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Global asymptotic stabilization for a nonlinear system on a manifold via a dynamic compensator

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Abstract: The purpose of this paper is to solve a global asymptotic stabilization problem for a nonlinear control system on a Riemannian manifold. As well known, a system on a noncontractible manifold is not globally asymptotically stabilizable via a C^1 feedback law. This problem results from the existence of multiple singular points of such a controlled system. It is shown that if all the singular points can be assigned to a subset of the extended state space using a dynamic compensator and a C^0 feedback, then the original system becomes globally asymptotically stable. Moreover, a method for stabilization is developed using a dynamic compensator and a global control Lyapunov function for an input-affine system. Finally, we propose a method for constructing the control Lyapunov function for a controllable system.

1. INTRODUCTION

We deal with a global asymptotic stabilization problem for a nonlinear control system on a Riemannian manifold of dimension n_x. For this problem, Sontag (1998) has shown that if the state space of a control system is not contractible, the system is not C^1 globally asymptotically stabilizable. One of the reasons why this problem occurs is that the system has multiple singular points (equilibria). We introduce a modified control Lyapunov function (CLF) and a dynamic compensator defined on $\mathbb{R}^{n_p}$ to resolve the problem.

CLF methods have often been used to solve stabilization problems for nonlinear systems; see Artstein (1983) and Sontag (1989). However, there are no CLFs for a system with noncontractible state space, because any function on a noncontractible manifold has either zero or more than one critical point while the CLF must have only one critical point. For any input-affine system, the existence of a CLF is equivalent to global asymptotic stabilizability via a C^1 feedback. This implies that all affine systems on a noncontractible manifold are not C^1 globally asymptotically stabilizable.

To solve the global asymptotic stabilization problem for systems on manifolds, we have proposed a control Lyapunov-Morse function (CLMF) based on the ideas of CLFs and Lyapunov-Morse functions in Franks (1979), and have derived a discontinuous global asymptotic stabilizer using a CLMF; see Tsuzuki and Yamashita (2006). CLMFs eixst for systems on manifolds and can have multiple critical points. So additional conditions at the critical points are necessary for stabilization. This paper shows a new approach for stabilizing such systems. We define a global control Lyapunov function extended from a CLMF, and then derive additional conditions for stabilization using a dynamic compensator.

The control system and the compensator are represented by an $(n_x + n_p)$-dimensional augmented system. The n_x-dimensional system that is controlled is called the original system. The point at which the original system is to be globally asymptotically stabilized is denoted by 0_x and is called the origin of the original system. The basic idea is that if all singular points of a controlled augmented system are on $\{0_x\} \times \mathbb{R}^{n_p}$ and the augmented system has a weak Lyapunov function, then any state of the original system converges to 0_x.

In this paper we present

(1) a C^0 global asymptotic stabilization method via a dynamic compensator in section 3,
(2) a definition of a GCLF and conditions for stabilization on a GCLF using a dynamic compensator for an input-affine system in section 4, and
(3) a method for constructing a GCLF satisfying the above conditions for a controllable system in section 5.

As for other studies for the stabilization problem of nonlinear systems on manifolds, Enomoto and Shima (1998) have proposed a stabilization method for a gradient-like Morse-Smale controlled systems, and Malisoff et al. (2006) have shown that global asymptotic controllability implies s-stabilizability.

2. CONTROL SYSTEM ON A MANIFOLD

The purpose of this study is to globally asymptotically stabilize a general nonlinear control system

$$\dot{x} = \tilde{f}(x, u), \quad x \in \mathbb{X},\ u \in \mathbb{U} = \mathbb{R}^m \tag{1}$$

where x is the state, u is the input, and where $\tilde{f}$ is assumed to be smooth with respect to x and u. $\mathbb{X}$ denotes an n_x-dimensional Riemannian manifold, and $\mathbb{R}^m$ an m-dimensional Euclidean space.

We consider a continuous feedback law:

$$u = k(x). \tag{2}$$

Throughout this paper, we assume that for any initial state x_0 there exists a unique solution of the controlled system

$$\dot{x} = \tilde{f}(x, k(x)), \quad x(0) = x_0. \tag{3}$$

We define a singular point of this controlled system.

Definition 1. For the system (3) and a point $x^* \in \mathbb{X}$, if

$$\tilde{f}(x^*, k(x^*)) = 0 \tag{4}$$

holds, then the point x^* is called a singular point of (3). ♦

Sontag (1998) has shown the following theorem.

Theorem 1. If $\mathbb{X}$ is not contractible, then the system (1) is not C^1 globally asymptotically stabilizable. ♦

For example, consider the system:

$$\dot{\theta} = u, \tag{5}$$

where u is the input and θ is the state on a circle S^1. Here, S^1 is a space $[-\pi, \pi]$ such that the boundary is identified to a single point, i.e., $\pi = -\pi$. To locally stabilize the system, one can use a smooth feedback $u = -\sin\theta$. Under this smooth feedback, its origin is locally stable, but the system is not globally asymptotically stable because of the existence of an unstable singular point at $\theta = \pm\pi$. The existence of multiple singular points for any C^1 feedback law is followed from the relative Poincaré-Hopf index formula or the Morse inequality; see Pugh (1968) and Milnor (1963).

The existence of singular points except at the origin is an obstacle to the global asymptotic stabilization of nonlinear systems on manifolds. The following section develops a method for making the system globally asymptotically stable using a dynamic compensator.

3. GLOBAL ASYMPTOTIC STABILIZATION VIA A DYNAMIC COMPENSATOR

We introduce a dynamic compensator for the system (1) and consider the following augmented system:

$$\begin{aligned} \dot{x} &= \tilde{f}(x, u) \\ \dot{p} &= v, \end{aligned} \tag{6}$$

where $p \in \mathbb{R}^{n_p}$ and $v \in \mathbb{R}^{n_p}$ are the state and the input of the compensator, respectively, and where n_p is any positive integer. Let $\bar{\mathbb{X}} := \mathbb{X} \times \mathbb{R}^{n_p}$, and $\bar{\mathbb{U}} := \mathbb{U} \times \mathbb{R}^{n_p}$. We call the point $(x, p) = (0_x, 0)$ the origin of the augmented system.

For a feedback $k : \bar{\mathbb{X}} \to \bar{\mathbb{U}}$, we consider the controlled augmented system

$$\begin{aligned} \dot{x} &= \tilde{f}(x, k^u(\bar{x})) \\ \dot{p} &= k^v(\bar{x}), \end{aligned} \tag{7}$$

in which $\bar{x} := (x, p) \in \bar{\mathbb{X}}$ and $k(\bar{x}) = (k^u(\bar{x}), k^v(\bar{x}))$.

It is obvious that if $\mathbb{X}$ is not contractible, then $\bar{\mathbb{X}}$ is also not contractible. This implies that for any C^1 feedback the system (7) must have multiple singular points when $\mathbb{X}$ is not contractible. However, the degree of freedom for placing the singular points can be increased by the compensator. So, we consider changing the positions of the singular points for stabilization.

We clarify the term "a global asymptotic stability" of (1) when controlled by a feedback k.

Definition 2. The original system (1) controlled by a feedback $k : \bar{\mathbb{X}} \to \bar{\mathbb{U}}$ is globally asymptotically stable, iff for the controlled augmented system (7) no solutions have finite-escape time and moreover the following conditions hold:

- (i) **(Lyapunov stability)** The origin $(0_x, 0)$ is globally stable in the sense of Lyapunov.
- (ii) **(Attraction)** The x-part of the solution $\bar{x}(t)$ for any initial state $\bar{x}_0 \in \bar{\mathbb{X}}$ converges to 0_x, i.e.,

$$\lim_{t\to\infty} \bar{x}(t) = (0_x, p^*) \tag{8}$$

holds for a $p^* \in \mathbb{R}^{n_p}$ that is a function of $\bar{x}_0$. ♦

In this paper, we say that a function $V : \bar{\mathbb{X}} \to \mathbb{R}$ is proper if for any positive constant a, the level set

$$\bar{\mathbb{X}}^a := \left\{ \bar{x} \in \bar{\mathbb{X}} \mid V(\bar{x}) \le a \right\} \tag{9}$$

is compact, and that V is a generalized Lyapunov function for (7) if V is smooth and

$$\dot{V}(\bar{x}) < 0 \tag{10}$$

for any point $\bar{x} \in \bar{\mathbb{X}}$ except the chain recurrent set. Note that the generalized Lyapunov function can exist even for unstable systems, since it may not be positive definite. In Robinson (1999), the generalized Lyapunov function is called the (global) Lyapunov function.

Then the following theorem is obtained.

Theorem 2. We assume that for a feedback $k : \bar{\mathbb{X}} \to \bar{\mathbb{U}}$ the system (7) satisfies the following conditions:

- (i) The chain recurrent set consists of isolated singular points.
- (ii) There exists a smooth, proper, and positive definite generalized Lyapunov function $V : \bar{\mathbb{X}} \to \mathbb{R}$.
- (iii) All singular points are on the subset

$$\bar{\mathbb{X}}_{x=0} := \left\{ (x, p) \in \bar{\mathbb{X}} \mid x = 0_x \right\} = \{0_x\} \times \mathbb{R}^{n_p}. \tag{11}$$

Then the original system (1) controlled by k is globally asymptotically stable. ♦

Proof. Note that condition (ii) of theorem 2 implies the Lyapunov stability of the system.

It follows from the invariance principle that the solution starting from any point converges to a singular point. Moreover, all singular points are on $\bar{\mathbb{X}}_{x=0}$ from the assumption. Hence, (8) holds, and thus the original system controlled by k is globally asymptotically stable. □

This theorem implies that if a C^0 feedback satisfying the conditions exists, the original system is globally asymptotically stabilizable via the compensator and the feedback.

All solutions of (7) satisfying the conditions of theorem 2 converge to the set of singular points. However, there exists a solution such that it converges to a singular point after it gets close to another singular point. For example, we consider the case where the controlled system (7) has a stable singular point e^S and a saddle singular point e^U. The solution starting from a point in a neighborhood of the stable manifold of e^U moves in a neighborhood of e^U. Then the solution may get away from the neighborhood of e^U and converge to e^S. In this case, the x-part of the solution once approaches 0_x, but then gets away from

0_x; finally, it tends to 0_x again. The convergence of the solution may require a much time, and so the control may be inefficient. Therefore, we consider a jump of the state p of the compensator.

Let $B(r, \bar{x}_c)$ be a closed ball centered at $\bar{x}_c$ with radius r. For a positive constant r and the singular points $e_1, \dots, e_s$ except the origin of the controlled augmented system, we consider the following system with a jump:

$$\begin{aligned} \dot{x} &= \tilde{f}(x, k^u(\bar{x})) \\ \dot{p} &= k^v(\bar{x}) \\ p(\tau^+) &= 0, \text{ if } \bar{x}(\tau) \in \bigcup_{i=1,\dots,s} B(r, e_i). \end{aligned} \tag{12}$$

See Ichikawa and Katayama (2001) and Engell et al. (2002) for jump systems.

We establish the following theorem.

Theorem 3. We assume that a feedback $k : \bar{\mathbb{X}} \to \bar{\mathbb{U}}$ satisfies the conditions of theorem 2. Then there exist positive numbers ε and r, and a function $T : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following hold:

(A) All solutions tends to $(0_x, 0)$, i.e.,
$$\lim_{t\to\infty} \bar{x}^J(t) = (0_x, 0) \tag{13}$$
for the solution $\bar{x}^J(t)$ of the jump system (12) with any initial state.

(B) For any initial state $\bar{x}_0$, and for the above ε and r,
$$\bar{x}^J(t) \in B(\varepsilon, (0_x, 0)), \quad \forall t \geq T(\|\bar{x}_0\|). \tag{14}$$
♦

Proof. Conditions (i) and (ii) of theorem 2 implies that the origin $(0_x, 0)$ is locally asymptotically stable. We can choose an $\varepsilon > 0$ such that for any initial state $\bar{x}_0 \in B(\varepsilon, (0_x, 0))$ the solution $\bar{x}(t)$ of (7) converges to $(0_x, 0)$ as $t \to \infty$, i.e., $B(\varepsilon, (0_x, 0))$ is a compact domain of attraction of $(0_x, 0)$.

For this ε and (12), we can choose an $r > 0$ such that
$$r \leq \varepsilon \tag{15}$$
and
$$B(\varepsilon, (0_x, 0)) \bigcap \left(\bigcup_{i=1,\dots,s} B(r, e_i) \right) = \emptyset. \tag{16}$$

It follows from (15) and (16) that for a time τ at which the jump of (12) happens
$$\bar{x}^J(t) \in B(\varepsilon, (0_x, 0)), \quad \forall t > \tau. \tag{17}$$
Then any solution of (12) has at most a one-time jump. Since any solution tends to the set of singular points, (A) holds for the ε and the r.

Because the system (7) is globally stable, for any initial state $\bar{x}_0$ there exists a positive M_1 such that
$$\|\bar{x}(t)\| \leq M_1. \tag{18}$$
The subset
$$B(M_1, (0_x, 0)) \setminus \text{int} \left(B(\varepsilon, (0_x, 0)) \bigcup \left(\bigcup_{i=1,\dots,s} B(r, e_i) \right) \right) \tag{19}$$
is compact and does not contain the singular points, i.e., the vector field of (7) is not zero and the generalized Lyapunov function V has no critical points on the subset (19). These mean that there exits an $M_2 > 0$ such that
$$\dot{V}(\bar{x}) \leq -M_2 \tag{20}$$
for any $\bar{x}$ in (19). If T does not exist, $V(\bar{x}(t))$ is not positive definite, which contradicts the assumption of V. Hence, T exists. □

The solution $\bar{x}^J(t)$ may be discontinuous with respect to t, and when it is so, only the state p of the compensator is discontinuous while the original state x is continuous.

In this section we have shown the conditions for global asymptotic stabilization via a dynamic compensator and a jump system. However, finding a feedback law satisfying the conditions of theorem 2 is very difficult. Thus, we shall use a CLF in the following section.

4. GLOBAL ASYMPTOTIC STABILIZATION VIA A GLOBAL CONTROL LYAPUNOV FUNCTION

In this section we assume that the control system (1) can be represented by the input-affine system:
$$\begin{aligned} \dot{x} = \tilde{f}(x, u) &= f(x) + G(x)u \\ G(x) &= (g_1(x), g_2(x), \dots, g_m(x)), \end{aligned} \tag{21}$$
where $f, g_1, \dots, g_m$ are smooth vector fields.

4.1 Definition of a global control Lyapunov function

We extend CLFs to ones having multiple critical points to use CLF methods for systems on manifolds. For any dynamical system with a flow, the existence of a generalized Lyapunov function is guaranteed from Conley's fundamental theorem of dynamical systems; see Conley (1978) and Robinson (1999). The definition of the generalized Lyapunov function is given in the previous section.

For a function $V : \mathbb{X} \to \mathbb{R}$ we say that a point $q \in \mathbb{X}$ is a critical point when
$$\text{grad}V(q) = 0 \tag{22}$$
holds.

We define a generalized control Lyapunov function using the idea of generalized Lyapunov functions.

Definition 3. Let $V : \mathbb{X} \to \mathbb{R}$ be a positive definite, proper, and smooth function. Moreover, we assume that V satisfies
$$\inf_{u\in\mathbb{U}} \{L_f V(x) + L_G V(x)u\} < 0 \tag{23}$$
for any $x \in \mathbb{X}$ except the critical points of V. Then the function V is called a generalized control Lyapunov function (GCLF) for the control system (21) ♦

A GCLF with only one critical point at the origin is identical to Sontag's global CLF. CLMF is defined as a GCLF such that the critical points are nondegenerate. The definition of GCLF can be also applied to systems with multiple locally asymptotically stable points.

For any controlled system satisfying the conditions of theorem 2, there exists a GCLF from Conley's fundamental theorem of dynamical systems.

Given GCLF V of (21), we use Sontag's feedback $k_S(x)$ as in the CLF case.

$$k_S(x) := \begin{cases} -\dfrac{L_fV(x) + \sqrt{|L_fV(x)|^2 + |L_GV(x)|^4}}{|L_GV(x)|^2} L_G^T V(x), & \\ & L_GV(x) \neq 0 \\ 0, & L_GV(x) = 0 \end{cases} \tag{24}$$

Sontag (1989) has shown that the control k_S is analytic on $\{x \in \mathbb{X} \mid L_GV(x) \neq 0 \text{ or } L_fV(x) < 0\}$. Then k_S for GCLF V is analytic on $\mathbb{X}$ except the set of critical points.

The continuity of k_S at the critical points is assured via the small control property as in the CLF case.

Definition 4. Let q be a critical point of GCLF V. We suppose that for each $\varepsilon > 0$, there is a $\delta > 0$ such that there exists a u satisfying $|u| < \varepsilon$ and

$$L_fV(x) + L_GV(x)u < 0, \tag{25}$$

for $x \neq q$ in $B_\delta(q)$. Then we say that V has the small control property at q. ♦

Given a GCLF with the small control property at any critical point, k_S is continuous on $\bar{\mathbb{X}}$.

4.2 Global asymptotic stabilization via a compensator and a GCLF

Theorem 2 is reconstructed in terms of a GCLF.

Theorem 4. For the augmented system (6) let $V : \bar{\mathbb{X}} \to \mathbb{R}$ be a GCLF such that the critical points of V are on $\bar{\mathbb{X}}_{x=0}$ and are isolated, and the number of the critical points is finite. Moreover, we assume that at any critical point q, $f(q) = 0, G(q) \neq 0$, and V has the small control property. Then the original system (21) controlled by Sontag's feedback k_S is globally asymptotically stable. ♦

Proof. First, we show Lyapunov stability. The derivative of V with respect to time is negative semi-definite.

$$\dot{V}(x) = -\sqrt{|L_fV(x)|^2 + |L_GV(x)|^4} \leq 0 \tag{26}$$

This means that the augmented system is globally stable.

The equality of (26) holds only at the critical points. So, all solutions converge to the maximum invariant set consisting of the critical points. From the assumption, all critical points are on $\bar{\mathbb{X}}_{x=0}$. Therefore, all solutions tend to $\bar{\mathbb{X}}_{x=0}$. □

Recall that k_S is analytic on $\bar{\mathbb{X}}$ except the critical points of the GCLF, and continuous on the whole space in theorem 4. Next, we show a method for constructing a GCLF satisfying the conditions of theorem 4 for a controllable system with a one-dimensional compensator.

5. CONSTRUCTION OF A GCLF FOR A CONTROLLABLE SYSTEM

In this section we restrict (1) to the following form:

$$\begin{aligned} \dot{x} &= G(x)u \\ G(x) &= (g_1(x), \ldots, g_m(x)), \end{aligned} \tag{27}$$

where for any $x \in \mathbb{X}$ we suppose

$$\mathrm{rank} G(x) = n_x, \tag{28}$$

and the dimension of compensator is fixed to 1, that is, $n_p = 1$. Note that any proper, positive definite, and smooth function is a GCLF with the small control property for (27).

Remark . Condition (28) requires $n \leq m$. In particular, if $n = m$, $\mathbb{X}$ should be a parallelizable manifold. We call $\mathbb{X}$ a parallelizable manifold when there exist vector fields $v_1, \ldots, v_n \in T\mathbb{X}$ such that for any $x \in \mathbb{X}$ the vector fields provide a basis of $T_x\mathbb{X}$. Equivalently, the tangent bundle $T\mathbb{X}$ is a trivial bundle. ♦

Assume that $V_0 : \mathbb{X} \to \mathbb{R}$ is a GCLF for (27) with the isolated critical points $x_c^1, \ldots, x_c^s \in \mathbb{X}$. This V_0 cannot stabilize (27) with a C^0 feedback because of the existence of multiple critical points. For the augmented system (6) of dimension $n + 1$, we consider the GCLF

$$V_1(\bar{x}) := V_0(x) + p^2, \tag{29}$$

where V_1 has the critical points $q_1^i := (x_c^i, 0) \in \bar{\mathbb{X}}$ for $i = 1, \ldots, s$. We show a method for constructing a GCLF satisfying the conditions of theorem 4 by modifying the GCLF V_1.

◇ *Step1*

We construct a GCLF V_2 by a coordinate transformation of p.

Let $c_0 : \mathbb{R} \to \mathbb{X}$ denote a smooth curve on $\mathbb{X}$ such that

- c_0 passes through all critical points x_c^i, and
- $c_0(0) = 0 \in \mathbb{X}$.

The curve c_0 may have self-intersections. We regard the state p of the compensator as the parameter of c_0. The curve c_0 induces a curve and points to which we assign the critical points of V_2:

$$\begin{aligned} c_1(p) &:= (c_0(p), p) \\ q_2^i &:= (x_c^i, p_c^i), \quad i = 1, \ldots, s, \end{aligned} \tag{30}$$

where p_c^i is any point on the set $c_0^{-1}(x_c^i)$. For any smooth function $P : \mathbb{X} \to \mathbb{R}$ such that

$$P(x_c^i) = p_c^i, \tag{31}$$

we can define the diffeomorphism $\bar{T}_1 : \bar{\mathbb{X}} \to \bar{\mathbb{X}}$ by

$$\bar{T}_1(\bar{x}) := (\mathrm{id}_x, p - P(x)), \tag{32}$$

where id_x is the identity map of $\mathbb{X}$. V_1 and $\bar{T}_1$ induce

$$V_2(\bar{x}) := V_1 \circ \bar{T}_1(\bar{x}). \tag{33}$$

The critical points of V_2 are identical with q_2^i defined by (30). It is obvious that c_0 and P satisfying the above conditions exist.

◇ *Step2*

We construct a GCLF V_3 satisfying the conditions of theorem 4 from V_2.

Suppose that the following map $\bar{T}_2$ be a diffeomorphism on $\bar{\mathbb{X}}$.

$$\bar{T}_2 := (T_2, \mathrm{id}_p) : \bar{\mathbb{X}} \to \bar{\mathbb{X}} \tag{34}$$

Hence, $T_2 : \bar{\mathbb{X}} \to \mathbb{X}$ is a map such that

$$T_2^p := T_2(\circ, p) : \mathbb{X} \to \mathbb{X} \tag{35}$$

is a diffeomorphism on $\mathbb{X}$ for any $p \in \mathbb{R}$.

If for each critical point $q_2^i = (x_c^i, p_c^i)$ of V_2

$$T_2(0_x, p_c^i) = T_2^{p_c^i}(0_x) = x_c^i \tag{36}$$

, i.e., $\bar{T}_2(0_x, p_c^i) = q_2^i$. Then

$$V_3(x,p) := V_2 \circ \bar{T}_2(x,p) = V_1 \circ \bar{T}_1 \circ \bar{T}_2(x,p) \tag{37}$$

has a critical point $q_3^i = (0_x, p_c^i)$. Because both $\bar{T}_1$ and $\bar{T}_2$ are only smooth coordinate transformations, it is clear that V_3 is proper, positive definite, and smooth, and that the number of isolated critical points of V_3 is equivalent to that of V_1. Then V_3 is a GCLF satisfying the conditions of theorem 4.

Lemma 5. There exists a T_2 satisfying (36) for (27). ♦

Proof. We can choose a curve c_0 without self-intersections. Then there exists a flow $\varphi : \mathbb{X} \times \mathbb{R} \to \mathbb{X}$ such that

$$\varphi(0_x, t) = c_0(t) \tag{38}$$

holds. At any p_c^i, $\varphi(0_x, p_c^i) = c_0(p_c^i) = x_c^i$.

If $T_2(x,t) := \varphi(x,t)$, then

$$T_2(0_x, p_c^i) = \varphi(0_x, p^i) = q_2^i. \tag{39}$$

Thus, the condition (36) holds. □

For (27), Sontag's stabilizer k_S is smooth on $\bar{\mathbb{X}}$, since $k_S(x) = -L_G V(x)$ for GCLF V.

In this section we have shown that GCLF V_3 of (6) for (27) satisfying the conditions of theorem 4 can be induced from any GCLF V_1 of (27). Therefore, the original system (27) controlled by Sontag's feedback k_S derived from V_3 is globally asymptotically stable.

6. EXAMPLES

6.1 A simple case

We recall the system (5) as a simple example of (27).

$$\dot{\theta} = u, \quad x \in S^1, \quad u \in \mathbb{R} \tag{40}$$

This system is not C^0-stabilizable. We shall design a global asymptotic C^∞ stabilizer to $\theta = 0$ by using a compensator and a GCLF.

The function

$$V_0(\theta) := 1 - \cos\theta \tag{41}$$

is a GCLF of (40) with the two critical points $\theta_c^1 = 0, \theta_c^2 = \pi$. We consider the following augmented system of (5):

$$\dot{\theta} = u \tag{42}$$

$$\dot{p} = v, \quad p, v \in \mathbb{R}. \tag{43}$$

In this case, the diffeomorphism

$$P(\theta) := 1 - \cos\theta \tag{44}$$

derives

$$V_2(\theta, p) := 1 - \cos\theta + (p - (1 - \cos\theta))^2. \tag{45}$$

The critical points of V_2 are $q_2^1 = (0,0)$ and $q_2^2 = (\pi, 2)$.

For the smooth map $T_2(\theta, p) := \theta - \frac{\pi}{2}p$, (36) holds, i.e.,

$$\bar{T}_2(0,0) = q_2^1, \ \bar{T}_2(0,2) = q_2^2. \tag{46}$$

Then the function

$$V_3(\theta, p) := 1 - \cos\left(\theta - \frac{\pi}{2}p\right) + \left\{p - \left(1 - \cos\left(\theta - \frac{\pi}{2}p\right)\right)\right\}^2 \tag{47}$$

is a GCLF satisfying the conditions of theorem 4. The critical points of V_3 are $q_3^1 = (0,0)$ and $q_3^2 = (0,2)$.

Figure 1 shows the contour plot of V_3 and trajectories of the controlled augmented system via k_S for V_3. We can see that all trajectories converge to q_3^1 or q_3^2.

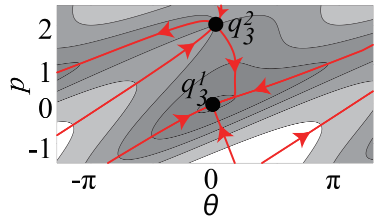

Fig. 1. Contour plot of V_3 and trajectories.

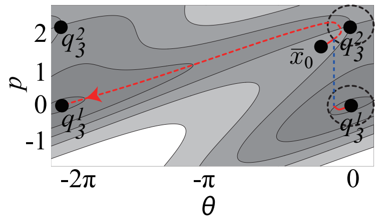

Fig. 2. Trajectories starting from $\bar{x}_0$.

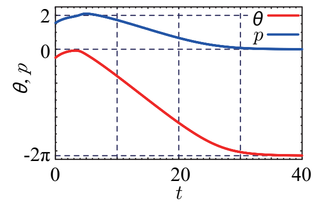

Fig. 3. Time responses of x_1 and p with no jump.

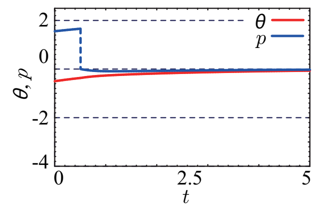

Fig. 4. Time responses of x_1 and p with jump.

In Fig.2, the dashed curve means the solution starting from $\bar{x}_0$. The solution converges to $q_3^1 = (0,0) = (-2\pi, 0)$ after it moves in a neighborhood of $q_3^2 = (0,2)$. Also, Fig.3 shows the time responses. The solution requires a much time for convergence to q_3^1. So, we use the jump system (12) for $\varepsilon = r = 0.5$.

The solution starting from $\bar{x}_0$ has jump $p(\tau^+) = 0$ when it enters $B(0.5, q_3^2)$. In Fig.2, the solid curve starting from $\bar{x}_0$ denotes the solution of the jump system, and Fig.4 illustrates the time responses. It follows from these figures that the solution converges to the origin quickly.

6.2 A three-dimensional case

We consider an attitude control of a satellite expressed as

$$\dot{R} = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} R, \tag{48}$$

where the attitude $R \in \mathrm{SO}(3)$ is the state, and the velocity $\omega = (\omega_1, \omega_2, \omega_3)$ is the input. SO(3) denotes the three-dimensional special orthogonal group, and R is a three-dimensional orthogonal matrix. This is an example of the controllable system (27). The control objective is to globally asymptotically stabilize the system to $R = I$, where I is the identity matrix.

By using the quaternion $x := (x_1, x_2, x_3, x_4)$ such that $|x| = 1$, (48) is transformed into

$$\dot{x} = \Sigma(x)\omega := \frac{1}{2}\begin{pmatrix} x_4 & -x_3 & x_2 \\ x_3 & x_4 & -x_1 \\ -x_2 & x_1 & x_4 \\ -x_1 & -x_2 & -x_3 \end{pmatrix}\omega, \tag{49}$$

where

$$R = R(x) = \begin{pmatrix} x_1^2-x_2^2-x_3^2+x_4^2 & 2(x_1x_2+x_3x_4) & 2(x_1x_3-x_2x_4) \\ 2(x_1x_2-x_3x_4) & -x_1^2+x_2^2-x_3^2+x_4^2 & 2(x_2x_3+x_1x_4) \\ 2(x_1x_3+x_2x_4) & 2(x_2x_3-x_1x_4) & -x_1^2-x_2^2+x_3^2+x_4^2 \end{pmatrix}. \tag{50}$$

Note that $R(\cdot)$ is not injective and $R(x) = R(-x)$. Then the attitude indicated by x is identical with that of $-x$. The origin $R = I$ corresponds to $x = (0, 0, 0, \pm 1)$.

As a GCLF of (49), we consider

$$V_0(x) = x_1^2 + 2x_2^2 + 3x_3^2. \tag{51}$$

The critical points are

$$x_c^1 := (\pm 1, 0, 0, 0), \qquad x_c^2 := (0, \pm 1, 0, 0),$$
$$x_c^3 := (0, 0, \pm 1, 0), \qquad x_c^4 := (0, 0, 0, \pm 1). \tag{52}$$

We have found the following:

$$c_0(p) := (\cos h_1 \cos h_2 \sin h_3,\ \cos h_1 \sin h_2,\ \sin h_1,\ \cos h_1 \cos h_2 \cos h_3) \tag{53}$$
$$P(x) := x_1^2 + 2x_2^2 + 3x_3^2 \tag{54}$$
$$T_2(x, p) := \mathcal{R}_1(h_1)\mathcal{R}_2(h_2)\mathcal{R}_3(h_3)x, \tag{55}$$

where

$$h_1 := \frac{\pi}{4}p(p-2)(p-3) \tag{56}$$
$$h_2 := -\frac{\pi}{4}p(p-1)(p-3) \tag{57}$$
$$h_3 := \frac{\pi}{12}p(p-1)(p-2), \tag{58}$$

and

$$\mathcal{R}_1(h) := \begin{pmatrix} \cos h & 0 & 0 & \sin h \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin h & 0 & 0 & \cos h \end{pmatrix}, \mathcal{R}_2(h) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos h & 0 & \sin h \\ 0 & 0 & 1 & 0 \\ 0 & -\sin h & 0 & \cos h \end{pmatrix},$$
$$\mathcal{R}_3(h) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos h & \sin h \\ 0 & 0 & -\sin h & \cos h \end{pmatrix}. \tag{59}$$

These induce $V_3(x, p) = V_1 \circ \bar{T}_1 \circ \bar{T}_2(x, p)$. The critical points are

$$q_3^1 = (x_c^4, 1), q_3^2 = (x_c^4, 2), q_3^3 = (x_c^4, 3), q_3^4 = (x_c^4, 0). \tag{60}$$

All solutions of the augmented system controlled by k_S derived from V_3 converge to the critical points, i.e., the original system controlled by the k_S is globally asymptotically stable.

Figure 5 shows a solution with an initial state of the augmented system controlled by the k_S. It tends to q_3^4, i.e., the corresponding solution $R(t)$ of (48) converges to I.

7. CONCLUSION

In this paper we have shown a global asymptotic stabilization method for a nonlinear system on a Riemannian manifold by using a dynamic compensator. Moreover, we have proposed a definition of GCLF for an input-affine system, and a method for constructing a GCLF for a controllable system.

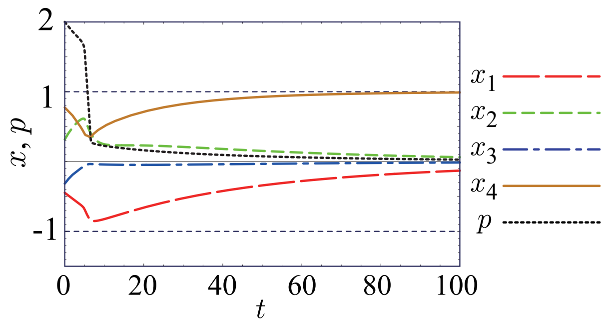

Fig. 5. The solution starting the initial state $(x, p) = (-\sqrt{0.2}, \sqrt{0.1}, -\sqrt{0.1}, \sqrt{0.6}, 2)$

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