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Relations Between Control Signal Properties and Robustness Measures

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Abstract: In this paper we consider control signal properties, such as maximum magnitude and activity, as well as system robustness measures. We derive an ideal controller and control signal for exponential disturbance rejection for a first order process with time delay. For the resulting closed-loop system, it is shown analytically that there are strong interconnections between robustness measures and control signal properties regarding load disturbance attenuation. The results imply that popular controller design methods implicitly take control signal properties into consideration.

Keywords: controller constraints and structure, disturbance rejection, constrained control, robust time-delay systems, robustness analysis.

1. INTRODUCTION

One of the main advantages of feedback is the ability to counteract unmeasured load disturbances acting on the process. The counteraction should normally be as fast as possible under specified constraints on e.g., robustness, control signal magnitude, and control signal activity. It is natural that a fast return to set-point demands a rapid controller response, and hence, the gain at high frequencies is required to be large for this property. On the other hand, as the control signal is actuated, there is an upper limit on how rapid the response can be due to e.g., actuator dynamics as well as wear. Often, rapid control signal changes are allowed as long as the amplitude is small compared to full control signal range. Additionally, the upper limit on high frequency gain is also affected by, for instance, output measurement noise and process variations. Thus, a certain robustness margin must be taken into consideration. There are hence clear trade-offs on how rapidly the controller should act, and how the control signal can behave, at load disturbances in practice.

Popular design methods of e.g., PID controllers, include minimizing the error at load disturbance, for instance integrated error (IE), with respect to controller parameters. Robustness is included by constraining the maximum of the sensitivity and complementary sensitivity function, see e.g., Åström and Hägglund (2005). However, these design methods do not explicitly take control signal properties into account, as is done in for instance linear quadratic control where a control signal weight is applied in the cost function. The control signal properties are instead assumed to be implicitly covered by the constrained robustness measures.

In this note, we will derive the ideal control signal and controller for exponential disturbance recovery for a first order process with time delay. The in practice limited properties,

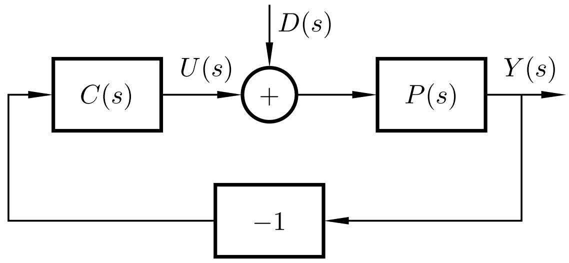

Fig. 1. Closed-loop system with process $P(s)$, controller $C(s)$, output $Y(s)$, and load disturbance $D(s)$.

maximum amplitude of the control signal and control signal activity, will be deduced as functions of process parameters and load disturbance response specifications. It will be shown that they can be closely connected to robustness measures, which indicates that the assumptions in the control design methods are correct. Additionally, the cost of fast load disturbance attenuation in terms of control signal magnitude, activity, and robustness, is visualized.

2. PROBLEM FORMULATION

In process control, the most common process model is the first order with time delay (FOTD), since it is easy to estimate with e.g., step response methods using a small amount of effort and time. Consider such a process,

$$P(s) = \frac{K}{sT+1}\, e^{-sL} = P_0(s)e^{-sL}, \tag{1}$$

where $P_0(s)$ is the the delay free part, L the time delay, K the static gain, and T is the process time constant. The process is in a feedback control loop with controller $C(s)$, and a load disturbance $D(s)$ acts on the process input, see Figure 1. It is assumed that the set-point is 0.

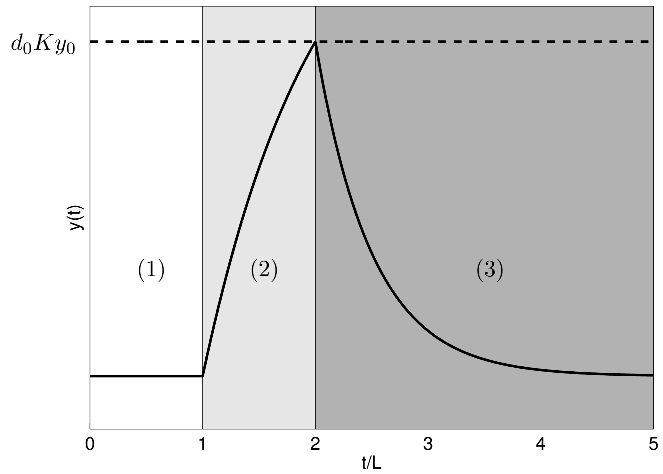

Fig. 2. Load response of an FOTD process in feedback divided into three parts.

Assume a load disturbance step $D(s) = d_0/s$ at time 0. The load response of the feedback system can be divided into three main parts as indicated in Figure 2,

(1) $t \in [0, L)$. The process output is identically zero because of the time delay in the process.
(2) $t \in [L, 2L)$. The load step begins to affect the output, which follows the first order response, i.e.,
$$y(t) = d_0K\left(1 - e^{-\frac{t-L}{T}}\right).$$
By this there is a control error and the controller generates a counteractive control signal. Due to the time delay L, the control signal will not affect the output until $t = 2L$.
(3) $t \in [2L, \infty)$. The output tends to set-point value. Note that the output does not have to decrease monotonically due to constraints on the system.

The first two intervals can not be affected by a feedback controller, while the third is a direct function of the control input on the interval $t \in [L, \infty)$. Thus, theoretically, for a FOTD process, the maximum deviation from set-point is at $t = 2L$, as shown in Figure 2. For convenience later on, we define
$$y_0 = 1 - e^{-\frac{L}{T}} > 0, \tag{2}$$
which gives $y(2L) = d_0Ky_0$.

With the control signal limitations discussed in Section 1, we know that the decay towards set-point can not be made arbitrarily fast. When tuning a control system, it would be practical to see how maximum amplitude and activity of control signal, as well as system robustness, depend on how fast the load disturbance is attenuated. This paper concerns the third part of the load response and hence these relations. In particular, the case when the response is chosen, by tractability, as an exponential decay with specified time constant T_d. In Shinskey (1994), the concept of how the controller should act, in order to get a fully attenuated disturbance after $3L$ seconds, was studied. The return to set-point was given by a piecewise constant control signal with zero magnitude except in the interval $t \in [L, 2L)$. In practice, this is not viable for systems with short dead-time, since the control signal must be within certain limits. Here, we will not use a piecewise constant control signal. Instead, the control signal will be derived as an explicit function of process parameters and load disturbance specification. This will give the relations searched for.

3. CONSTRUCTION OF IDEAL CONTROL SIGNAL

Without choosing controller structure, the load disturbance response is specified as
$$y(t) = d_0Ky_0e^{-\frac{t-2L}{T_d}}, \quad t \in [2L, \ \infty),$$
that is, exponentially decaying with time constant T_d. The constants d_0Ky_0 gives a continuous output, see Figure 2. Denote by $H(t)$ the Heaviside function
$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$
The specified output $y(t)$ at a load disturbance can then be expressed for all t as
$$\begin{aligned} y(t) = & \underbrace{H(t-L)d_0K\left(1 - e^{-\frac{t-L}{T}}\right)}_{I} \\ & - \underbrace{H(t-2L)d_0K\left(1 + e^{-\frac{t-2L}{T}}(y_0 - 1)\right)}_{II} \\ & + \underbrace{H(t-2L)d_0Ky_0e^{-\frac{t-2L}{T_d}}}_{III}, \end{aligned} \tag{3}$$
where y_0 was defined in (2). The three parts of the expression have the following interpretations,

I. Response of load disturbance without control action. Note that this is the only part that is not 0 in the interval $t \in [L, 2L)$.
II. Undesired part of the load response is removed, i.e., for $t \geq 2L$ we have $I - II = 0$.
III. Desired part of the load response for $t \geq 2L$, i.e., exponential decay with time constant T_d.

Taking the Laplace transform of (3) yields
$$\begin{aligned} Y(s) = & \frac{d_0K}{s(sT+1)}\, e^{-Ls} - d_0K\left(\frac{1}{s(sT+1)}\right. \\ & \left. + y_0\frac{T}{sT+1} - y_0\frac{T_d}{sT_d+1}\right)e^{-2Ls}. \end{aligned} \tag{4}$$
From the process dynamics in (1) we have
$$U(s) + D(s) = \frac{sT+1}{K}\, e^{sL}\, Y(s).$$
Thus, the control signal has the Laplace transform
$$\begin{aligned} U(s) &= -D(s) + \frac{sT+1}{K}\, e^{sL}\, Y(s) \\ &= -\frac{d_0}{s}e^{-sL} - d_0y_0Te^{-Ls} + d_0y_0T_d\frac{sT+1}{sT_d+1}e^{-sL} \\ &= -\frac{d_0}{s}\, e^{-sL} - d_0y_0\frac{T-T_d}{sT_d+1}e^{-Ls}, \end{aligned} \tag{5}$$
which in time domain is equal to
$$u(t) = -H(t-L)d_0\left(1 + y_0\left(\frac{T}{T_d} - 1\right)e^{-\frac{t-L}{T_d}}\right). \tag{6}$$
The control signal is hence an explicit function of T, L, d_0, and T_d, and can be divided in two parts as suggested in the last parenthesis. The first part is a direct removal of the load disturbance while the second part adds control action to give arbitrary time constant T_d. Note that if $T_d \geq T$, the

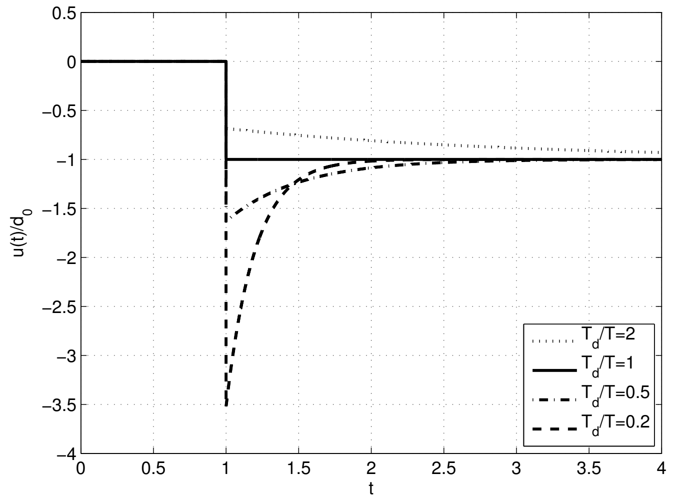

Fig. 3. Control signals for different values of T_d and the process $P(s) = K/(s+1)\,e^{-s}$.

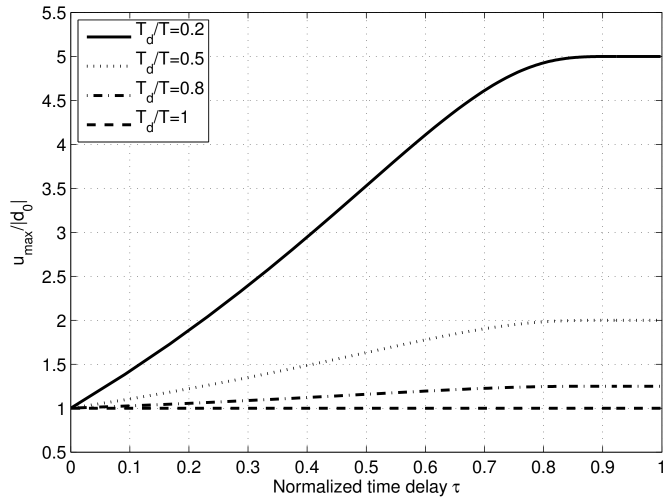

Fig. 4. Maximum magnitude $u_{\max}$ of control signal as function of normalized time delay τ.

control signal will not go further away from 0 than $-d_0$, while if $T_d < T$ the control signal will have an overshoot. Examples of control signals with different T/T_d–ratios are found in Figure 3.

4. IDEAL CONTROLLER

From the specified output $Y(s)$ and the control signal $U(s)$, one can derive a controller realization. Using the expressions from (4) and (5) and the fact that $C(s) = -U(s)/Y(s)$ when set-point is 0, yields

$$\begin{aligned} C(s) &= \frac{(1+sT)\,(1+s\,(T_d + y_0\,(T-T_d)))}{K\,(1+sT_d - (1+s\,(T_d+y_0\,(T-T_d)))\,e^{-sL})} \\ &= \frac{(1+sT)(1+sT_1)}{K(1+sT_d-(1+sT_1)\,e^{-sL})}, \end{aligned} \tag{7}$$

where we have defined

$$T_1 = T_d + y_0(T - T_d). \tag{8}$$

Note that the controller has integral action since as $s \to 0$, we have

$$C(s) \approx \frac{1}{sK\,(L + y_0(T_d - T))}.$$

The resulting controller can be compared to known control structure by rewriting (7) as

$$C(s) = \frac{Q(s)P_0^{-1}}{1 - Q(s)P_0^{-1}P}, \quad Q(s) = \frac{sT_1+1}{sT_d+1},$$

which is the common internal model controller (IMC), see e.g., Morari and Zafiriou (1989) and references therein. For the special case of $T_d = T$, we have

$$C(s) = \frac{sT+1}{K(1-e^{-sL})},$$

which is equal to the PD$_\tau$ controller without derivative filter, see e.g., Shinskey (1994). Thus, the exponential decay specification results in well known controllers, justifying the undertaken approach.

5. CONTROL SIGNAL PROPERTIES

Control signal properties such as maximum magnitude and activity can now be defined as functions of process parameters and specified load attenuation.

5.1 Maximum Control Signal Magnitude

The maximum of the control signal magnitude is defined as

$$u_{\max} = \sup_t |u(t)|,$$

and it can easily be verified from (6) that

$$\frac{u_{\max}}{|d_0|} = \begin{cases} 1, & T_d \geq T \\ 1 + y_0\left(\dfrac{T}{T_d} - 1\right), & T_d < T. \end{cases} \tag{9}$$

For the case $T_d \geq T$, the maximum is achieved as $t \to \infty$.

Introducing the normalized time delay

$$\tau = \frac{L}{L+T},$$

we can write

$$\frac{u_{\max}}{|d_0|} = \begin{cases} 1, & T_d \geq T \\ \left(1 + \left(1 - e^{\frac{\tau}{\tau-1}}\right)\left(\dfrac{T}{T_d} - 1\right)\right), & T_d < T. \end{cases}$$

In Figure 4, $u_{\max}/|d_0|$ is shown as a function of τ and specified time constant T_d. We see that a delay dominated process (large τ) in general requires a larger control signal. This is due to the fact that the second part of the step response, illustrated in Figure 2, is longer and hence, the output is able to go further away from set-point prior control action response. The exponential decay specification yields that the further away from set-point $y(t)$ is, the steeper will the return be, and thus the larger will u_{max} be. Figure 4 can hence be seen as a cost of fast disturbance rejection in terms of maximum control signal magnitude.

5.2 Control Signal Activity

Studying Figure 3 one can find that a reasonable definition of control signal activity is the magnitude of the initial step of $u(t)$. Thus, by setting $t = L$ in the modulus of (6), we define

$$Activity = |u(L)| = |d_0|\left(1 + y_0\left(\frac{T}{T_d} - 1\right)\right).$$

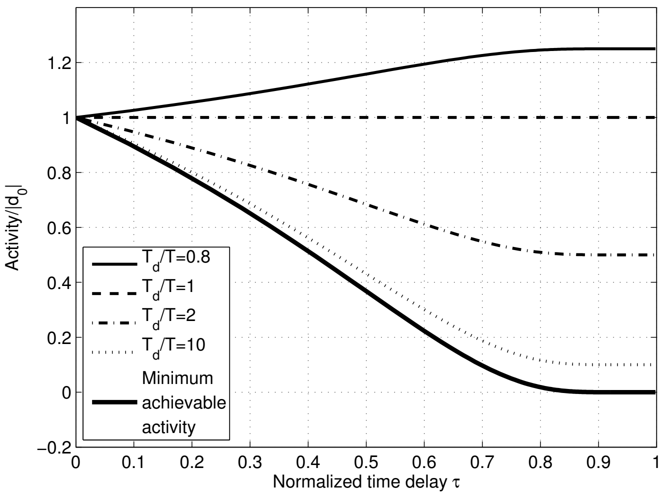

Fig. 5. $Activity/|d_0|$ as function of normalized time delay τ for different values of T_d/T.

Note that by this definition, we have $Activity = u_{\max}$ for $T_d < T$. It is easy to understand that the activity will be smallest as $T_d \to \infty$. This can also be seen from the state space realization of the system. Assume that $t = 2L$ and $d_0 > 0$, then

$$T\dot{y}(2L) = -y(2L) + Kd_0 + Ku(L).$$

To stop $y(t)$ going further away from set-point, we must have

$$0 \geq -Kd_0(1 - e^{-\frac{L}{T}}) + Kd_0 + Ku(L),$$

which gives the lower bound for the *Activity*

$$Activity \geq |d_0|e^{-\frac{L}{T}} = |d_0|e^{\frac{\tau}{\tau-1}},$$

that is independent of T_d. The inequality also holds for $d_0 < 0$.

Using the normalized time delay, the activity can explicitly be written as

$$Activity = |d_0| \left(1 + \left(1 - e^{\frac{\tau}{\tau-1}}\right) \left(\frac{T}{T_d} - 1\right)\right).$$

In Figure 5, which can be seen as an extension of Figure 4, *Activity* and its lower bound are shown as functions of the normalized time delay τ. As intuition implies, if we require a response that is faster than the process time constant, i.e., $T_d < T$, we must have a control signal activity that is larger than the magnitude of the load disturbance entering. The analogous holds for $T_d \geq T$. An interesting observation can be made for processes with no time delay, $\tau = 0$. Due to the specified exponential decay, the controller will directly give a step of size d_0 that has opposite sign to the load and hence the activity will be $|d_0|$ irrespective of T_d.

The simplicity of exponential decay, i.e., that we demand the output derivative to change sign instantly, gives an initial step in the control signal for all parameter sets. That is, the activity will always be greater than 0. Although, this response specification gives a fair picture of parameter relations for a non-ideal controller.

Analogous to $u_{\max}$, we can see *Activity* as a cost of fast load disturbance attenuation in undesired rapid control signal change.

6. ROBUSTNESS MEASURES

Two common measures used to show robustness of a closed loop system are the maximum of the complementary sensitivity and sensitivity function, $T(s)$ and $S(s)$, respectively, defined as

$$M_T = \max_{\omega} |T(i\omega)|$$
$$M_S = \max_{\omega} |S(i\omega)|.$$

M_T gives for instance an estimate of how large relative error that can be accepted in the process model while maintaining stability and M_S shows e.g. the worst case amplification of measurement noise.

The measures can be illustrated in a Nyquist diagram as two circles with centers at $-M_T^2/(M_T^2 - 1)$ and -1, respectively, and radii equal to

$$M_T/(M_T^2 - 1), \qquad 1/M_S, \tag{10}$$

respectively. For the closed loop system to fulfill the robustness margins, the Nyquist curve is not allowed inside the circles.

For the controller in (7) we have that

$$P(s)C(s) = \frac{1 + sT_1}{1 + sT_d - (1 + sT_1)\, e^{-sL}}\, e^{-sL}$$

and hence

$$T(s) = \frac{1 + sT_1}{1 + sT_d}\, e^{-sL}$$
$$S(s) = 1 - \frac{1 + sT_1}{1 + sT_d}\, e^{-sL},$$

where T_1 was defined in (8).

M_T can now easily be obtained by the following,

$$M_T^2 = \max_{\omega} |T(i\omega)|^2 = \max_{\omega} \left(1 + \alpha \frac{\omega^2 T_d^2}{1 + \omega^2 T_d^2}\right), \tag{11}$$

where

$$\alpha = \underbrace{\frac{y_0\left((2 - y_0)\, T_d + y_0 T\right)}{T_d^2}}_{>0} (T - T_d).$$

Since $T, T_d > 0$, α has the properties

$$\alpha \leq 0 \text{ if } T_d \geq T$$
$$\alpha > 0 \text{ if } T_d < T,$$

and the maximum value is hence determined by the last term in (11) being 0 or 1 for $\omega = 0$ and ∞, respectively.

To simplify, notice that

$$1 + \alpha = \left(1 + y_0 \left(\frac{T}{T_d} - 1\right)\right)^2.$$

Thus, the following expression for M_T is obtained

$$M_T = \begin{cases} 1, & T_d \geq T \\ 1 + y_0 \left(\dfrac{T}{T_d} - 1\right), & T_d < T. \end{cases} \tag{12}$$

Figure 6 shows the relationship between M_T and T_d for different values of τ. We see that for a lag dominant process (small τ), the permissible relative process model error is larger than for a delay dominant process (large τ). And also, for faster disturbance rejection, we need a better process model. Thus, Figure 6 yields a ballpark estimate

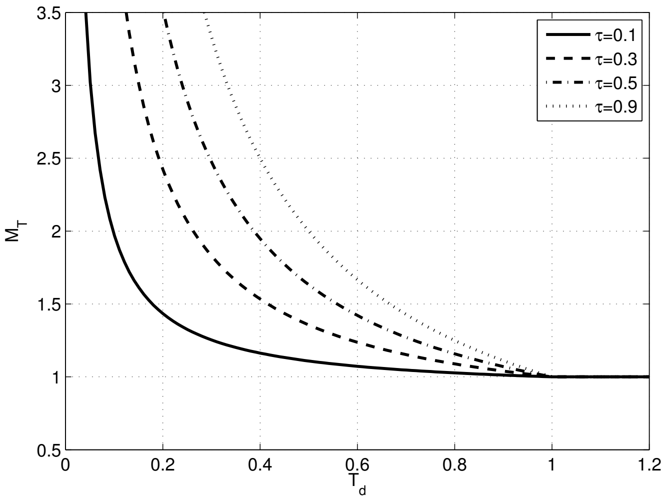

Fig. 6. The trade-off between M_T and T_d for different values of the normalized time delay τ for the process $P(s) = 1/(s+1)\, e^{-sL}$. Note that choosing $T \neq 1$ gives slightly changed curves.

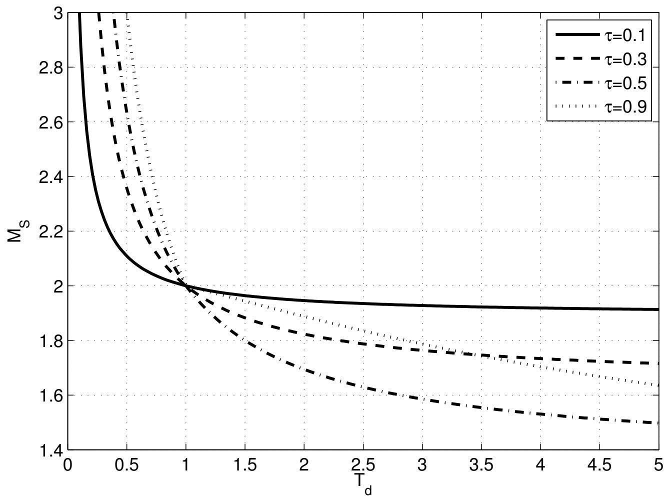

Fig. 7. The trade-off between M_S and T_d for different values of the normalized time delay τ for the process $P(s) = 1/(s+1)\, e^{-sL}$. Note that choosing $T \neq 1$ gives slightly changed curves.

of the trade-off between process uncertainty and fast load disturbance rejection when designing a controller.

The maximum of the sensitivity function, M_S can partially be given an explicit expression. By definition,

$$M_S = \max_{\omega} \left| 1 - \frac{1 + i\omega T_1}{1 + i\omega T_d}\, e^{-i\omega L} \right|.$$

When $T_d < T$,

$$T_1 = T_d + y_0(T - T_d) > T_d$$

and hence

$$\frac{1 + sT_1}{1 + sT_d}$$

is a lead filter. The maximum of the sensitivity function will now be obtained when

$$\frac{1 + i\omega T_1}{1 + i\omega T_d}\, e^{-i\omega L}$$

is real valued and as small as possible. Using the fact that $e^{-i\omega L} = -1$ for $\omega L = \pi + 2\pi k,\ k \in \mathbb{Z}$, and that

$$\lim_{\omega\to\infty} \frac{1 + i\omega T_1}{1 + i\omega T_d} = \frac{T_d + y_0(T - T_d)}{T_d},$$

we have

$$M_S = 1 + \frac{T_d + y_0(T - T_d)}{T_d} = 1 + M_T, \quad T_d < T. \quad (13)$$

For $T_d \geq T$, no explicit solution for M_S can be found, although it is easy to show that

$$\lim_{T_d\to\infty} M_S = 1 + e^{-\frac{L}{T}}. \quad (14)$$

The explicit expression in (13) together with numerical estimates of M_S for $T_d \geq T$ are shown in Figure 7. It is clear that for $T_d < T$, M_S follows the same pattern as M_T while for $T_d \geq T$ it is a more complex function, although decreasing with T_d. Note that the convergence to the limit value in (14) is dependent on process properties.

7. ROBUSTNESS AND MAXIMUM CONTROL SIGNAL MAGNITUDE

Robustness measures and control signal magnitude can now be related to each other. Comparing the expressions in (9) and (12) we see that the following holds,

$$\frac{u_{\max}}{|d_0|} = M_T.$$

This simple relation can be seen from the fact that the transfer function from $D(s)$ to $U(s)$ is $-T(s)$ and $T(s)$ can be written as

$$\begin{aligned} e^{sL}\, T(s) &= \frac{s\,(T_d + y_0\,(T - T_d)) + 1}{sT_d + 1} \\ &= 1 + y_0 \left(\frac{T}{T_d} - 1 \right) + y_0 \left(1 - \frac{T}{T_d} \right) \frac{1}{sT_d + 1}, \end{aligned}$$

where the direct term is $u_{\max}$ as $T_d < T$, and $u_{\max} = T(0)$ as $T_d \geq T$.

An analogous expression can be derived for M_S even though there is no explicit expression for M_S as $T_d \geq T$. For this case, it is easy to see that

$$\begin{aligned} M_S &= \max_{\omega} \left| 1 - \frac{1 + i\omega T_1}{1 + i\omega T} e^{-i\omega L} \right| \\ &\leq 1 + \max_{\omega} \left| \frac{1 + i\omega T_1}{1 + i\omega T} e^{-i\omega L} \right| = 2. \end{aligned}$$

Thus, an upper conservative bound on M_S is found. Combining this with the knowledge that $u_{\max} = |d_0|$ for this case, the following relationship is found,

$$\frac{u_{\max}}{|d_0|} = \begin{cases} 1, & M_S \leq 2 \\ M_S - 1, & M_S > 2. \end{cases}$$

The relations between M_T, M_S, and $u_{\max}$ are shown in Figure 8. Note that M_S has lower limit of 1 in the figure, corresponding to $L \to \infty$ in (14).

8. ROBUSTNESS AND CONTROL SIGNAL ACTIVITY

Bounds on the activity of the control signal is, as mentioned in the introductory section, present in practice. We can see that, since *Activity* was equal to $u_{\max}$ for $T_d < T$, it follows that *Activity*$/|d_0| = M_T$ for this case. When decreasing *Activity* to be less than $|d_0|$, that is $T_d \geq T$,

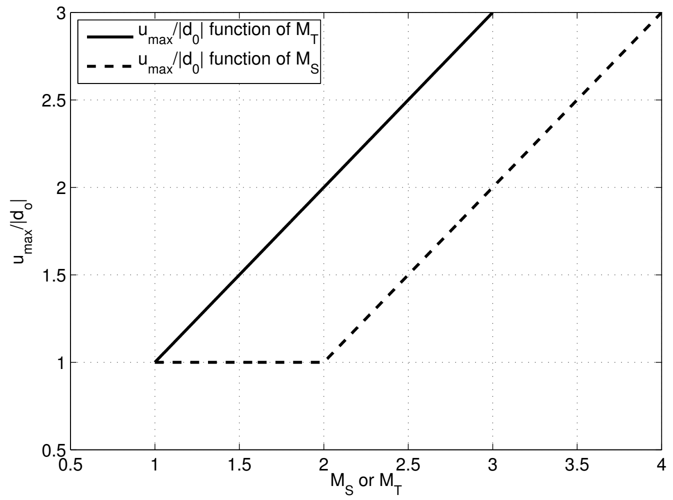

Fig. 8. The trade-off between $u_{\max}/|d_0|$ and M_T or M_S.

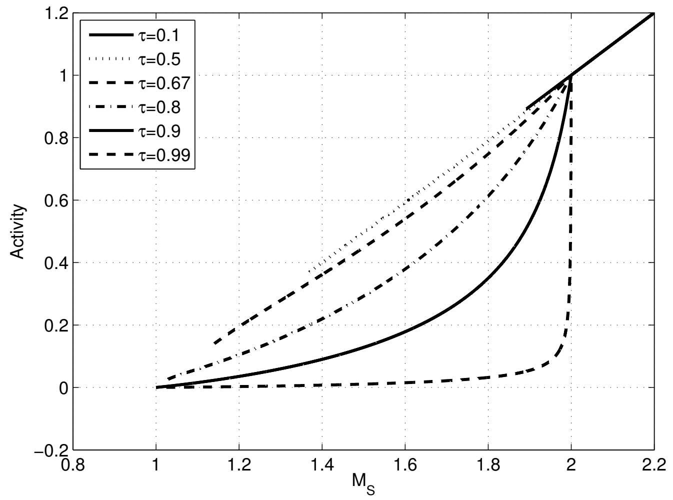

Fig. 9. The trade-off between *Activity* and M_S for different normalized time delays τ for the process $P(s) = K/(s+1)\,e^{-sL}$. Note that choosing $T \neq 1$ gives slightly changed curves.

M_T will be 1. Hence, the relationship between these two parameters becomes non-informative in this case.

For M_S, an analogous discussion yields $Activity/|d_0| = M_S - 1$ for $T_d < T$. For the other case, $T_d \geq T$, no explicit expression can be found, although, simulations can by ease give the relation. Figure 9 shows, by simulation, how *Activity* depends on M_S. It implies that specifying an activity bound also gives a rough estimate of the robustness measure. A controlled delay dominated process will be more sensitive to measurement noise than a lag dominated one when requiring the same control signal activity level. The fact that the activity do not tend to zero for lag dominated processes is, as mentioned in Section 5.2, due to the exponential disturbance attenuation

The fact that M_T does not give any information about activity of the control signal when $T_d \geq T$ can easily be understood as follows. We see from (10) that as $M_T \to 1$, the circle radius tends to infinity and the circle becomes the closed half plane $\{s \in \mathbb{C} \,\big|\, \mathrm{Re}\{s\} \leq -1/2\}$. This implies that

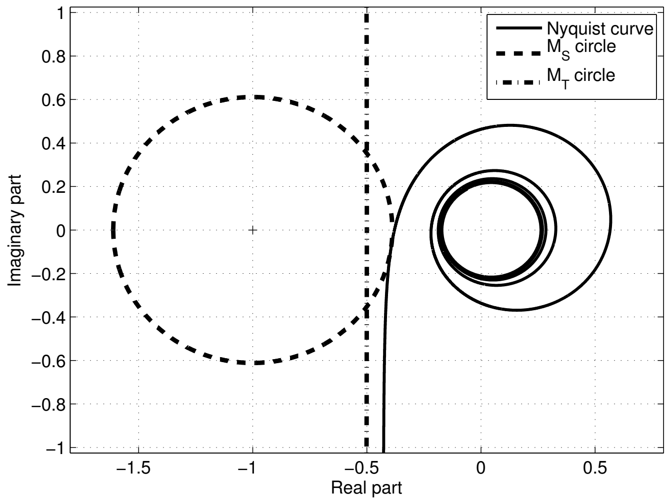

Fig. 10. Nyquist curve for $P(s) = K/(sT+1)\,e^{-sL}$ with $T_d = 5T = 5$ and $L = 9$ and corresponding circles for $M_S = 1.64$ and $M_T = 1$.

$$\mathrm{Re}\{P(i\omega)C(i\omega)\} \geq -\frac{1}{2},\ \forall\omega,\ T_d \geq T,$$

where we have equality for $T_d = T$. The only way we can affect the high frequency gain of the controller in this situation is to use the M_S circle, i.e., decrease or increase M_S. In Figure 10, an example of the Nyquist curve and the robustness circles are presented for this case.

9. CONCLUSIONS

In this paper, we have considered relations between control signal properties and robustness measures. We have derived explicit expressions for maximum control signal magnitude and activity as functions of FOTD process parameters and load disturbance rejection specification. The resulting closed-loop system's robustness in terms of maximum sensitivity and complementary sensitivity has been derived. It has been shown that the control signal properties are well correlated with the robustness measures. This implies that limitations on control signal magnitude and activity regarding load disturbances are covered by the constrained robustness measures in common controller design methods.

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