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Boundary Control of flexible marine risers ⋆

K. D. Do * **J. Pan** **

* *School of Mechanical Engineering, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia, Tel: +61864883125, Fax: +61864881024, Email: duc@mech.uwa.edu.au*
** *School of Mechanical Engineering, The University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia, Email: pan@mech.uwa.edu.au*

Abstract: A method to design a boundary controller for global stabilization of three-dimensional nonlinear dynamics of flexible marine risers is presented. Equations of motion of the risers are first developed in a vector form. The boundary controller at the top end of the risers is then designed based on Lyapunov's direct method. It is shown that when there are no environmental disturbances, the proposed boundary controller is able to force the riser to be globally exponentially stable at its equilibrium position. When there are environmental disturbances, the riser is stabilized in the neighborhood of its equilibrium position by the proposed boundary controller.

1. INTRODUCTION

A typical configuration of an offshore platform is depicted in Figure 1. The riser is considered in this paper as a slender thin walled circular beam because of its large length to diameter ratio. In general, the riser is subject to nonlinear deformation dependent hydrodynamic loads induced by waves, ocean currents, tension exerted at the top, distributed/concentrated buoyancy from attached modules, its own weight, inertia forces and distributed/concentrated torsional couples. Since the riser dynamics is essentially a distributed system and its motion is governed by a set of partial differential equations (PDE) in both time and space variables, modal control and boundary control approaches are often used to control the riser in the literature.

In the modal control approach, see Meirovitch [1997], distributed systems are controlled by controlling their modes. As a result, many concepts developed for lumped-parameter systems in Khalil [2002] can be used for controlling the distributed ones, since both types can be described in terms of modal coordinates. The main difficulty is computation of infinite dimensional gain matrices. This difficulty can be avoided by using the independent modal-space control method, but this method requires a distributed control force, which can be problematic to implement.

The boundary control approach is more practical and efficient than the modal control approach since it excludes the effect of both observation and control spillover phenomenon, and the use of distributed actuators and sensors. Design of boundary controllers for distributed systems has been usually based on functional analysis and semi-group theory, see Chen et al. [2001] and Curtain and Zwart [1995], and the Lyapunov's direct method, see Queiroz et al. [2000] and Junkins and Kim [1993]. Using Lyapunov's direct method, various boundary controllers

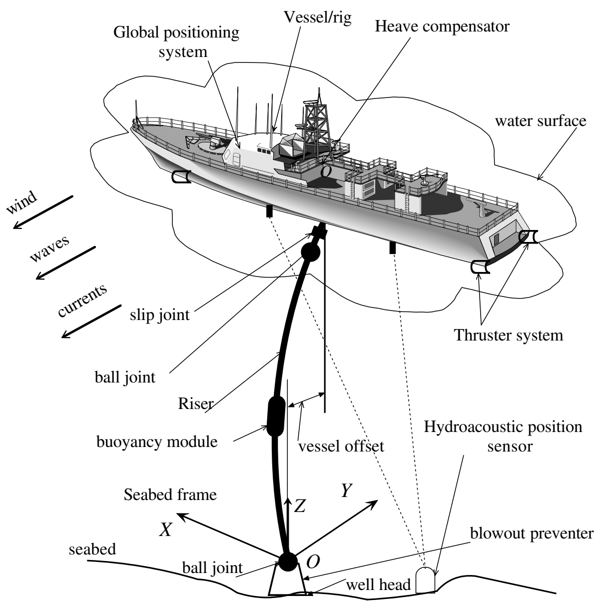

Fig. 1. A typical riser system.

have been proposed for flexible beam-like systems. In Fung et al. [1999] and Fung and Tseng [1999], asymptotic and exponential stability of an axially moving string is proven by using a linear and nonlinear state feedback boundary control, respectively.

In this paper, we consider a problem of global stabilization of three-dimensional nonlinear flexible marine risers. A set of partial differential equations and boundary conditions describing motion of the risers is presented. Using the Lyapunov's direct method, a boundary controller at the top end of the risers is designed. The environmental disturbances induced by waves, wind and ocean currents are also considered. This paper is a short version of Do and Pan [2007].

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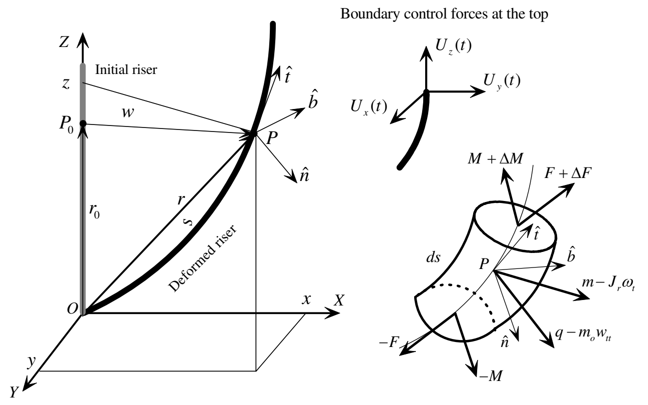

Fig. 2. Riser coordinates.

2. MATHEMATICAL MODEL AND CONTROL OBJECTIVES

2.1 Mathematical Model

In developing the equations of motion of the riser, we make the following assumption:

Assumption 1:
1) The riser can be modeled as a beam rather than a shell since the diameter-to-length of the riser is small, i.e. we consider the riser as a slender structure.
2) Plane sections remain plane after deformation, i.e. warping is neglected.
3) The riser is locally stiff, i.e. cross sections do not deform and Poisson effect is neglected.
4) The riser material is homogeneous, isotropic and linearly elastic, i.e. it obeys Hookes's law.
5) The riser is initially straight and vertical.
6) Torsional and distributed moments induced by environmental disturbances are neglected.

Remark 1. Items 1) - 4) mean that the riser will be modeled as a Bernoulli-type of beam and not a Timoshenko-type, and that the extension of the riser axis small. Bernoulli-Euler models are satisfactory for modeling low frequency vibrations of beams. Item 5) generally holds in practice, and is made to simplify the development of the mathematical model and boundary controller. This item can be readily removed. Item 6) implies that we consider fluid/gas transportation risers rather than drilling risers, and that moment induced by the asymmetry of the relative flow due to vortex shedding is ignored.

Preliminaries The riser coordinates are presented in Figure 2. In this figure, we have two coordinate systems. The earth-fixed system is $(OXYZ)$, where O is the bottom ball-joint of the riser, and the OZ axis is along the initial riser. Let $r^0(s_0, t_0) = [x_0, y_0, z_0]$ be the position vector of the point P_0 of the initial riser centerline at the time t_0 and the arc length s_0 from the point O. Hence at the time $t > t_0$, the point P_0 moves to the point P of the deformed riser centerline, whose position is denoted by $r(s,t) = [x(s,t), y(s,t), z(s,t)]$ at the arc length s from the point O. Moreover, let $w(s,t) = [w_x(s,t), w_y(s,t), w_z(s,t))]^T$ be the vector from the point P_0 to the point P. Then we have

$$r = r^0 + w \tag{1}$$

where from now onward whenever it is not confusing, we drop the arguments (t, s) and (t_0, s_0) of r, w and r^0, respectively for clarity. The body-fixed system is $(\hat{t}, \hat{n}, \hat{b})$, whose axes are the tangent, principal normal and binormal and unit vectors. These vectors can be expressed in terms of the fixed system as

$$\hat{t} = r_s, \qquad \hat{n} = \hat{t}_s/\kappa, \qquad \hat{b} = \hat{t} \times \hat{n} \tag{2}$$

where the subscript s denotes the partial derivative with respect to the arc-length s, and κ is curvature of the riser center line at s depicting the rate of change of the orientation of the normal plane $(\hat{n}, \hat{b})$ defined by $\kappa = \|r_{ss}\|$. The above definition of the body-fixed coordinate system means that $(\hat{t}, \hat{n}, \hat{b})$ form a right handed orthonormal triad. The derivatives of the unit body-fixed vectors are given by the well-known Frenet-Serret relations:

$$\hat{n}_s = \tau\hat{b} - \kappa\hat{t}, \qquad \hat{b}_s = -\tau\hat{n}, \qquad \hat{t}_s = \kappa\hat{n} \tag{3}$$

where τ is the geometric torsion of the riser centerline depicting the rate of change of the orientation of the osculating plane $(\hat{n}, \hat{t})$ defined by $\tau = r_s.(r_{ss} \times r_{sss})/\kappa^2$. Now from the right hand side sub-figure of Figure 2, balancing the forces and moments on a component ds of the deformed riser results in

$$\begin{aligned} m_o w_{tt} &= F_s + q \\ J\omega_t &= M_s + \hat{t} \times F + m \end{aligned} \tag{4}$$

where from now onward, we use the subscript t to denote the partial derivative with respect to the time t, $m_o = \rho A$ is the oscillating mass of the riser per unit length with A being the riser cross section area, and ρ being the density of the riser, $J = \rho I$ with I being the second moment of the riser cross section area about the $\hat{b}$ axis, F and M are internal force and moment vectors, q and m are the external distributed force and moment vectors, and $\omega_t = \hat{n} \times \hat{n}_{tt} + \hat{b} \times \hat{b}_{tt}$ is the angular acceleration of a point on the centerline. The distributed moment vector m is induced by the asymmetry of the relative flow due to vortex shedding. Let $(F_{\hat{t}}, F_{\hat{n}}, F_{\hat{b}})$ and $(M_{\hat{t}}, M_{\hat{n}}, M_{\hat{b}})$ be the components of F and M along the $\hat{t}$, $\hat{n}$, $\hat{b}$ axes of the body-fixed system, respectively. We then can write F and M as

$$\begin{aligned} F &= F_{\hat{t}}\hat{t} + F_{\hat{n}}\hat{n} + F_{\hat{b}}\hat{b}, \\ M &= M_{\hat{t}}\hat{t} + M_{\hat{n}}\hat{n} + M_{\hat{b}}\hat{b}. \end{aligned} \tag{5}$$

Since the riser is assumed to be straight at the initial time t_0, we have the following constitutive relations, see Love [1920] and Bernitsas [1982]:

$$\begin{aligned} F_{\hat{t}} &= EA\epsilon + T_0 + \rho_w g\frac{\pi D_o^2}{4}(H_w - z) - \rho_m g\frac{\pi D_i^2}{4}(H_m \\ &-z), \quad M_{\hat{b}} = B\kappa, \qquad M_{\hat{n}} = 0, \qquad M_{\hat{t}} = G\tau + H \end{aligned} \tag{6}$$

where E is Young's modulus, T_0 is the initial tension in the riser; H_w and H_m are the vertical coordinates of the free surface of the water and mud, respectively; ρ_w and ρ_m are the density of the water and mud, respectively; D_o and D_i are the external and internal diameters of the riser; z is the vertical coordinate of the point P; $B = EI$ is the bending rigidity of the riser; H is the initial torsional moment around the $\hat{t}$ axis; $G = 2\mu I$ is the torsional rigidity of the riser with μ being the shear modulus, ϵ is the extension of the riser centerline given by Dill [1992]

$$\epsilon = \frac{ds}{ds_0} - 1 = \sqrt{\frac{dr}{ds_0}.\frac{dr}{ds_0}} - 1. \qquad (7)$$

It is noted that since we assumed that extension of the riser centerline is small and the riser centerline is stretched, hence $0 \le \epsilon \le 1$. The case where $\epsilon = 0$ corresponds to an inextensible riser. Moreover, $F_{\hat{t}}$ in (6) is referred to as the effective tension, while the actual tension is $EA\epsilon$.

Remark 2. In Bernitsas [1982], the constitutive equation for the moment in the normal direction, $M_{\hat{n}}$, is misgiven, since $M_{\hat{n}}$ is always zero for the riser under consideration.

Equations of motion From (5) and the second equation of (6), we have

$$M_s = (B\kappa\hat{b})_s + (\bar{H}\hat{t})_s = \hat{t} \times \big((B\kappa\hat{n})_s - \bar{H}\kappa\hat{b}\big) + \bar{H}_s\hat{t} \qquad (8)$$

where $\bar{H} = H + G\tau$. Now substituting (8) into the second equation of (4) results in

$$J\omega_t = \hat{t} \times \big((B\kappa\hat{n})_s - \bar{H}\kappa\hat{b} + F\big) + \bar{H}_s\hat{t} + m. \qquad (9)$$

Now producting vector both sides of (9) with $\hat{t}$ gives

$$\hat{t}.(J\omega_t) = \hat{t}.\big(\hat{t} \times \big((B\kappa\hat{n})_s - \bar{H}\kappa\hat{b} + F\big)\big) + \bar{H}_s\hat{t}.\hat{t} + m.\hat{t}$$
$$\Rightarrow \quad r_s.(J\omega_t) = \bar{H}_s + m.r_s \qquad (10)$$

where we have used the definition of $\hat{t}$ in (2). On the other hand, vectoring both sides of (9) with $\hat{t}$ gives

$$\hat{t} \times (J\omega_t) = \hat{t} \times (\hat{t} \times (B\kappa\hat{n})_s) - \hat{t} \times (\hat{t} \times \bar{H}\kappa\hat{b})) + \hat{t} \times (\hat{t} \times F) + \hat{t} \times (\bar{H}_s\hat{t}) + \hat{t} \times m. \qquad (11)$$

Let us calculate the first three terms of the right hand side of (11) using the definitions of $\hat{t}$, $\hat{n}$ and $\hat{b}$ in (2) as follows:

$$\begin{aligned} \hat{t} \times (\hat{t} \times (B\kappa\hat{n})_s) &= -(Br_{ss})_s - B\kappa^2 r_s, \\ \hat{t} \times (\hat{t} \times \bar{H}\kappa\hat{b})) &= -\bar{H}r_s \times r_{ss} \\ \hat{t} \times (\hat{t} \times F) &= -F + (F.r_s)r_s. \end{aligned} \qquad (12)$$

Substituting (12) into (11) gives

$$r_s \times (J\omega_t) = -(Br_{ss})_s - B\kappa^2 r_s + \bar{H}r_s \times r_{ss} - F + (F.r_s)r_s + r_s \times m. \qquad (13)$$

Now substituting F from (13) into the first equation of (6) and combining the second equation of (10) result in the equations of motion of the riser as follows:

$$\begin{aligned} m_o w_{tt} &= -(Br_{ss})_{ss} + (F_{\hat{t}} - B\kappa^2)r_s)_s \\ &\quad + (\bar{H}r_s \times r_{ss})_s + (r_s \times m)_s - (r_s \times (J\omega_t))_s + q, \\ r_s.(J\omega_t) &= \bar{H}_s + m.r_s. \end{aligned} \qquad (14)$$

It is noted that we have assumed the torsional moment $\bar{H}$ and the distributed moment m are negligible, and that the riser has a constant cross section. Furthermore, since the riser is initially straight, we have $r_{ss} = w_{ss}$, $r_{ssss} = w_{ssss}$ and $w_s = r_s - r_s^0$ where we take $s \simeq s_0$ due to the small extension assumption, see Dill [1992]. With these in mind, we now have the equations of motion of the riser from (14) for the boundary control design in the next section:

$$\begin{aligned} m_o w_{tt} &= -Bw_{ssss} + (F_{\hat{t}} - B\kappa^2)_s(w_s + r_s^0) + \\ &\quad (F_{\hat{t}} - B\kappa^2)w_{ss} + q, \quad \kappa = \|w_{ss}\|. \end{aligned} \qquad (15)$$

Initial and boundary conditions The initial conditions of the riser consist of the initial position and velocity functions. They are

$$w(s, t_0) = g_1(s), \quad w_t(s, t_0) = g_2(s), \quad \forall s \in (0, L) \qquad (16)$$

where $g_1(s)$ and $g_2(s)$ are sufficiently smooth and bounded function vectors of s, and compatible with the boundary conditions. Next, we will apply Hamilton's principle to derive the boundary conditions for the riser under consideration. We first provide the kinetic and potential energies, then use the first variation of the Lagrangian of the system to derive the boundary conditions. As such, the kinetic energy K_E and the potential energy P_E of the riser with a length of L are

$$\begin{aligned} K_E &= \frac{1}{2}\int_0^L m_o r_t.r_t ds, \\ P_E &= \frac{1}{2}\int_0^L Br_{ss}.r_{ss} ds - \int_0^L qr ds + F(0)r(0) \\ &\quad - F(L)r(L) + M(0)r_s(0) - M(L)r_s(L) \end{aligned} \qquad (17)$$

where we have used $r_t = w_t$ and $r_{ss} = w_{ss}$. The Lagrangian L_A of the riser is

$$L_A = \int_{t_1}^{t_2} (K_E - P_E) dt \qquad (18)$$

where t_1 and t_2 denote time. Moreover, the riser response must satisfy the kinetic constraint of the unit tangent vector $\hat{t}$. In terms of deformation, this constraint is

$$r_s.r_s = 1. \qquad (19)$$

The above constraint is applied along the riser by modifying the Lagrangian of the riser and by embedding a continuous multiplies $(F_{\hat{t}} - B\kappa^2)/2$. As such, the modified Lagrangian L_{MA} is

$$L_{MA} = \int_{t_1}^{t_2} \left[K_E - P_E + \frac{(F_{\hat{t}} - B\kappa^2)}{2}\int_0^L (r_s.r_s - 1) ds\right] dt. \qquad (20)$$

Including the term $\int_{t_1}^{t_2} \big[\frac{(F_{\hat{t}} - B\kappa^2)}{2}\int_0^L (r_s.r_s - 1) ds\big] dt$ in the modified Lagrangian physically means that the modified Lagrangian takes the contribution of the axial deformation into account in the potential energy. From (20), the first variation of L_{MA} is given by

$$\begin{aligned} \delta L_{MA} &= \int_{t_1}^{t_2}\int_0^L [-(Br_{ss})_{ss} + ((F_{\hat{t}} - B\kappa^2)r_s)_s + q \\ &\quad - m_o r_{tt}]\delta r\, ds\, dt + \int_{t_1}^{t_2} (r_s \times M - Br_{ss})\delta r_s\Big|_0^L dt \\ &\quad + \int_{t_1}^{t_2} (-(Br_{ss})_s + (F_{\hat{t}} - B\kappa^2)r_s - F)\delta r\Big|_0^L dt. \end{aligned} \qquad (21)$$

Since δr is arbitrary over the domain $0 < s < L$, letting $\delta L_{MA} = 0$ results in

$$-(Br_{ss})_{ss} + ((F_{\hat{t}} - B\kappa^2)r_s)_s + q - m_o r_{tt} = 0, \quad \forall s \in [0, L],\ t \in \mathbb{R}^+ \tag{22}$$

and

$$r_s \times M - Br_{ss} = 0 \quad \text{or } r_s = 0 \text{ at } s = 0 \text{ and } s = L\ \forall t \in \mathbb{R}^+ \tag{23}$$

and

$$-(Br_{ss})_s + (F_{\hat{t}} - B\kappa^2)r_s - F = 0 \quad \text{or } r = 0 \text{ at } s = 0 \text{ and } s = L\ \forall t \in \mathbb{R}^+. \tag{24}$$

The equation (22) is exactly the same as (15). The equations (23) and (24) specify the boundary conditions of the riser at top and bottom ends. Choosing proper conditions from (23) and (24) depends on the riser configuration. For the riser considered in this paper, ball joints at both ends imply that the moments acting at both ends are zero, i.e. $M(L,t) = M(0,t) = 0$, and the force vector $U(t)$ as the boundary control inputs at the top end. With this observation in mind, the boundary conditions (23) and (24) for the riser considered in this paper become:

$$\begin{aligned}
&w_{ss}(0,t) = 0, \quad w_{ss}(L,t) = 0, \quad w(0,t) = 0 \\
&-Bw_{sss}(L,t) + F_{\hat{t}}(L,t)[w_s(L,t) + r_s^0(L)] = F(L,t) \\
&:= U(t)
\end{aligned} \tag{25}$$

where $F_{\hat{t}}(L,t)$ is calculated from (6) as follows:

$$\begin{aligned}
F_{\hat{t}}(L,t) =\ & EA\epsilon(L,t) + T_0 + \rho_w g \frac{\pi D_o^2}{4}(H_w - z(L,t)) \\
& -\rho_m g \frac{\pi D_i^2}{4}(H_m - z(L,t)).
\end{aligned} \tag{26}$$

Environmental disturbance vector q The external disturbance vector q per unit length consists of fluid drag force, any concentrated forces exerted on the riser by attached cables and/or buoys modeled by dirac distributions, and effective riser weight defined as the weight of the riser plus contents in water. It is noted that the effective rather than the actual riser weight is used because the effective tension is used instead of the actual tension. In this paper, we do not consider cables or buoys attached to the riser. The fluid drag force is found by the use of a generalization of Morison's formula to account for cylinders, which are not oriented normal to the relative flow Borgman [1958]. Taking the effective riser weight into account, we have

$$q(s,t) = \hat{t}\times(W_{re}\times\hat{t}) + \frac{\rho_w C_{LD} D_H V_n}{2} + \frac{\rho_w C_{ND} D_H \|V_n\| V_n}{2} \tag{27}$$

where C_{LD} and C_{ND} are the linear and nonlinear drag coefficients, respectively; D_H is the local riser hydrodynamic diameter; $W_{re} = -[0\ 0\ w_{re}]^T$ with w_{re} is the effective riser weight per unit length; V_n is the component of the relative flow velocity normal to the riser centerline. Letting V be the (bounded) liquid flow velocity due to waves and currents. Then taking the riser motion into account, the relative flow velocity normal to the riser centerline, V_n, is given by

$$V_n = \hat{t} \times ((V - w_t) \times \hat{t}) = (I_{3\times3} - r_s r_s^T)(V - w_t) \tag{28}$$

where $I_{3\times3}$ is the three dimensional identity matrix. Substituting (28) into (27) results in the equation for external disturbance vector q as follows:

$$\begin{aligned}
q(s,t) =\ & (I_{3\times3} - r_s r_s^T)W_{re} + \frac{1}{2}\rho_w C_{LD} D_H (I_{3\times3} \\
& -r_s r_s^T)(V - w_t) + \frac{1}{2}\rho_w C_{ND} D_H \|(I_{3\times3} \\
& -r_s r_s^T)(V - w_t)\|(I_{3\times3} - r_s r_s^T)(V - w_t).
\end{aligned} \tag{29}$$

2.2 Control objectives

Under Assumption 1, design the boundary control $U(t)$ for the riser dynamics given by (15) subject to the boundary conditions given by (25) to globally stabilize the riser at its vertical position, i.e. finding the boundary control $U(t)$ of the form $U(t) = \Omega(w_s(L,t), w_t(L,t))$ such that:

(1) when the external disturbance vector q is ignored, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t).w_s(s,t)ds$, $\int_0^L w_t(s,t).w_t(s,t)ds$ and $\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds$ exponentially converge to zero for all $s \in [0, L]$ and $t \geq t_0$,
(2) when the external disturbance vector q is present, all the terms $\|w(s,t)\|$, $\int_0^L w_s(s,t).w_s(s,t)ds$, $\int_0^L w_t(s,t).w_t(s,t)ds$ and $\int_0^L w_{ss}(s,t).w_{ss}(s,t)ds$ exponentially converge to some small positive constants for all $s \in [0, L]$ and $t \geq t_0$.

3. BOUNDARY CONTROL DESIGN

Consider the following Lyapunov function candidate

$$\begin{aligned}
W =\ & \frac{m_o}{2}\int_0^L w_t.w_t ds + \frac{B}{2}\int_0^L w_{ss}.w_{ss} ds + \\
& \frac{\lambda}{2}\int_0^L w_s.w_s ds + \alpha\int_0^L s w_t.w_s ds
\end{aligned} \tag{30}$$

where λ and α are positive constants to be specified later. Since for all $t \geq t_0$

$$\begin{aligned}
-L\rho_0 \int_0^L w_t.w_t ds - \frac{L}{4\rho_0}\int_0^L w_s.w_s ds \leq \int_0^L s w_t.w_s ds \\
\leq L\rho_0 \int_0^L w_t.w_t ds + \frac{L}{4\rho_0}\int_0^L w_s.w_s ds
\end{aligned} \tag{31}$$

where ρ_0 is a positive constant, the function W satisfies

$$\begin{aligned}
W \geq\ & \left(\frac{m_o}{2} - \alpha L\rho_0\right)\int_0^L w_t.w_t ds + \frac{B}{2}\int_0^L w_{ss}.w_{ss} ds \\
& + \left(\frac{\lambda}{2} - \frac{\alpha L}{4\rho_0}\right)\int_0^L w_s.w_s ds, \\
W \leq\ & \left(\frac{m_o}{2} + \alpha L\rho_0\right)\int_0^L w_t.w_t ds + \frac{B}{2}\int_0^L w_{ss}.w_{ss} ds \\
& + \left(\frac{\lambda}{2} + \frac{\alpha L}{4\rho_0}\right)\int_0^L w_s.w_s ds.
\end{aligned} \tag{32}$$

Hence if we choose λ, α and ρ_0 such that

$$\frac{m_o}{2} - \alpha L\rho_0 = c_1, \qquad \frac{\lambda}{2} - \frac{\alpha L}{4\rho_0} = c_2 \tag{33}$$

where c_1 and c_2 are strictly positive constants, then the function W defined in (30) is a proper function of

$\int_0^L w_t.w_t ds$, $\int_0^L w_{ss}.w_{ss} ds$, and $\int_0^L w_s.w_s ds$. We do not detail the conditions (33) at the moment, but deal with them after the boundary control $U(t)$ is designed since the constants λ, α and ρ_0 need to satisfy some more conditions later. Differentiating both sides of (30) with respect to the time t, along the solutions of the riser dynamics (15) results in

$$\dot{W} = \Delta_1 + \Delta_2 \tag{34}$$

where

$$\begin{aligned}\Delta_1 &= \int_0^L w_t.\Big(-Bw_{ssss} + (F_{\hat{t}} - B\kappa^2)_s(w_s + r_s^0) \\ &\quad + (F_{\hat{t}} - B\kappa^2)w_{ss} + q\Big)ds + B\int_0^L w_{ss}.w_{sst}ds \\ &\quad + \lambda\int_0^L w_s.w_{st}ds + \alpha\int_0^L sw_t.w_{ts}ds, \\ \Delta_2 &= \frac{\alpha}{m_o}\int_0^L sw_s.\Big(-Bw_{ssss} + (F_{\hat{t}} - B\kappa^2)_s(w_s + r_s^0) \\ &\quad + (F_{\hat{t}} - B\kappa^2)w_{ss} + q\Big)ds. \end{aligned} \tag{35}$$

Using integration by part rules, we have

$$\begin{aligned}\Delta_1 &= -B\Big(w_{sss}.w_t\big|_0^L - w_{ss}.w_{st}\big|_0^L\Big) \\ &\quad + (F_{\hat{t}} - B\kappa^2)(w_s + r_s^0).w_t\big|_0^L - \\ &\quad \int_0^L (F_{\hat{t}} - B\kappa^2)(w_{ss}.w_t + (w_s + r_s^0).w_{st}) \\ &\quad + \int_0^L (F_{\hat{t}} - B\kappa^2)w_{ss}.w_t ds + \int_0^L w_t.q ds + \lambda w_s.w_t\big|_0^L \\ &\quad - \lambda\int_0^L w_{ss}.w_t ds + \frac{\alpha}{2}sw_t.w_t\big|_0^L - \frac{\alpha}{2}\int_0^L w_t.w_t ds. \end{aligned} \tag{36}$$

Since $r_s.r_s = 1$, we have $(w_s + r_s^0).w_{st} = 0$, which is substituted into (36) to yield

$$\begin{aligned}\Delta_1 &= -B\Big(w_{sss}.w_t\big|_0^L - w_{ss}.w_{st}\big|_0^L\Big) + (F_{\hat{t}} - B\kappa^2)(w_s \\ &\quad + r_s^0).w_t\big|_0^L + \lambda w_s.w_t\big|_0^L + \frac{\alpha}{2}sw_t.w_t\big|_0^L + \\ &\quad \int_0^L w_t.q ds - \lambda\int_0^L w_{ss}.w_t ds - \frac{\alpha}{2}\int_0^L w_t.w_t ds. \end{aligned} \tag{37}$$

We now focus on the term Δ_2. Expanding this term gives

$$\Delta_2 = \Delta_{21} + \Delta_{22} + \frac{\alpha}{m_o}\int_0^L sw_s.q ds \tag{38}$$

with

$$\begin{aligned}\Delta_{21} &= -\frac{\alpha B}{m_o}\int_0^L sw_s.w_{ssss}ds, \\ \Delta_{22} &= \frac{\alpha}{m_o}\int_0^L (F_{\hat{t}} - B\kappa^2)_s(w_s + r_s^0)ds \\ &\quad + \frac{\alpha}{m_o}\int_0^L (F_{\hat{t}} - B\kappa^2)w_{ss}ds. \end{aligned}$$

Using integration by part rules, we can calculate the term Δ_{21} as

$$\begin{aligned}\Delta_{21} &= -\frac{\alpha B}{m_o}sw_s.w_{sss}\big|_0^L + \frac{\alpha B}{2m_o}sw_{ss}.w_{ss}\big|_0^L \\ &\quad + \frac{\alpha B}{m_o}w_s.w_{ss}\big|_0^L - \frac{3\alpha B}{2m_o}\int_0^L w_{ss}.w_{ss}ds. \end{aligned} \tag{39}$$

Similarly, the term Δ_{22} is calculated as

$$\begin{aligned}\Delta_{22} &= \frac{\alpha}{m_o}(F_{\hat{t}} - B\kappa^2)sw_s.(w_s + r_s^0)\big|_0^L \\ &\quad - \frac{\alpha}{2m_o}\int_0^L F_{\hat{t}}w_s.w_s ds - \frac{\alpha}{2m_o}\int_0^L F_{\hat{t}}(1 - r_s^0.r_s^0)ds \\ &\quad + \frac{\alpha B}{m_o}\int_0^L w_{ss}.w_{ss}\,w_s.(w_s + r_s^0)ds \end{aligned} \tag{40}$$

where we have used $(w_s + r_s^0).w_{ss} = 0$ and $w_s.w_s + 2r_s^0.w_s + r_s^0.r_s^0 = 1$ since $r_s.r_s = 1$ and $r_{ss}^0 = 0$ due to the riser is initially straight. Now substituting (40) and (39) into (38), then substituting (38) and (37) into (34) results in

$$\begin{aligned}\dot{W} &= -B\Big(w_{sss}.w_t\big|_0^L - w_{ss}.w_{st}\big|_0^L\Big) + (F_{\hat{t}} - B\kappa^2)(w_s \\ &\quad + r_s^0).w_t\big|_0^L + \lambda w_s.w_t\big|_0^L + \frac{\alpha}{2}sw_t.w_t\big|_0^L \\ &\quad - \frac{\alpha B}{m_o}sw_s.w_{sss}\big|_0^L + \frac{\alpha B}{2m_o}sw_{ss}.w_{ss}\big|_0^L \\ &\quad + \frac{\alpha B}{m_o}w_s.w_{ss}\big|_0^L + \frac{\alpha}{m_o}(F_{\hat{t}} - B\kappa^2)sw_s.(w_s + r_s^0)\big|_0^L \\ &\quad - \lambda\int_0^L w_{ss}.w_t ds - \frac{\alpha}{2}\int_0^L w_t.w_t ds \\ &\quad - \frac{3\alpha B}{2m_o}\int_0^L w_{ss}.w_{ss}ds - \frac{\alpha}{2m_o}\int_0^L F_{\hat{t}}w_s.w_s ds \\ &\quad - \frac{\alpha}{2m_o}\int_0^L F_{\hat{t}}(1 - r_s^0.r_s^0)ds + \frac{\alpha B}{m_o}\int_0^L w_{ss}.w_{ss} \\ &\quad \times w_s.(w_s + r_s^0)ds + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L sw_s.q ds. \end{aligned} \tag{41}$$

Before going further, we find maximum and minimum values of $F_{\hat{t}}$, and maximum value of $w_s.(w_s + r_s^0)$ and $r_s^0.r_s^0$. From (6), we have

$$F_{\hat{t}} \le F_{\hat{t}}^{max}, \quad F_{\hat{t}} \ge F_{\hat{t}}^{min} \tag{42}$$

where we have used $0 \le \epsilon(s,t) \le 1$ and $0 \le z(s,t) \le L$ for all $s \in [0, L]$ and $t \ge t_0 \ge 0$. On the other hand, from (7) we have

$$w_s.(r_s^0 + w_s) =\le 1, \quad r_s^0.r_s^0 =\le 1 \tag{43}$$

where we have used the fact that the angle θ between the vectors r and r^0 is in the range $[-\pi/2, +\pi/2]$ due to the initial straight and vertical position of the riser. Using (42) and (43), and $F_{\hat{t}}^{min} > 0$, which holds when T_0 is sufficiently large, i.e.

$$T_0 \ge -\rho_w g\frac{\pi D_o^2}{4}(H_w - L) + \rho_m g\frac{\pi D_i^2}{4}H_m + \bar{T}_0 \tag{44}$$

where $\bar{T}_0$ is a strictly positive constant. Now using the boundary conditions (25), we can write $\dot{W}$ as

$$
\begin{aligned}
\dot{W} \leq\ & U(t).w_t(L,t) + \lambda w_s(L,t).w_t(L,t) \\
& + \frac{\alpha L}{2} w_t(L,t).w_t(L,t) - \frac{\alpha L B}{m_o} w_s(L,t).w_{sss}(L,t) \\
& + \frac{\alpha L}{m_o} F_{\hat{t}}(L,t) w_s(L,t).[w_s(L,t) + r_s^0(L)] \\
& + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L s w_s.q ds - \Big(\frac{\alpha}{2} - \lambda\rho_1\Big) \\
& \times \int_0^L w_t.w_t ds - \Big(\frac{\alpha B}{2m_o} - \frac{\lambda}{4\rho_1}\Big)\int_0^L w_{ss}.w_{ss} ds \\
& - \frac{\alpha F_{\hat{t}}^{min}}{2m_o}\int_0^L w_s.w_s ds
\end{aligned} \tag{45}
$$

where ρ_1 is a positive constant to be specify later. From (45), we choose the boundary control $U(t)$ as follows

$$
U(t) = -k_1 w_t(L,t) - k_2 w_s(L,t) \tag{46}
$$

where k_1 and k_2 are positive constants to be specified later. It is recalled from (25) that $U(t) = -Bw_{sss}(L,t) + F_{\hat{t}}(L,t)[w_s(L,t) + r_s^0(L)]$. Hence from (46), we have

$$
\begin{aligned}
-Bw_{sss}(L,t) =\ & -k_1 w_t(L,t) - k_2 w_s(L,t) \\
& - F_{\hat{t}}(L,t)[w_s(L,t) + r_s^0(L)].
\end{aligned} \tag{47}
$$

Now substituting (46) and (47) into (45) gives

$$
\begin{aligned}
\dot{W} \leq\ & -\Big(k_1 - \frac{\alpha L}{2}\Big) w_t(L,t).w_t(L,t) \\
& - \frac{\alpha L k_2}{m_o} w_s(L,t).w_s(L,t) + \Big(\lambda - k_2 - \frac{\alpha L k_1}{m_o}\Big) \\
& \times w_s(L,t).w_t(L,t) - \Big(\frac{\alpha}{2} - \lambda\rho_1\Big)\int_0^L w_t.w_t ds \\
& - \Big(\frac{\alpha B}{2m_o} - \frac{\lambda}{4\rho_1}\Big)\int_0^L w_{ss}.w_{ss} ds - \frac{\alpha F_{\hat{t}}^{min}}{2m_o} \\
& \times \int_0^L w_s.w_s ds + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L s w_s.q ds.
\end{aligned} \tag{48}
$$

From (48), we specify the positive constants ρ_1, λ, α, k_1 and k_2 such that

$$
\begin{aligned}
& k_1 - \frac{\alpha L}{2} = c_3,\ \lambda - k_2 - \frac{\alpha L k_1}{m_o} = 0,\ \frac{\alpha}{2} - \lambda\rho_1 = c_4, \\
& \frac{\alpha B}{2m_o} - \frac{\lambda}{4\rho_1} = c_5
\end{aligned} \tag{49}
$$

where c_3, c_4 and c_5 are strictly positive constants. Using the conditions given in (49) and the upper bound of W given in (32), we can write (48) as follows:

$$
\begin{aligned}
\dot{W} \leq\ & -c_3 w_t(L,t).w_t(L,t) - \frac{\alpha L k_2}{m_o} w_s(L,t).w_s(L,t) \\
& - cW + \int_0^L w_t.q ds + \frac{\alpha}{m_o}\int_0^L s w_s.q ds
\end{aligned} \tag{50}
$$

where

$$
c = \frac{\min\Big(c_4, c_5, \frac{\alpha \bar{T}_0}{2m_o}\Big)}{\max\Big(\big(\frac{m_o}{2} + \alpha L\rho_0\big), \frac{B}{2}, \big(\frac{\lambda}{2} + \frac{\alpha L}{4\rho_0}\big)\Big)} \tag{51}
$$

where $\bar{T}_0$ is the strictly positive constant in (44). Before going further, we show that there always exist constants ρ_0, ρ_1, λ, α, k_1 and k_2 such that the conditions specified in (33) and (49) hold with c_i, $i = 1, ..., 5$ strictly positive constants. For simplicity, we choose $\rho_0 = L\sqrt{\frac{m_o}{B}}$ and $\rho_1 = \sqrt{\frac{m_o}{4B}}$. A calculation shows that as long as the positive constants λ, α, k_1 and k_2 are chosen such that the following inequalities strictly hold:

$$
\begin{aligned}
& \alpha < \frac{1}{2L^2}\sqrt{\frac{B}{m_o}},\quad \frac{\alpha}{2}\sqrt{\frac{B}{m_o}} < \frac{\alpha L k_1}{m_o} + k_2 < \alpha\sqrt{\frac{B}{m_o}}, \\
& k_1 > \frac{\alpha L}{2},\quad \lambda = \frac{\alpha L k_1}{m_o} + k_2
\end{aligned} \tag{52}
$$

then there exist strictly positive constants c_i, $i = 1, ..., 5$ satisfying the conditions specified in (33) and (49). We are ready to state the main result of our paper in the following theorem whose proof is omitted due to space limitation.

Theorem 1. Under Assumption 1, the boundary control $U(t)$ given in (46) solves the control objective provided that the initial tension T_0 is sufficiently large, i.e. the condition (44) holds, and the design constants k_1 and k_2 are chosen such that the conditions given in (52) hold.

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