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A New Leakage Term in the Adaptive Law

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Abstract: The hyperstability theory of adaptive control systems is extended to encompass plants with unmodelled dynamics and disturbances. The analysis not only shows that leakage in the adaptive law is a natural way to avoid robustness problems (which is a known result in adaptive control theory), but also provides a new adaptive law that is a sort of signal-dependent σ-modification. The proposed adaptive law is less conservative than the σ-modification, but still ensures the global stability of the system, which is formally proven in the paper. Since it is shown that the design parameter σ' of the proposed adaptive law is directly related to the H_∞ norm of the parasitic dynamics, the criteria for system stability are derived. Based on these, some guidelines for choosing the leakage parameter σ' and the bandwidth of the reference model are presented.

Keywords: adaptive law; leakage; direct model reference adaptive control; parasitic dynamics.

1. INTRODUCTION

At the end of the 1970s much effort was put into proving the stability of adaptive algorithms. Several stability proofs were derived showing the stability of the adaptive system based on certain assumptions (knowledge of the plant order, no disturbances, noises etc.). Rohrs et al. [1985] showed that a relatively slight violation of these assumptions can result in system instability. In the 1980s there was a lot of work on robust adaptive control in order to try to minimise the negative influence of parasitics and disturbances in the system. One of the best known approaches to robust adaptive control is the use of leakage in the adaptive law. Well-known types of leakage are σ-modification, e_1-modification, switching σ-modification, etc. The general idea of the leakage is developed further by Wu et al. [1993], where the leakage parameter switches to some signal-dependent term when the estimates are far from the a-priori guess, and by Nascimento and Sayed [1999], where a so-called "circular-leaky" algorithm is introduced. Another possibility for preventing parameter bias is the use of directional leakage by Hovd and Bitmead [2006] where the leakage is applied only in the directions in parameter space in which the exciting signal is not informative.

To prove the stability of the classical adaptive systems several mathematical tools were used such as Lyapunov functions and Popov hyperstability theory. In this paper the latter will be extended to the case where there are parasitics and disturbances present in the system. The analysis will show the very well-known fact that the inclusion of leakage into the adaptive law is a natural way to stabilise the system in the presence of parasitics. This analysis will also lead to a new adaptive law with leakage, which is similar to the σ-modification but less conservative and more robust to different amplitudes of the signals in the system. A very important property of the proposed adaptive law is that the leakage parameter is not dependent on any signals present in the system.

One of the main drawbacks of adaptive control algorithms is the lack of guidelines for choosing the design parameters in such systems. Some issues on tuning the σ parameter have been discussed by Rey et al. [1989], especially those that influence the onset of bursting. Another route was taken by Kamenetsky and Widrow [2004]; they proposed an algorithm for adjusting the σ parameter on line, based on the a-posteriori errors obtained using the algorithm with and without leakage. One of the important results of the analysis carried out in this paper is the fact that some criteria on ensuring the stability of the system in the presence of parasitic dynamics will be obtained. Consequently, some guidelines on choosing the design parameters in direct model reference adaptive control (MRAC) systems will be obtained.

2. PROBLEM FORMULATION

Consider the minimum phase plant with input u and output y_p

$$y_p = \frac{B(s)}{A(s)}\left(1+\Delta(s)\right)(u+d_u) \qquad (1)$$

where $B(s)$ and $A(s)$ are the following polynomials

$$\begin{aligned} A(s) &= s^n + a_1 s^{n-1} + \ldots + a_n \\ B(s) &= b_0 s^m + b_1 s^{m-1} + \ldots + b_m \end{aligned} \qquad (2)$$

and $\Delta(s)$ is a transfer function that describes parasitic (unmodelled) dynamics in multiplicative form. Since the plant is linear and minimum phase, all bounded disturbances in the plant can be replaced by one bounded disturbance at the plant input – denoted by d_u.

The desired behaviour of the closed-loop system is defined by a reference model

$$y_m = G_m(s)w = \frac{B_m(s)}{A_m(s)}w = \frac{b_{m0}s^{m_m} + \ldots + b_{mm}}{s^{n_m} + \ldots + a_{mn}}w \qquad (3)$$

where w represents the reference.

If the plant parameters are unknown, but $\Delta(s)$ and d_u are absent in Eq. (1), the problem can be solved by direct MRAC. The following control law is used:

$$u = \psi^T \hat{\theta} \tag{4}$$

where

$$\psi^T = \left[\tfrac{\alpha^T(s)}{\Lambda(s)} u \;\; \tfrac{\alpha^T(s)}{\Lambda(s)} y_p \;\; y_p \;\; w \right] \in \mathbb{R}^{2n}$$
$$\alpha^T(s) = \begin{cases} [\, s^{n-2} \;\; s^{n-3} \;\; \dots \;\; s \;\; 1\,] & n \geq 2 \\ 0 & n < 2 \end{cases} \tag{5}$$

and $\Lambda(s)$ is the arbitrary Hurwitz polynomial that includes the numerator polynomial of the reference model as a factor. The control parameter vector $\hat{\theta}$ is obtained with the adaptive law. The parameter error $\tilde{\theta}$ is defined as

$$\tilde{\theta} = \hat{\theta} - \theta \tag{6}$$

where θ is the true control parameter of the tuned system:

$$\theta^T = \left[\, \theta_1^T \;\; \theta_2^T \;\; \theta_3 \;\; \theta_4 \,\right], \quad \theta_1, \theta_2 \in \mathbb{R}^{n-1} \tag{7}$$

If the control law of the tuned system (with adaptation switched off) is interpreted as a pole placement control law, the feedback controller can be obtained:

$$\frac{u}{-y_p} = \frac{-(\theta_2^T \alpha(s) + \theta_3 \Lambda(s))}{\Lambda(s) - \theta_1^T \alpha(s)} \tag{8}$$

The numerator and the denominator of the above feedback control are denoted by $Q(s)$ and $P(s)$, respectively:

$$\begin{aligned} Q(s) &= -(\theta_2^T \alpha(s) + \theta_3 \Lambda(s)) \\ P(s) &= \Lambda(s) - \theta_1^T \alpha(s) \end{aligned} \tag{9}$$

In the case of pole placement control, the Bezout identity is obtained, which in the case of model reference control where the plant zeros are cancelled becomes (see Isermann et al. [1992])

$$A_m \mathcal{O} = b_0 Q + P' A \tag{10}$$

where P' is the controller denominator polynomial after cancelling the plant zeros ($P = P' \frac{B}{b_0}$) and $\mathcal{O}$ is interpreted as the observer polynomial whose roots are the roots of $\Lambda(s)$ that are not included in $B_m(s)$:

$$B_m(s)\mathcal{O}(s) = b_{m0} \Lambda(s) \tag{11}$$

The problem arises if parasitic dynamics and disturbances are present in the system. The question is: What adaptive law should be used in combination with control law (4)?

3. THE ERROR MODEL OF THE ADAPTIVE SYSTEM

It is fairly easy to obtain the following equation that occurs often in classical adaptive literature (see Isermann et al. [1992]):

$$\varepsilon = y_p - y_m = \frac{b_0}{b_{m0}} G_m(s) \left[\psi^T \tilde{\theta} \right] \tag{12}$$

This equation is the error model of the system. If parasitic dynamics and disturbances are present in the plant, similar error model is obtained after some calculations:

$$\begin{aligned} \varepsilon = \overbrace{\overbrace{\tfrac{b_0}{b_{m0}} G_m}^{H_\varepsilon^0} \frac{1+\Delta}{1+\Delta \frac{b_0 Q}{A_m \mathcal{O}}}}^{H_\varepsilon} \left(\psi^T \tilde{\theta} \right) + \\ \underbrace{+ \frac{P' A \Delta}{A_m \mathcal{O} + b_0 Q \Delta} y_m + \tfrac{b_0}{b_{m0}} G_m \frac{1+\Delta}{1+\Delta \frac{b_0 Q}{A_m \mathcal{O}}} d_u}_{d_e} \end{aligned} \tag{13}$$

where $H_\varepsilon(s)$ is the transfer function $\varepsilon/(\psi^T \tilde{\theta})$ and $H_\varepsilon^0(s)$ denotes the nominal part of this transfer function (that can be seen in Eq. 12), and d_e is equivalent disturbance that combines the contributions of all external signals. More compact version of the error model:

$$\varepsilon = H_\varepsilon(s) \left(\psi^T \tilde{\theta} \right) + d_e \tag{14}$$

The error model described by Eq. (13) or (14) will be analysed. It will be particularly important if the transfer function $H_\varepsilon(s)$ is strictly positive real (SPR). The latter demand in the nominal case requires that $G_m(s)$ in Eq. (12) is SPR. That would limit the use of direct MRAC to plants with relative degree 1. Many adaptive schemes were designed in the late 1970s that overcome this problem and modify the design so that $H_\varepsilon^0(s)$ is SPR even if $G_m(s)$ is not. We assume that $H_\varepsilon^0(s)$ is indeed SPR and that the direct MRAC applied on the plant without parasitics would result in globally stable system. Our question is: How do parasitic dynamics affect the SPR property of the error model and consequently stability of the system?

The poles of all transfer functions on the right hand-side of Eq. (13) are the same – they are equal to the closed-loop poles of the tuned system, but with unmodelled dynamics taken into account. Hence, the closed-loop poles are not exactly the same as the desired ones. If there exists the controller parametrisation θ that stabilises the whole class of plants (1) where $\Delta(s)$ is bounded in some way, then all transfer functions on the right hand-side of Eq. (13) are stable.

To analyse the error model further, the "additive uncertainties" in $H_\varepsilon(s)$ will be introduced:

$$\tilde{H}_\varepsilon^a = H_\varepsilon - H_\varepsilon^0 = H_\varepsilon^0 \frac{P' A}{A_m \mathcal{O} + \Delta b_0 Q} \Delta \tag{15}$$

Our task is to show the contribution of $\tilde{H}_\varepsilon^a$ to the violation of the SPR property, i.e. we will try to estimate $\inf_\omega \Re\{\tilde{H}_\varepsilon^a(j\omega)\}$. In order to do so, the frequency characteristics of the function in Eq. (15) will be analysed next.

Frequency response of $\Delta(s)$ *at low frequencies.* It has to be noted that the meaning of parasitic dynamics in adaptive control is slightly different than in robust linear control. In the latter the parasitic dynamics cater for the differences between the plant and its nominal model in the whole frequency interval. Adaptive control itself is very convenient for controlling the plants with structured uncertainties. When there are also unstructured uncertainties in the form of $\Delta(s)$, then the nominal parameters values θ are chosen so that the perfect tracking is obtained at low frequencies while there is some tracking error at higher frequencies. Practically all relevant adaptive schemes (the direct and the indirect ones) are capable of adapting to arbitrary (but of the same sign) DC-gain of the plant:

$$\begin{aligned} \Delta(j\omega)|_{\omega=0} &= 0 \\ |\Delta(j\omega)|\big|_{\omega \ll \omega_{par}} &\ll 1 \end{aligned} \tag{16}$$

where ω_{par} is the angular frequency of the dominant pole or zero in parasitic dynamics $\Delta(s)$. It is obvious that ω_{par} is higher than the bandwidth of the nominal plant model $B(s)/A(s)$. Transfer function $\Delta(s)$ is differential according to Eq. (16).

Frequency response of $\Delta(s)$ *at high frequencies.* It is very well known that it is much easier to construct adaptive controller when the relative degree (the difference between the number of poles and zeros) is small. This is why in all practical cases of designing an adaptive controller there are more poles in $(1 + \Delta(s))$ than there are zeros, or at least their number is equal. Consequently, the frequency response $|\Delta(j\omega)| \to K < \infty$ as $\omega \to \infty$. Merging the low frequency and the high frequency characteristics, $\Delta(s)$ has similar frequency response like the high-pass filter with the cut-off frequency ω_{par}.

Frequency response of $\frac{P'A}{A_m\mathcal{O}+\Delta b_0 Q}$. This transfer function can be rewritten as follows:

$$\frac{P'A}{A_m\mathcal{O}+\Delta b_0Q} = \frac{P'A}{P'A+b_0Q+\Delta b_0Q} = = \frac{1}{1+\frac{b_0Q}{P'A}(1+\Delta)} = \frac{1}{1+\frac{Q}{P}\frac{B}{A}(1+\Delta)} = S \quad (17)$$

Eq. (17) gives (output) sensitivity function $S(s)$ of the closed-loop system with plant $\frac{B}{A}(1+\Delta)$ and controller $\frac{Q(s)}{P(s)}$ where the latter was obtained based on the nominal plant model and desired closed-loop poles equal to roots of $A_m(s)\mathcal{O}(s)$. The sensitivity function $S(s)$ approaches 1 at high frequencies. Actually this happens at the bandwidth of the closed-loop system that is near the G_m bandwidth ω_{ref}. At low frequencies the sensitivity is not 0 as model reference controller in its original form is not capable of completely eliminate the effect of constant disturbance. Actually, quick analysis shows that $S(0)$ is greater than 1 if the closed-loop system is slower compared to the open-loop plant, and lower than 1 if the controller is designed to quicken the system. The latter case is met much more often.

Frequency response of H^0_ε. It can easily be shown that this transfer function is equal to the following:

$$H^0_\varepsilon(s) = T_0(s) = 1 - S(s)|_{\Delta(s)=0} \quad (18)$$

where $T_0(s)$ is the inverse sensitivity function of the system under the assumption that the plant is without parasitics and the controller is also the nominal one $\frac{Q(s)}{P(s)}$. As such, $H^0_\varepsilon(s)$ acts as a low-pas filter with cut-off frequency near ω_{ref} while the gain at low frequencies is finite and usually lower than 1 (in cases where the bandwidth of the closed-loop system is greater than the open-loop one).

By introducing Eqs. (17) and (18) into Eq. (15) we obtain

$$\tilde{H}^a_\varepsilon(s) = T_0(s)S(s)\Delta(s) \quad (19)$$

The analysis in this subsection has shown that $T_0(s)$ (low-pass filter with cut-off frequency near ω_{ref}) significantly defines the behaviour of $\tilde{H}^a_\varepsilon(s)$ at high frequencies while $\Delta(s)$ (high-pass filter with cut-off frequency near ω_{par}) is the most important for the behaviour at low frequencies. Based on the relation between the two cut-off frequencies, two possible approximations of $|\tilde{H}^a_\varepsilon(j\omega))|$ are shown in Fig. 1. Note that the gain in the flat area of the frequency response in Fig. 1b is around 1 (0 dB).

If the poles play the dominant role in the parasitic dynamics (which is the usual case), the majority of the polar diagram of the $\tilde{H}^a_\varepsilon(j\omega)$ resides in the left hand-side half-plane. Approximate analysis of the simple frequency

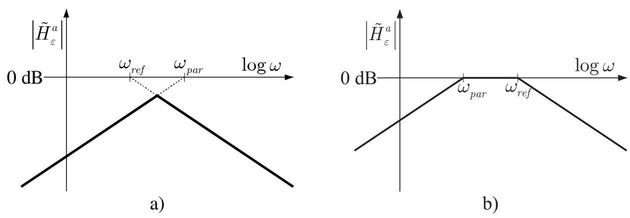

Fig. 1. Approximate frequency response of $|\tilde{H}^a_\varepsilon(j\omega))|$

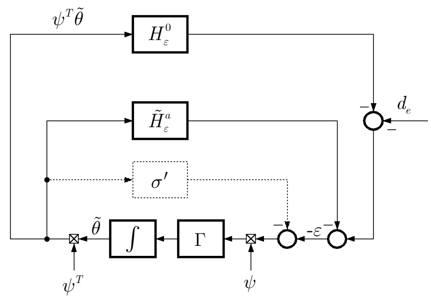

Fig. 2. The modified scheme of the adaptive system

responses (like those in Fig. 1) also shows that the shape of the polar diagram is similar to the circle in the left hand-side half-plane. We are particularly interested in the point that is the farthest to the left in the polar diagram. This usually happens around phase shift of $-180°$ which nearly coincides with the point that is the farthest from the origin of the complex plane:

$$\inf_\omega \Re\{\tilde{H}^a_\varepsilon(j\omega)\} \approx -\sup_\omega |\tilde{H}^a_\varepsilon(j\omega)| \quad (20)$$

Many proofs of adaptive system stability in the 1970s relied on the Popov hyperstability theory. Applying this theory on the adaptive system in Fig. 2 (without dotted block) leads to the following corollary: *If the operator* $H^0_\varepsilon + \tilde{H}^a_\varepsilon$ *is SPR and* $d_e(t) \in \mathcal{L}_2$*, then* $\psi^T(t)\tilde{\theta}(t) \in \mathcal{L}_2$. It was taken into account that the feedback operator in Fig. 2 (due to adaptive law) is positive real (PR).

Since H^0_ε is SPR by assumption, we are only interested in properties of $\tilde{H}^a_\varepsilon$. A new block (dotted) is added to come up with a new scheme in Fig. 2. The direct operator (H^0_ε) is SPR. The system in Fig. 2 is therefore hyperstable if the system in feedback is PR and $d_e \in \mathcal{L}_2$. The system in feedback is again a feedback connection of two systems and as such it is PR if both subsystems – $\tilde{H}^a_\varepsilon + \sigma'$ and the nonlinear subsystem due to adaptation – are PR. The latter is PR as before while the other has to satisfy:

$$\Re\{\tilde{H}^a_\varepsilon(j\omega)\} \geqslant -\sigma' \;\; \forall\omega \in \mathbb{R} \quad (21)$$

By fulfilling (21) and $d_e \in \mathcal{L}_2$, the adaptive system is hyperstable. The former demand is met by properly choosing design parameter σ' while the latter one means that only disturbances of finite energy are allowed. This is very unrealistic requirement and it will be shown in the next section how to circumvent it.

4. THE DEVELOPMENT OF THE ADAPTIVE LAW

The results of the adaptive system analysis in the previous section will be used for designing a new adaptive law. By including σ' into the scheme of the adaptive system in Fig. 2, the adaptive law has been modified as follows:

$$\dot{\hat{\theta}} = \dot{\tilde{\theta}} = \Gamma\psi\left(-\varepsilon - \sigma'\psi^T\tilde{\theta}\right) = -\Gamma\psi\varepsilon - \Gamma\sigma'\psi\psi^T\tilde{\theta} \tag{22}$$

The obvious problem of the adaptive law given by Eq. (22) is that it is not realisable as an adaptive law since it contains the unknown $\tilde{\theta} = \hat{\theta} - \theta$. But we can use any possible a-priori estimate for θ, which will be denoted θ^* (even 0 if there is no information available) to produce the following adaptive law:

$$\dot{\hat{\theta}} = -\Gamma\psi\varepsilon - \Gamma\sigma'\psi\psi^T\left(\hat{\theta} - \theta^*\right) \tag{23}$$

It can be seen that the adaptive law (23) uses leakage. The leakage parameter is $\sigma'\psi\psi^T \in \mathbb{R}^{2n}$. Such a leakage term only acts in the direction of the regressor vector ψ and is not successful in preventing parameter drift in other directions of the parameter space. This ascertainment is linked to the demand that the disturbance has to be of finite energy to ensure global stability. To overcome this problem, $\psi\psi^T$ is substituted by its trace $\psi^T\psi \in \mathbb{R}$ in the adaptive law, and the leakage will act in all dimensions of the parameter space:

$$\dot{\hat{\theta}} = -\Gamma\psi\varepsilon - \Gamma\sigma'\psi^T\psi\left(\hat{\theta} - \theta^*\right) \tag{24}$$

Theorem 1. Applying the control law given by Eq. (4) and the adaptive law given by Eq. (24) to the plant given by Eq. (1) results in a stable system (in the sense that all the signals in the system are bounded) provided that the following conditions have been met:

- the parameter of the adaptive law σ' satisfies
$$\sigma' > -\inf_{\omega\in\mathbb{R}} \Re\{\tilde{H}^a_\varepsilon(j\omega)\} \tag{25}$$
where $\tilde{H}^a_\varepsilon$ is defined in Eq. (15),
- the equivalent disturbance d_e defined in Eq. (13) is bounded,
- there exists $t_0 \geq 0$, such that for each $t \geq t_0$ the following inequality is satisfied:
$$\left|\tilde{\theta}(t)\right| \geq \frac{|d_e(t)|}{(\sigma' - \underline{\sigma}')|\psi(t)|} + \frac{\sigma'|\theta - \theta^*|}{\sigma' - \underline{\sigma}'} \tag{26}$$
where
$$\underline{\sigma}' = -\inf_{\omega\in\mathbb{R}} \Re\{\tilde{H}^a_\varepsilon(j\omega)\} < \sigma' \tag{27}$$

Proof. The error model given by Eq. (14) can be decomposed into three parts based on Eq. (15):

$$\varepsilon = \underbrace{H^0_\varepsilon\left(\psi^T\tilde{\theta}\right)}_{\varepsilon_0} + \underbrace{\tilde{H}^a_\varepsilon\left(\psi^T\tilde{\theta}\right)}_{\varepsilon_a} + d_e \tag{28}$$

As discussed in section 3, the two transfer functions $H^0_\varepsilon(s)$ and $\tilde{H}^a_\varepsilon(s)$ have zero-gain at high frequencies, and are therefore strictly proper. Their minimum state-space realisations are:

$$\begin{aligned} \dot{x}_0 &= A_0x_0 + b_0\psi^T\tilde{\theta} \\ \varepsilon_0 &= c_0^Tx_0 \end{aligned} \tag{29}$$

and

$$\begin{aligned} \dot{x}_a &= A_ax_a + b_a\psi^T\tilde{\theta} \\ \varepsilon_a &= c_a^Tx_a \end{aligned} \tag{30}$$

The systems described by Eqs. (29) and (30) are stable, as revealed in section 3. The system (29) is also SPR, while the system (30) is not in general positive real. But due to Eq. (27), the system $(\tilde{H}^a_\varepsilon + \underline{\sigma}')$ is PR and can be rewritten in the state-space form:

$$\begin{aligned} \dot{x}_a &= A_ax_a + b_a\psi^T\tilde{\theta} \\ \varepsilon_a + \underline{\sigma}'\psi^T\tilde{\theta} &= c_a^Tx_a + \underline{\sigma}'\psi^T\tilde{\theta} \end{aligned} \tag{31}$$

Applying Kalman-Yakubovich lemma on the SPR system (29) and the PR system (31) leads to the following result: For any $L_0 = L_0^T > 0$ there exist a scalar $\nu_0 > 0$, matrices $P_0 = P_0^T > 0$ and $P_a = P_a^T > 0$, and vectors q_0 and q_a such that

$$\begin{aligned} A_0^TP_0 + P_0A_0 &= -q_0q_0^T - \nu_0L_0 \\ P_0b_0 &= c_0 \end{aligned} \tag{32}$$

and

$$\begin{aligned} A_a^TP_a + P_aA_a &= -q_aq_a^T \\ P_ab_a - c_a &= \pm q_a\sqrt{2\underline{\sigma}'} \end{aligned} \tag{33}$$

The following Lyapunov function is proposed for the stability analysis:

$$V = x_0^TP_0x_0 + x_a^TP_ax_a + \tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{34}$$

Its derivative with respect to time can be calculated taking into account Eqs. (29), (31), and (24):

$$\begin{aligned} \dot{V} = {}& x_0^TA_0^TP_0x_0 + x_0^TP_0A_0x_0 + 2x_0^TP_0b_0\psi^T\tilde{\theta} + \\ &+ x_a^TA_a^TP_ax_a + x_a^TP_aA_ax_a + 2x_a^TP_ab_a\psi^T\tilde{\theta} + \\ &+ 2\tilde{\theta}^T\Gamma^{-1}\left(-\Gamma\psi\varepsilon - \Gamma\sigma'\psi^T\psi(\hat{\theta} - \theta^*)\right) \end{aligned} \tag{35}$$

The error ε can be rewritten based on Eqs. (28), (29), and (30):

$$\varepsilon = \varepsilon_0 + \varepsilon_a + d_e = c_0^Tx_0 + c_a^Tx_a + d_e \tag{36}$$

Introducing Eqs. (32), (33), (36) and (6) into Eq. (35) and cancelling the equal terms yields:

$$\begin{aligned} \dot{V} = {}& -\left(x_0^Tq_0\right)^2 - \nu_0x_0^TL_0x_0 - \left(x_a^Tq_a\right)^2 \pm 2x_a^Tq_a\sqrt{2\underline{\sigma}'}\psi^T\tilde{\theta} - \\ &- 2\psi^T\tilde{\theta}d_e - 2\sigma'\psi^T\psi\tilde{\theta}^T\tilde{\theta} - 2\sigma'\psi^T\psi\tilde{\theta}^T(\theta - \theta^*) \end{aligned} \tag{37}$$

Applying Cauchy's inequality on the sixth term on the right-hand side of Eq. (37) gives:

$$\begin{aligned} \dot{V} \leq {}& -\left(x_0^Tq_0\right)^2 - \nu_0x_0^TL_0x_0 - \\ &- \left(\left(x_a^Tq_a\right)^2 \mp 2\left(x_a^Tq_a\right)\sqrt{2\underline{\sigma}'}\left(\psi^T\tilde{\theta}\right) + 2\underline{\sigma}'\left(\psi^T\tilde{\theta}\right)^2\right) - \\ &- 2(\sigma' - \underline{\sigma}')|\psi|^2\left|\tilde{\theta}\right|^2 - 2\psi^T\tilde{\theta}d_e - 2\sigma'\psi^T\psi\tilde{\theta}^T(\theta - \theta^*) = \\ &= -\left(x_0^Tq_0\right)^2 - \nu_0x_0^TL_0x_0 - \\ &- \left(\left(x_a^Tq_a\right) \mp \sqrt{2\underline{\sigma}}\left(\psi^T\tilde{\theta}\right)\right)^2 - 2(\sigma' - \underline{\sigma}')|\psi|^2\left|\tilde{\theta}\right|^2 - \\ &- 2\psi^T\tilde{\theta}d_e - 2\sigma'\psi^T\psi\tilde{\theta}^T(\theta - \theta^*) \end{aligned} \tag{38}$$

The first three terms on the right-hand side of Eq. (38) are always negative semi-definite, the fourth term is also negative semi-definite and has to make the derivative of the Lyapunov function negative semi-definite. The only critical terms in Eq. (38) are the last two terms. Applying Cauchy's inequality to them results in:

$$\begin{aligned} \dot{V} \leq {}& -2(\sigma' - \underline{\sigma}')|\psi|^2\left|\tilde{\theta}\right|^2 + 2|\psi|\left|\tilde{\theta}\right||d_e| + 2\sigma'|\psi|^2\left|\tilde{\theta}\right|\left|\theta - \theta^*\right| = \\ &= -2(\sigma' - \underline{\sigma}')|\psi|^2\left|\tilde{\theta}\right|\left(\left|\tilde{\theta}\right| - \frac{|d_e|}{(\sigma' - \underline{\sigma}')|\psi|} - \frac{\sigma'\left|\theta - \theta^*\right|}{\sigma' - \underline{\sigma}'}\right) \end{aligned} \tag{39}$$

If there exists $t_0 \geq 0$ such that the condition given by inequality (26) is satisfied for each $t \geq t_0$, then $V(t) \leq V(t_0)$ for each $t \geq t_0$ due to Eq. (39). Consequently, it follows from Eq. (34) that the signals x_0, x_a, and $\tilde{\theta}$ are bounded. The error ε is then also bounded (see Eq. (36) and the assumption on bounded d_e). The plant output is also bounded ($y_p = y_m + \varepsilon$) and due to the minimum-phase plant, the same is true for the plant input u.

Remark 1. The proposed adaptive law (24) is similar to the adaptive law with the σ-modification:

$$\dot{\hat{\theta}} = -\Gamma\psi\varepsilon - \Gamma\sigma\left(\hat{\theta} - \theta^*\right) \tag{40}$$

The proposed approach has certain advantages. When the excitation is no longer present ($\psi = 0$), the adaptation stops completely, while in the case of adaptive law (40) only the leakage term is active – moving $\hat{\theta}$ towards θ^*, which results in "forgetting" of the current estimates.

Remark 2. It is very well known that the adaptive law with the σ-modification in the form of Eq. (40) produces very inaccurate results if the amplitudes of the signals change drastically. One of the main reasons for this is that the ratio between the nominal adaptation and the leakage also changes drastically. If the signals (this includes ψ and ε) change by a factor of β, then the first term in Eq. (40) changes by a factor of β^2, while the second term remains the same. In the case of the proposed adaptive law (24), both terms change by a factor of β^2, keeping the ratio constant. Most of the existing adaptive laws involving leakage do not share this property, e.g., in the e_1-modification the leakage term changes by a factor of β, the switching σ-modification is equivalent to the σ-modification when leakage is activated. The exception is the adaptive law, proposed by Wu et al. [1993]. These are of course just very approximate estimates, but this simple analysis shows the roots of the problem. It has been known for a long time how to overcome this problem, i.e., by using normalisation (static or dynamic). The interesting fact about the proposed adaptive law without normalisation (24) is that not only is the stability of the system achieved at different amplitudes of the signals, but also the dynamic properties do not change much. This is due to a very subtle mechanism introduced through a signal-dependent leakage term (ψ is present in the leakage).

Remark 3. If at some point in time $\psi(t)$ becomes 0, the condition (26) becomes impossible to satisfy. The problem lies in the nature of the adaptive law in the vicinity of $\psi = 0$. Note that the "classical term" in the adaptive law (24) is linearly dependent on ψ, while the "leakage term" is quadratically dependent on ψ and vanishes compared to the "classical term" when ψ is very small. So, effectively the leakage is turned off when small signals are driving the adaptation. When ψ is exactly 0, the adaptation is also switched off and no problems are encountered. To overcome the above-mentioned problems, an extra linear term is added to the leakage, which is treated in the next theorem.

Theorem 2. The next adaptive law is proposed:

$$\dot{\hat{\theta}} = -\Gamma\psi\varepsilon - \Gamma(\sigma'\psi^T\psi + \sigma_2'|\psi|)\left(\hat{\theta} - \theta^*\right) \tag{41}$$

It again results in a stable system where the last condition from Theorem 1 is replaced by:

- there exists $t_0 \geq 0$ such that for each $t \geq t_0$ the following inequality is satisfied:

$$\left|\tilde{\theta}(t)\right| \geq \max\left\{\frac{|d_e(t)|}{\sigma_2'} + |\theta - \theta^*|, \frac{\sigma'|\theta - \theta^*|}{\sigma' - \underline{\sigma}'}\right\} \tag{42}$$

Proof. The proof of the theorem follows the same lines as the proof of theorem 1 (A part of the theorem and complete proof are left out due to space limitations).

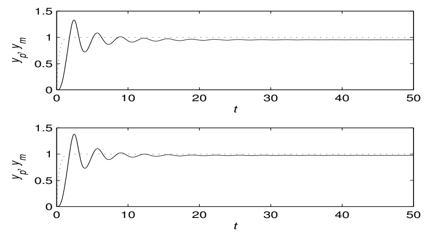

Fig. 3. The performance of the system in the easy conditions (proposed approach vs. σ-modification)

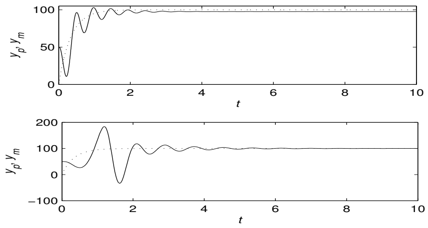

Fig. 4. The performance of the system in the difficult conditions (proposed approach vs. σ-modification)

Remark 4. The feature that makes the last condition of theorems 1 and 2 problematic is that it is practically impossible to fulfil it. The parameter error is almost never high from some time to infinity. Rather, it is high for a shorter period which is often referred to as a burst (bursting phenomena are quite common in many adaptive schemes). During that time, the derivative of Lyapunov function is negative and the system is "stabilised" during that time. Excitation is high during a burst and the parameter error decreases. Due to parasitic dynamics and disturbances parameter error may increase again and another burst may occur after a period of time. Note that all signals are still bounded during bursts and that bursting phenomenon occurs quite rarely or even not at all.

Example 1. Let us take the famous Rohrs' example with plant $G_p(s)$ – the nominal part is $G_0(s)$:

$$G_p(s) = G_0(s)\left(1 + \Delta(s)\right) = \frac{2}{s+1} \cdot \frac{229}{s^2 + 30s + 229} \tag{43}$$

The following parameters have been chosen: $\Gamma = I$, $\sigma = 0.1$, $\sigma' = 0.1$. The first experiment was done with unit step on reference signal and without disturbances. The results are shown in Figs. 3. The comparison shows that the performance of both approaches is similar. The second experiment was done in much harder conditions: $w(t) = 100$, $d_e(t) = 50$. The results of the proposed approach are shown in the upper part of Fig. 4. The same experiment was conducted on the adaptive system using σ-modification and it resulted with instability. The system was still unstable after reducing the adaptive gain for the factor 100. After reducing it for the factor 1000 the system was stabilised as shown in the lower part of Fig. 4. Even after changing the design parameters, the system with σ-modification still has much worse transient compared to the proposed approach.

5. GUIDELINES FOR CHOOSING THE DESIGN PARAMETERS IN DIRECT MRAC

The transfer function $\tilde{H}^a_\varepsilon(s)$ that was analysed in the preceding sections will play the central role in this section. Especially important is the frequency response $\tilde{H}^a_\varepsilon(j\omega)$. It should not be necessary to emphasise that the supremum of $|\tilde{H}^a_\varepsilon(j\omega)|$ should be as low as possible to approach the ideal case where $H_\varepsilon(s)$ is SPR. In view of Fig. 1 this can be achieved by fulfilling the following condition:

$$\omega_{par} \gg \omega_{ref} \tag{44}$$

that can be interpreted in the following way: Since in the real systems it is not possible to ensure that $H_\varepsilon(s)$ is SPR, it is good for the robustness to come close to this requirement. This is done by selecting a reasonably low bandwidth for the reference model – in any case lower than the dominant part in the unmodelled dynamics. This demand is in accordance with robust linear control and is intuitively clear – when one wants good performance from the system (high bandwidth), it is necessary to cope with reduced robustness.

One of the requirements of theorems 1 and 2 is that σ' has to satisfy the following inequality:

$$\sigma' > -\inf_{\omega\in\mathbb{R}} \Re\{\tilde{H}^a_\varepsilon(j\omega)\} \tag{45}$$

If the design parameter σ' satisfies the following inequality

$$\left|\tilde{H}^a_\varepsilon(j\omega)\right| < \sigma' \quad \forall\omega\in\mathbb{R} \tag{46}$$

then the inequality (45) will definitely be satisfied. The parameter σ' will not be chosen based on the estimation of $(-\inf_\omega \Re\{\tilde{H}^a_\varepsilon(j\omega)\})$, but rather based on the estimation of $\sup_\omega |\tilde{H}^a_\varepsilon(j\omega)|$. The estimation based on $\sup_\omega |\tilde{H}^a_\varepsilon(j\omega)|$ provides a higher value for σ', although the two estimates are similar. The exact values of $(-\inf_\omega \Re\{\tilde{H}^a_\varepsilon(j\omega)\})$ and $\sup_\omega |\tilde{H}^a_\varepsilon(j\omega)|$ for the case treated in example 1 are 0.2361 and 0.2369, respectively.

From inequality (46) it follows that the estimate of $\|\tilde{H}^a_\varepsilon\|_\infty$ is needed for the choice of σ'. It is obvious that the estimate of σ' according to inequality (46) is different from the one in (21), but on the other hand unmodelled dynamics are not known (even the dominant dynamics are unknown in adaptive control), and we need a very raw estimation. The next question is: How is $\|\tilde{H}^a_\varepsilon\|_\infty$ related to ω_{par} and ω_{ref}? We have three possibilities:

- $\omega_{ref} > \omega_{par}$: the frequency response of $|\tilde{H}^a_\varepsilon(\omega)|$ has the shape as shown in Fig. 1b; the leakage parameter that fulfils inequality (46) is too large for the system to have acceptable performance; for these reasons, choosing the reference model bandwidth so high is not preferable.
- $\omega_{ref} \approx \omega_{par}$: the frequency response of $|\tilde{H}^a_\varepsilon(\omega)|$ has the shape as shown in Fig. 1a with the supremum around 0 dB; σ' should be around 1 so that the inequality (46) is fulfilled.
- $\omega_{ref} < \omega_{par}$: this is the preferred choice of the reference model bandwidth that results in a system with acceptable robustness and performance.

When $\omega_{ref} < \omega_{par}$, the graphical analysis of the diagram in Fig. 1a gives:

$$\sigma' = \sqrt{\frac{\omega_{ref}}{\omega_{par}}} \tag{47}$$

Since the frequency response is rounded off, the estimate in Eq. (47) is a little conservative and results in a value for σ' that is too high. Usually, a choice of σ' between 0.1 and 1 is a good idea. When more robustness is desired, the parameter is increased, while in the case of a high-performance demand the value can be decreased especially when the condition $\omega_{par} \gg \omega_{ref}$ is fulfilled with a high probability.

6. CONCLUSIONS

An analysis of the direct MRAC with parasitics has shown that the hyperstability of the system can be ensured by adding an extra term to the adaptive law. This term can be interpreted as the leakage in the adaptive law. The algorithm is further developed so that the global stability in the presence of disturbances is proven. The proposed adaptive law has signal-dependent leakage, and is therefore more suitable for controlling systems where the amplitudes of signals change drastically. It has to be pointed out that this is achieved without using signal normalisation. It is shown that under some circumstances a version of the proposed adaptive law is similar to the σ-modification with the static normalisation of signals. For this reason the two approaches are compared in the sense of how they are able to control the famous Rohrs' plant. It turned out that the proposed approach gives better results, especially in the case of very high or very low signals in the regressor ψ.

A great problem in adaptive control design is the choice of several design parameters. Since it is shown that the design parameter σ' of the proposed adaptive law is directly related to the H_∞ norm of the parasitic dynamics, the criteria for system stability are derived. Based on these, some guidelines for choosing the leakage parameter σ' and the bandwidth of the reference model are also presented.

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