{aligned} \tag{21}$$

2.4 Computational Considerations

In this method, the choice of (T_1, T_2, M, M') is very important. The frequencies retained in the pair (U, Y) should cover the system bandwidth while excluding higher frequency noise, it is clear that the highest frequencies in the modulating functions, $(Mw_0, M'w_0')$, should be comparable to the system bandwidth (W_c, W_c') (Pearson *et al.*, 1985). If (W_c, W_c') is approximately known, a quantitative statement of this is

$$M\, w_0 \cong 1.25\, W_c \quad , \quad M'\, w_0' \cong 1.25\, W_c' \tag{22}$$

In the case of one-shot estimation, the equation (21) implies another limitation. The value of (T_1, T_2) should be chosen sufficiently large so as to assure reasonable resolution in distinguishing system modes of the 2-D transfer function (Pearson *et al*., 1985).

The most important computational aspect of this method is the direct frequency domain interpretation afforded by the vectors (*U*, *Y*) and the efficiency with which these vectors can be computed by a FFT algorithm. In order to clarify this point, let $z(t_1,t_2)$ denote a 2-D data function on $[0,T_1]\times[0,T_2]$ and assume uniform sampling in generation the discrete samples as follows:

$$z_{i_1,i_2} = z(i_1 h_1, i_2 h_2) \;\; ; \;\; h_1 = \frac{T_1}{N_1} \;\; h_2 = \frac{T_2}{N_2} \qquad (23)$$
$$i_1 = 0,1,\dots,N_1 \;\; i_2 = 0,1,\dots,N_2$$

Then equation (7) and (14) imply determination the following integrals (complex form)

$$Z_1 = \int_0^{T_1}\int_0^{T_2} z(t_1,t_2)\, e^{-jmw_0t_1}\, e^{jm'w'_0t_2}\, dt_1\, dt_2 \qquad (24a)$$

$$Z_2 = \int_0^{T_1}\int_0^{T_2} z(t_1,t_2)\, e^{-jmw_0t_1}\, e^{-jm'w'_0t_2}\, dt_1\, dt_2 \qquad (24b)$$

where $m = 0,1,\dots,M$, $m' = 0,1,\dots,M'$. The above numerical integrations can be evaluated by using well known digital approximation. For example, the two-dimensional Simpson's rule yields (Bregains *et al*., 2004)

$$\begin{aligned}
&\int_0^{T_1}\int_0^{T_2} z(t_1,t_2)\, e^{jmw_0t_1} e^{jm'w'_0t_2}\, dt_1\, dt_2 \\
&= \frac{1}{9} h_1 h_2 [z_{0,0} + z_{0,N_2} + z_{N_1,0} + z_{N_1,N_2} + 4\sum_{i_2=1,3,\dots}^{N_2-1} z_{0,i_2} W_2^{m'i_2} \\
&\quad + 2\sum_{i_2=2,4,\dots}^{N_2-2} z_{0,i_2} W_2^{m'i_2} + 4\sum_{i_2=1,3,\dots}^{N_2-1} z_{N_1,i_2} W_2^{m'i_2} + 2\sum_{i_2=2,4,\dots}^{N_2-2} z_{N_1,i_2} W_2^{m'i_2} \\
&\quad + 4\sum_{i_1=1,3,\dots}^{N_1-1} z_{i_1,0} W_1^{mi_1} + 2\sum_{i_1=2,4,\dots}^{N_1-2} z_{i_1,0} W_1^{mi_1} + 4\sum_{i_1=1,3,\dots}^{N_1-1} z_{i_1,N_2} W_1^{mi_1} \\
&\quad + 2\sum_{i_1=2,4,\dots}^{N_1-2} z_{i_1,N_2} W_1^{mi_1} + 16\sum_{i_2=1,3,\dots}^{N_2-1}\sum_{i_1=1,3,\dots}^{N_1-1} z_{i_1,i_2} W_1^{mi_1} W_2^{m'i_2} \\
&\quad + 8\sum_{i_2=2,4,\dots}^{N_2-2}\sum_{i_1=1,3,\dots}^{N_1-1} z_{i_1,i_2} W_1^{mi_1} W_2^{m'i_2} + 8\sum_{i_2=1,3,\dots}^{N_2-1}\sum_{i_1=2,4,\dots}^{N_1-2} z_{i_1,i_2} W_1^{mi_1} W_2^{m'i_2} \\
&\quad + 4\sum_{i_2=2,4,\dots}^{N_2-2}\sum_{i_1=2,4,\dots}^{N_1-2} z_{i_1,i_2} W_1^{mi_1} W_2^{m'i_2}] + o(h_1^4) + o(h_2^4)
\end{aligned} \qquad (25)$$

where $W_1 = e^{-j\frac{2\pi}{N_1}}$, $W_2 = e^{-j\frac{2\pi}{N_2}}$ and $o(h_1^4), o(h_2^4)$ are the order of the error as functions of the sampling interval h_1 and h_2 . Simpson's rule is a Newton-Cotes formula for approximating the integral of a function using quadratic polynomials (Bregains *et al*., 2004).

Assuming N_1 and N_2 is power of 2, the usual 2-D FFT algorithm can be used to evaluate the DFT of the sum on the RHS of the above yielding the Fourier series coefficients for $m = 0,1,\dots,N_1 - 1$ and $m' = 0,1,\dots,N_2 - 1$, i.e. ,

$$\begin{aligned}
Z_1 = \frac{1}{9} h_1 h_2\, FFT[&(z_{0,0} + z_{0,N_2} + z_{N_1,0} + z_{N_1,N_2}), \\
&4z_{0,1}, 2z_{0,2}, \dots, 4z_{0,N_2-1}, \\
&4z_{N_1,1}, 2z_{N_1,2}, \dots, 4z_{N_1,N_2-1}, \\
&4z_{1,0}, 2z_{2,0}, \dots, 4z_{N_1-1,0}, \\
&4z_{1,N_2}, 2z_{2,N_2}, \dots, 4z_{N_1-1,N_2}, \\
&16z_{1,1}, 8z_{1,3}, \dots, 16z_{1,N_2-1}, \\
&8z_{2,1}, 4z_{2,2}, \dots, 8z_{2,N_2-1}, \dots, \\
&16z_{N_1-1,1}, 8z_{N_1-1,2}, \dots, 16z_{N_1-1,N_2-1}\,]
\end{aligned} \qquad (26)$$

The computational saving of this algorithm for large N_1 or/and N_2 are well known.

3. NUMERICAL SIMULATIONS

In this section, a number of simulated examples are presented to provide verification of the theoretical results. In these simulations, the unknown parameters of two-dimensional linear continuous-time systems by using trigonometric functions as modulating functions are estimated. Analysis and simulation results demonstrate the applicability of the proposed method in parameter identification of two-dimensional linear differential systems.

The normalized error criterion for the estimated parameters is defined by

$$\|\Delta\theta\| = \left[\frac{1}{K} \sum_{i=1}^{K} \left[\frac{\hat{\theta}_i - \theta_i^*}{\theta_i^*} \right]^2 \right]^{\frac{1}{2}} \times 100 \qquad (27)$$

where θ_i^* is the actual parameter value and *K* is the number of unknown parameters.

In these examples, the data length is $N_1 \times N_2$; $N_1 = 100, N_2 = 100$.

Example 1: Consider the 2-D linear continuous-time system that governed by transfer function as

$$H_1(s_1,s_2) = \frac{b_0}{s_1 s_2 + a_1 s_1 + a_2 s_2 + a_3}$$

where $a_1 = 3, a_2 = 2, a_3 = 6, b_0 = 6$.

The initial or/and boundary conditions are arbitrary and input signal is a two-dimensional white Gaussian noise. The objective of continuous-time system identification is to estimate the parameter coefficients $\theta = \{a_1, a_2, a_3; b_0\}$ using finite time-series of input data $u(t_1, t_2)$ and output data $y(t_1, t_2)$ ranging from $t_1 = 0$ to $t_1 = 2\pi$ and $t_2 = 0$ to $t_2 = 2\pi$.

In these simulations, a minimum error is reached around $M = 2$ and $M' = 3$. Note that if the system bandwidth (W_c, W_c') is approximately known, the values of *M* and *M′* can be determined by equation (22). In the case of one-shot estimation, the equation (21) implies another limitation on *M* and *M′*.

Example 2: Consider as a second example a two-dimensional system with order $n_1 = 2$, $n_2 = 2$ and $n_1' = 0$, $n_2' = 0$ defined as follows:

$$H_2(s_1, s_2) = \frac{b_0}{s_1^2 s_2^2 + a_1 s_1^2 s_2 + a_2 s_1^2 + a_3 s_1 s_2^3 + a_4 s_1 s_2 + a_5 s_1 + a_6 s_2^2 + a_7 s_2 + a_8}$$

where
$a_1 = 2, a_2 = 4, a_3 = 2, a_4 = 4, a_5 = 8, a_6 = 4, a_7 = 8, a_8 = 16$,
and $b_0 = 16$.

In this example, the initial or/and boundary conditions are arbitrary and input signal is a two-dimensional white Gaussian noise.

Results of the identification using sine-cosine functions as modulating functions are summarized in figures 1-4 and Tables 1-2. Since the system output and the model output are not distinguishable in figures 1- 4, the error signals are shown in figures 5 and 6.

Analysis and simulation results demonstrate the applicability of Fourier based modulation in parameter identification of two-dimensional linear continuous-time systems.

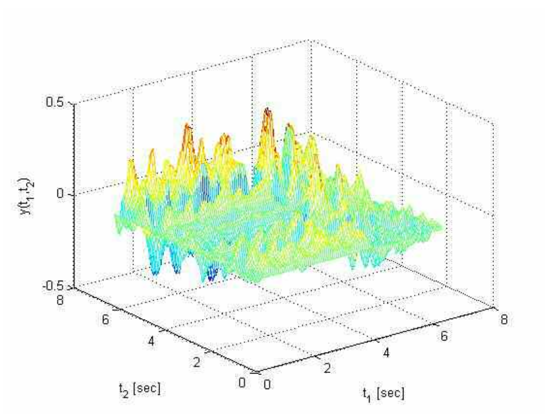

Fig. 1. Actual output in Example 1

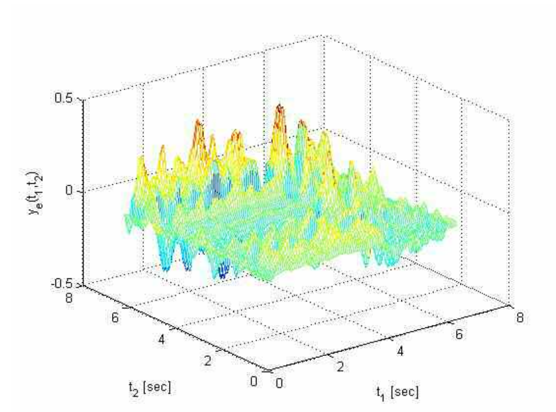

Fig. 2. Estimated output (Example 1)

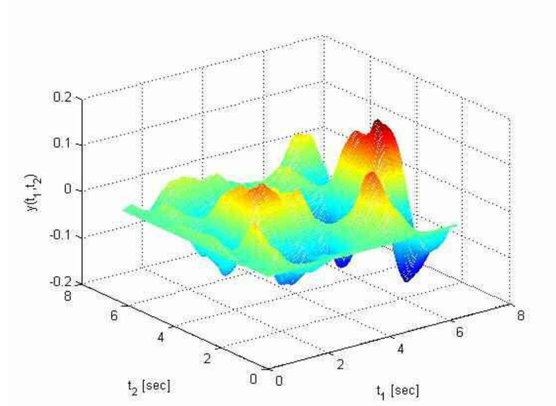

Fig. 3. Actual output in Example 2

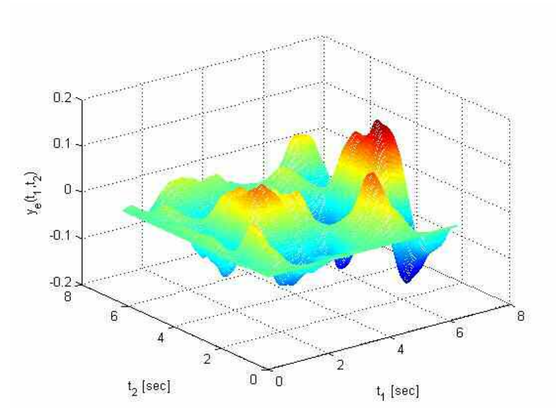

Fig. 4. Estimated output (Example 2)

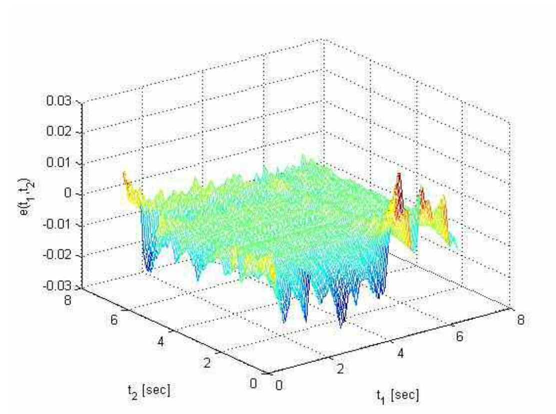

Fig. 5. The error signal of Fig. 2.

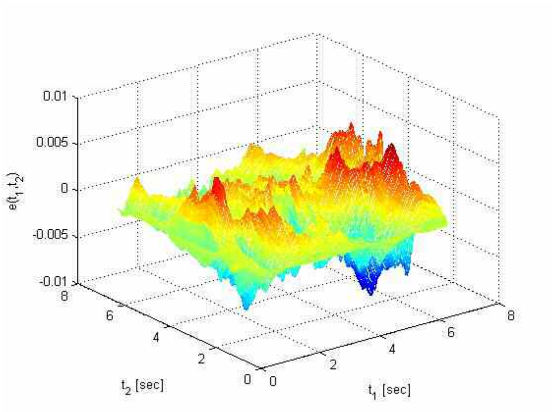

Fig. 6. The error signal of Fig. 4.

Table 1. Estimated parameters using trigonometric functions as modulating functions (Example 1)

Unknown parameters θ_i^*	Estimated parameters $\hat{\theta}_i$
a_1	2.9887
a_2	1.9840
a_3	5.9316
b_0	5.9268
$\|\Delta\theta\|$	%0.9449

Table 2. Estimated parameters using trigonometric functions as modulating functions (Example 2)

Unknown parameters θ_i^*	Estimated parameters $\hat{\theta}_i$
a_1	2.0155
a_2	4.0048
a_3	2.0040
a_4	4.0391
a_5	8.0257
a_6	4.0025
a_7	8.0669
a_8	16.0294
b_0	16.0416
$\|\Delta\theta\|$	%0.5290

4. CONCLUSIONS

In this paper, parameter identification of two-dimensional linear differential systems using Fourier based modulation function is proposed. In this method, a linear differential equation on the finite time intervals converts into an algebraic equation in the parameters. This equation can be solved using the least squares algorithm. Analysis and simulation results demonstrate the applicability of the proposed method in parameter identification of two-dimensional (2-D) linear continuous-time systems.

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