Proceedings of the 17th World Congress
The International Federation of Automatic Control
Seoul, Korea, July 6-11, 2008

Control Lyapunov Functions: New Framework for Nonlinear Controller Design

Yuqing He* Jianda Han**

{ State Key Laboratory of Robotics, Shenyang Institute of Automation, Chinese Academy of Sciences, Shenyang, 110016, China ; The Graduate School of CAS. (Tel: 86-24-23970126; e-mail: heyuqing@sia.cn).*
*** State Key Laboratory of Robotics, Shenyang Institute of Automation, Chinese Academy of Sciences, Shenyang, 110016, China (e-mail: jdhan@sia.cn)*

Abstract: Control Lyapunov function (CLF) is a successful attempt to directly use of the Lyapunov function stability analysis technique of nonlinear systems in the synthesis problem. In this paper, on the basis of Freeman's work (1996), the concept of CLF are re-analyze through using the method of set-valued analysis. And then, a new CLF based nonlinear controller design framework, called generalized pointwise min-norm (GPMN), is proposed. Simultaneously, three robust GPMN controllers are introduced with respect to respectively parameter uncertainties, external disturbance, and the combining cases. Actually, the framework provides us a new idea of nonlinear controller design since within which many other controller design indexes can be combined without re-considering the closed loop stability. Finally, a simple simulation is conducted to show one of the typical applications.

1. INTRODUCTION

Nonlinear controller design has been playing a more and more important role in control sciences. This is mainly due to the characteristics of complexity and absence of commonness of nonlinear systems, which make most nonlinear control strategies only applicable for certain kind of systems.

As far as the stability analysis of nonlinear systems is concerned, Lyapunov function has undoubtedly become one of the most successful tools since its debut in the early of last century. However, the use of Lyapunov function in nonlinear controller design is often passive, because in most cases, control strategy is presented firstly, and the stability of the closed loop is subsequently proven by heuristically searching a Lyapunov function of it.

Control Lyapunov function (CLF) is a new given concept in the 1980s in order to directly use of the Lyapunov function to the nonlinear system synthesis. A CLF of the nonlinear system is a positively definite function of states such that there always are some control inputs to make its derivative, along with the trajectory of the system, less than zero. Nonlinear controller design method based on CLF was firstly researched in the year of 1983 by Artstein (Artstein, 1983), where the equivalence between the continuous stabilization of a nonlinear system and the existence of a CLF was firstly proven. Although Artstein did not give any methods to obtain such a continuous stable controller, it still had been a milestone in the nonlinear controller design because several famous formulas of strategies appeared not long after that.

In 1986, Sontag firstly gave a 'universal' construction method for Artstein's theorem (Sontag, 1989). And then, Freeman introduced a so called pointwise min-norm control (PMN) based on a known CLF (Freeman, 1996). And recently, in 2004, Curtis (Curtis, 2004) proposed another strategy by combining the concept 'satisficing' with CLF to obtain a new controller design strategy.

Unfortunately, all of the preceding controllers have a common drawback--lack of flexibility. That means, the controller's parameters are not enough, and which make these methods difficult to be used in most of practical plants.

In Freeman's work, the nonlinear controller design is analyzed by set-valued analysis technique, within which the concept of CLF can be explained more clearly and naturally: in every 'feasible' state, a 'permitted' control action set can be found, and thus, the CLF itself defines a set-valued map from the states to the inputs. Thus, the controller design based on CLF is just to select a proper single-valued function from the set-valued map.

In this paper, we will generalize Freeman's work and obtain a new nonlinear controller design framework based on CLF – called generalized pointwise min-norm control (GPMN). And the continuity of the controller is proved by using the corresponding results in set-valued analysis. And then, the robust versions with respect to the parameters uncertainty and the H∞ version with respect to the external disturbances are presented.

2. CLF AND SET VALUED MAPS

Considering the following input affine system,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ u &\in U(x) \subset R^m \end{aligned} \tag{1}$$

where $x \in R^n$ is the state vector; u is the control input vector; $U(x)$ is the input constraint, which is possible different from

978-1-1234-7890-2/08/$20.00 © 2008 IFAC

10.3182/20080706-5-KR-1001.0260

state to state; $f(*)$ and $g(*)$ are both smooth functions. A CLF of system (1) can be defined as follows,

Definition 1 (Artstein, 1983):

A CLF of system (1) is a C^1 and positively definite function $V(x)$ with $V(0)=0$, which is defined on a neighbourhood W of 0 and satisfies the following inequality,

$$\inf_{u \in U(x)} [V_x(x) f(x) + V_x(x) g(x) u] < 0 \tag{2}$$

Furthermore, $V(x)$ is called a global CLF if W can be chosen R^n with $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. ■

Given $c \in R^+$, we denote Ω_c as follows,

$$\Omega_c := \{x \in R^n : V(x) \le c\} \tag{3}$$

$$cm := \max_c \{c \in R_+ : \Omega_c \subseteq W\} \tag{4}$$

Then, Ω_{cm} is used to approximate the maximum stabilizable region of system (1). With a known CLF, Freeman's PMN controller can be denoted as Eq. (5),

$$u_{PMN}(x) = \arg\min_{u \in U(x)} [V_x(x) f(x) + V_x(x) g(x) u \le -\sigma(x)] \tag{5}$$

where $\sigma(x)$ is a positively definite and continuous function such that $\sigma(0) = 0$. And the continuity of controller (5) can be obtained from the Lemma I (Freeman, 1996),

Lemma I,

If the input constraint $U(x)$ is lower semi-continuous set–valued map (Aubin, 1990) with convex and closed values; Graph(U) is closed; simultaneously, $V(x)$ is a CLF of system (1) on Ω_{cm}. Then controller (5) is continuous in $\Omega_{cm}\backslash\{0\}$. Furthermore, if $V(x)$ satisfies the following small control property (scp),

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < ||x|| < \delta$ implies,

$$\inf_{u \in U(x)} \{V_x(x) f(x) + V_x(x) g(x) u : \|u\| < \varepsilon\} < 0 \tag{6}$$

Then, controller (5) is continuous everywhere in Ω_{cm}. ■

Before proving Lemma I, we first give Theorem II,

Theorem II,

If $U(x)$ is lsc with convex closed values; Graph(U) is closed; $V(x)$ is a CLF of system (1). Thus, the set valued map $K_V(x)$, defined as,

$$K_V(x) = \{u \in U(x) : V_x(x) f(x) + V_x(x) g(x) u \le -\sigma(x)\} \tag{7}$$

is lsc with nonempty closed convex values in $\Omega_{cm}\backslash\{0\}$. And, Graph($K_V$) is closed. ■

Proof of Theorem II can be found in Appendix, and with Theorem II, we can give the proof of Lemma I as follows,

Proof of Lemma I: From Theorem II, $K_V(x)$ is lsc with nonempty closed convex values in $\Omega_{cm}\backslash\{0\}$, and, Graph($K_V(x)$) is closed. Thus, from Lemma 9.3.1 (Aubin, 1990), $x \rightarrow ||u_{PMN}(x)||$ is usc, which implies $u_{PMN}(x)$ is locally bounded. Because the input space is finite dimensional, the closeness of Graph($u_{PMN}(x)$) and the local boundedness of $u_{PMN}(x)$ imply the continuity of $u_{PMN}(x)$ (Theorem 2.3.2, Li, 2003). Next suppose $V(x)$ satisfies the scp, then, from Arstrein's theorem, there exists a continuous controller $k(x)$ such that $k(0) = 0$. And from the definition of $u_{PMN}(x)$, $0 \le ||u_{PMN}(x)|| \le ||k(x)||$, therefore $u_{PMN}(x)$ is continuous at $x = 0$. ■

3. GENERALIZED POINTWISE MIN-NORM CONTROL

In this section, we will give the generalized version of Freeman's PMN controller by introducing a guide function. The main results can be denoted in the following theorem.

Theorem III:

If $V(x)$ is a CLF of system (1) on Ω_c, and the following conditions are satisfied,

(1) $U(x)$ is lsc with closed and convex values;
(2) Graph(U) is closed;
(3) $\xi(x)$: $R^n \rightarrow R^m$ is continuous with $\xi(0)=0$;
(4) $\sigma(x)$ is positively definite and continuous.

Then the following controller (8), called GPMN,

$$u_\xi(x) = \arg\min_{u \in K_V(x)} \{\|u - \xi(x)\|\} \tag{8}$$

can stabilize system (1). And, it is continuous on Ω_c except possibly at $x = 0$. Furthermore, if $V(x)$ satisfies the scp, its continuity can also be ensured at zero state. ■

Before proving the Theorem III, we will first give the following Theorem IV.

Theorem IV:

If $V(x)$ is a CLF of system (1) on Ω_{cm}, $V(x)$ is a CLF of the following system (9) on Ω_{cm},

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi(x) + g(x)u \\ u &\in U(x) - \xi(x) \subset R^m \end{aligned} \tag{9}$$

Furthermore, if $\xi(x)$ is continuous, then

$$\begin{aligned} \overline{K}_V(x) = \{u \in U(x) - \xi(x) : V_x(x)[f(x) + g(x)\xi(x)] + \\ V_x(x) g(x) u \le -\sigma(x)\} \end{aligned} \tag{10}$$

is lsc with nonempty closed convex values on $\Omega_{cm}\backslash\{0\}$. And, Graph($\overline{K}_V$) is closed.

Proof of Theorem IV: From the definition of CLF, $V(x)$ is a CLF of system (1) on Ω_{cm} means,

$$\inf_{u \in U(x)} [V_x(x) f(x) + V_x(x) g(x) u] < 0 \tag{11}$$

Eq.(11) is equivalent to the following inequality,

$$\inf_{u \in U(x) - \xi(x)} [V_x(x) f(x) + V_x(x) g(x)\xi(x) + V_x(x) g(x) u] < 0 \tag{12}$$

That means $V(x)$ is a CLF of system (9).

From Theorem A-1, we know that since $U(x)$ is lsc with convex closed values, so does $U(x)$ - $\xi(x)$. And from Theorem II and continuity of $\xi(x)$, the second part of Theorem IV can also be proved. ■

Proof of Theorem III: Let $V(x)$ be a Lyapunov function candidate, and the derivative of $V(x)$ is

$$\dot{V}(x) = V_x f(x) + V_x g(x) u_\xi$$

From Eq. (7) and Eq. (8), we have

$$\dot{V}(x) = V_x f(x) + V_x g(x) u_\xi \le -\sigma(x)$$

Thus, the positively definite $\sigma(x)$ ensures the derivative of $V(x)$ is negative, i.e., controller (8) can stabilize system (1).

From the conditions of Theorem III and Theorem IV, $\bar{K}_V(x)$ is lsc with nonempty closed convex values in $\Omega_{cm}\backslash\{0\}$, and, Graph($\bar{K}_V$) is closed. Simultaneously, controller (9) is the minimal selection of $\bar{K}_V(x)$. Thus, by Theorem I - Theorem III, the continuity of controller (9) can be proved. ■

Remark: GPMN control is a direct generalization of Freeman's PMN controller by introducing a guide function $\xi(x)$. And this will bring great advantages and flexibilities to the CLF based nonlinear controller design. Up to known, at least two applications can be found: first, with GPMN structure, we can enforce the stability results of some heuristic controller design method, such as local linearization (He, Y.Q, 2007a), SDRE, MPC (He, Y.Q, 2007b); secondly, it can be used to improve the performance of some known controller design methods. We can combine some other controller design strategies with a known CLF which is obtained from some others controller design strategies.

4. ROBUST GPMN CONTROLLER

In this section, three different robust GPMN controllers will be given, including Parameters Uncertainty Robust GPMN; Disturbance Robust GPMN; and Combining Robust GPMN.

4.1 Parameter Uncertainties Robust GPMN

Considering the following uncertainty system with unknown parameters θ,

$$\begin{aligned} &\dot{x} = f(x,\theta) + g(x,\theta)u \\ &\theta \in \Xi(x) \subset R^p \\ &u \in U(x) \subset R^m \end{aligned} \tag{13}$$

where $\Xi(x)$ is the parameter uncertainty set; $U(x)$ is the input constraint; $f(*,*)$ and $g(*,*)$ are known smooth functions.

Freeman has given the following definition of Robust CLF (RCLF) to system (13). Here we call it parameter uncertainty - robust control Lyapunov function (PU-RCLF).

Definition II,

A C^1 and positively definite function $V(x)$ is a PU-RCLF for system (13) if there exist two positive numbers c_1 and c_2 ($c_1 > c_2$) such that,

$$\inf_{u\in U(x)} \sup_{\theta\in\Theta(x)} [V_x f(x,\theta) + V_x g(x,\theta)u] < 0 \tag{14}$$

for all $x \in \Omega_{c1}\backslash\Omega_{c2}$. Furthermore, $V(x)$ is called a global PU-RCLF if c_1 can be chosen $+\infty$ with $V(x)\rightarrow\infty$ as $|x|\rightarrow\infty$. ■

Freeman's PMN controller can also deal with system (13). In this paper, we will not describe more about it, and directly give the PU-GPMN in the following theorem.

Theorem V:

If $V(x)$ is a PU-RCLF of system (13) on $\Omega_{c1}\backslash\Omega_{c2}$, and the following conditions are satisfied,

(1) $U(x)$ is lsc with closed and convex values;
(2) Graph(U) is closed;
(3) $\Xi(x)$ is usc (upper semi-continuous) with nonempty compact values;
(4) $\xi(x)$: $R^n \rightarrow R^m$ is a continuous function such that $\xi(0)=0$;
(5) $\sigma(x)$ is a positively definite continuous function.

Then controller (15)-(16), called PU-RGPMN controller, can stabilize system (13), and is continuous in $\Omega_{c1}\backslash\Omega_{c2}$.

$$u_\xi^{PU}(x) = \arg\min_{u\in K_V^{PU}(x)} \{\|u - \xi(x)\|\} \tag{15}$$

$$\begin{aligned} &K_V^{PU}(x) = \\ &\{u \in U(x) : \max_{\theta\in\Theta(x)} [V_x f(x,\theta) + V_x g(x,\theta)u] \le -\sigma(x)\} \end{aligned} \tag{16}$$

Furthermore, if c_2 equals to zero, and $V(x)$ satisfies the following PU-scp,

for every ε>0 there exists δ>0 such that 0<||x||<δ implies

$$\inf\{\max_{\theta\in\Theta(x)} [V_x f(x,\theta) + V_x g(x,\theta)u] : u \in U(x), \|u\| < \varepsilon\} < 0$$

the continuity can be ensured in every point of Ω_{c1}. ■

4.2 H_∞ Robust GPMN

Another important robust control problem is to design disturbance attenuation controller for the following systems with external disturbances,

$$\begin{aligned} &\dot{x} = f(x) + g(x)u + l(x)\omega \\ &y = h(x) \\ &u \in U(x) \subset R^m \end{aligned} \tag{17}$$

where ω is external disturbance signal, y is output vector. $f(*)$, $g(*)$, $l(*)$, and $h(*)$ are all known smooth functions. One of the well-known methods to design a disturbance attenuation controller of system (17) is to use the concept of finite gain L stability and H_∞ control (Khalil, 2002).

In order to construct robust GPMN controller for system (17), we define the following H_∞-RCLF,

Definition III,

A C^1 and positively definite function $V(x)$ is an H∞-RCLF for the system (13) if there exist positive numbers c such that,

$$\inf_{u\in U(x)} \{V_x[f(x) + g(x)u] + \frac{1}{2\gamma^2} V_x l(x) l^T(x) V_x^T + \frac{1}{2} h^T(x) h(x)\} < 0 \tag{18}$$

For all $x \in \Omega_c$. Furthermore, $V(x)$ is called a global H∞ CLF if c can be chosen $+\infty$ with $V(x)\rightarrow\infty$ as $|x|\rightarrow\infty$. ■

And the corresponding H∞-RGPMN control is given in theorem VI,

Theorem VI

If $V(x)$ is a H∞-RCLF of system (18) in Ω_c, and the following conditions are satisfied,

(1) $U(x)$ is lsc with closed and convex values;
(2) Graph(U) is closed;
(3) $\xi(x)$: $IR^n \rightarrow IR^m$ is continuous with $\xi(0)=0$;
(4) $\sigma(x)$ is positive definite and continuous.

Then controller (19)-(20), called H∞-RGPMN controller, make system (18) finite gain L stable with the gain that is less than or equal to γ. And, it is continuous in Ω_c.

$$u_\xi^{H\infty}(x) = \arg\min_{u\in K_V^{H\infty}(x)} \{\|u - \xi(x)\|\} \tag{19}$$

$$K_V^R(x) = \{u \in U(x): V_x[f(x)+g(x)u] + \frac{1}{2\gamma^2}V_x l(x) l^T(x) V_x^T + \frac{1}{2}h^T(x)h(x) \le -\sigma(x)\} \tag{20}$$

Furthermore, if $V(x)$ satisfies the following H∞-scp,

for every ε > 0 there exists a δ > 0 such that 0 < ||x|| < δ implies,

$$\inf_{u\in U(x)}\{V_x[f(x)+g(x)u] + \frac{1}{2\gamma^2}V_x l(x) l^T(x) V_x^T + \frac{1}{2}h^T(x)h(x) : \|u\| < \varepsilon\} < 0 \tag{21}$$

its continuity can be ensured in Ω_c. ∎

Proof of Theorem VI: Firstly, the finite gain L stability of the closed loop with $u_\xi^{H\infty}(x)$ is obviously by taking $V(x)$ as a Lyapunov function.

Just like Theorem A-1, $K_V^{H\infty}(x)$ can be proved to be lsc with convex closed values. Also, its graph can be proved to be closed. Thus, the same process in the proof of Theorem III can be used to prove the continuity results of the theorem. ∎

4.3 Combining Robust GPMN

For the combining case with both parameter uncertainties and external disturbances, the uncertainty system can be denoted as follows,

$$\begin{aligned} &\dot{x} = f(x,\theta) + g(x,\theta)u + l(x,\theta)\omega \\ &y = h(x,\theta)u \\ &\theta \in \Xi(x) \subset R^p \\ &u \in U(x) \subset R^m \end{aligned} \tag{22}$$

where, ω is external disturbance signal, y is output vector; $\Theta(x)$ is the parameter uncertainty set; $U(x)$ is the input constraint; $f(*,*)$, $g(*,*)$, $l(*,*)$ and $h(*,*)$ are all known smooth functions.

We give the following definition of Combined RCLF (C-RCLF) and the corresponding C-RGPMN controller in Theorem VI.

Definition IV

A C^1 and positively definite function $V(x)$ is an C-RCLF for system (22) if there exist two positive numbers c_1 and c_2 ($c_1 > c_2$) such that,

$$\inf_{u\in U(x)}\sup_{\theta\in\Theta(x)}\{V_x[f(x,\theta)+g(x,\theta)u] + \frac{1}{2\gamma^2}V_x l(x,\theta) l^T(x,\theta) V_x^T + \frac{1}{2}h^T(x,\theta)h(x,\theta)\} < 0 \tag{23}$$

is satisfied for all $x \in \Omega_{c1}\backslash\Omega_{c2}$. Furthermore, $V(x)$ is called global RCLF if c_1 can be chosen ∞ with $V(x)\rightarrow\infty$ as $|x|\rightarrow\infty$. ∎

Theorem VII

If $V(x)$ is a C-RCLF of system (22) in $\Omega_{c1}\backslash\Omega_{c2}$, and the following conditions are satisfied,

(1) $U(x)$ is lsc with closed convex values;
(2) Graph(U) is closed;
(3) $\Xi(x)$ is usc with nonempty conpact values;
(4) $\xi(x)$: $IR^n\rightarrow IR^m$ is a continuous function with $\xi(0)=0$;

$\sigma(x)$ is a positive definite and continuous function.

Then controller (24)-(25), called C-RGPMN controller, can stabilize system (22). And, it is continuous in $\Omega_{c1}\backslash\Omega_{c2}$.

$$u_\xi^C(x) = \arg\min_{u\in K_V^C(x)}\{\|u-\xi(x)\|\} \tag{24}$$

$$K_V^C(x) = \{u \in U(x): \max_{\theta\in\Theta(x)}[V_x f(x,\theta) + V_x g(x,\theta)u + 1/(2\gamma^2)V_x l(x,\theta)l^T(x,\theta)V_x^T + 1/2h^T(x,\theta)h(x,\theta)] \le -\sigma(x)\} \tag{25}$$

Furthermore, if $V(x)$ satisfies the following C-scp,

for every ε>0 there exists δ>0 such that 0<||x||<δ implies

$$\inf\{\max_{\theta\in\Theta(x)}[V_x f(x,\theta) + V_x g(x,\theta)u + 1/(2\gamma^2)V_x l(x,\theta)l^T(x,\theta)V_x^T + 1/2h^T(x,\theta)h(x,\theta)] : u \in U(x), \|u\| < \varepsilon\} < 0$$

and c_1 is zero, its continuity can be ensured in Ω_{c2}. ∎

The proof of Theorem VII is similar to the proof of Theorem V and Theorem VI, and we will not repeat it.

5. SIMULATION

As known, in most of real applications, a widely used nonlinear controller design strategy is to design a linear controller based on a local linearization model. One of the advantages is that it will greatly reduce the complexity of controller design. However, the small and uncontrollable stability region is one of the bottlenecks of its application. In this section, we will give an example to show that with the idea of GPMN framework, this problem can be partly solved. Considering the following system equations,

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = -9.8\sin x_1 - kx_2 + 10u \\ &k \in [0, 0.5],\ u \in [-10, 10] \end{aligned} \tag{26}$$

Select a PU-RCLF of system (27) as:

$$V(x) = x^T\begin{bmatrix}1.5 & 0.5\\ 0.5 & 1\end{bmatrix}x \tag{27}$$

and $\sigma(x) = 0.1x_1^2+0.1x_2^2$.

By simple computation, Ω_c (c=15) is a stabilizability region of system (26). On the other hand, the local linearization model near origin can be denoted as:

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = -9.8x_1 - kx_2 + 10u \end{aligned} \tag{28}$$

Taking k = 0.25 in (26), we can design a linear H∞ controller,

$$u = -0.1332x_1 - 0.1435x_2 \tag{29}$$

Then, the PU-GPMN controller can be denoted as,

$$\begin{aligned} &m(x) = \min_{u\in[-10,10]}|u + 0.1332x_1 + 0.1435x_2| \\ &s.t.\ \max_{k\in[0,0.5]}[(1.5x_1+0.5x_2)x_2 - (0.5x_1+x_2)(9.8\sin x_1 + kx_2 - 10u) + 0.05x_1^2 + 0.05x_2^2] \le 0 \end{aligned} \tag{30}$$

Fig.1 denotes its simulation. Where initial state is (2,-4), and dash-dot line is the case when k=0.5; solid line is the case of k=0. From which, we can see that the PU-GPMN controller keeps the robustness of H∞ controller near zero state and ensures the closed loop stability in a comparatively large region Ω_{15}.

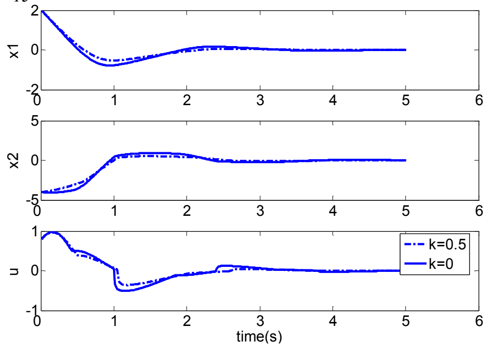

Fig.1. Time response

6. CONCLUSIONS

The purpose of this paper is to summarize and extend some of author's previous works about the nonlinear control based on a known CLF. It is shown that, by introducing a guide function, the Freeman's PMN controller can be generalized to a more flexible form. Also, it can be seen that two typical uncertainties, including the parameter uncertainties and external disturbances, can both be dealt with in the new framework. Finally, it is pointed out that two applications can be referred to from the new framework: First, it can be used to enforce the stability results of some heuristic controller design method; secondly, it can be used to improve the performance of some known controller design method. And the simulation results verify the feasibility of the new framework.

REFERENCES

Artstein, Z (1983). Stabilization with relaxed controls. *Nonlinear Analysis, Theory, Methods & Applications*, **7(11)**, 1163-1173.

Aubin, J.P. and A. Cellina (1984), Differential inclusions: set-valued maps and viability theory, Springer-Verlag, Berlin.

Aubin, J.P. and H. Frankowska (1990). *Set-valued analysis*, Birhauser, Boston.

Curtis, J.W. and R.W. Beard (2004). Satisficing: a new approach to constructive nonlinear control. *IEEE Transactions on Automatic Control*, **49(7)**, 1090-1102

Freeman, R.A. and P.V. Kokotovic (1996). Inverse optimality in robust stabilization. *SIAM Journal of Control & Optimization*, **34(4)**, 1365-1391.

He, Y.Q., and J.D. Han (2007a), Generalized pointwise min-norm control based on control Lyapunov function, Proceedings of the 26th Chinese Control Conference, Zhangjiajie, China, v2, 404-408

He, Y.Q., and J.D. Han (2007b), Nonlinear model predictive control enhanced by generalized pointwise min-norm scheme, Proceedings of the 46th Conference on Decision and Control, accepted manuscript

Khalil, H.K. (2002), *Nonlinear systems*, 3rd edition, Prentice Hall, NJ

Li, L. and C.X. Wu (2003). *Set – valued analysis*, Science Press, Beijing.

Sontag, E.D (1989). A 'universal' construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, **13(2)**, 117-123.

Appendix

Proof of Theorem II: That $K_V(x)$ is nonempty closed and convex valued can be directly proved by the definition of set valued map (Aubin, 1990).

In order to prove the lower semi-continuity of $K_V(x)$, defining the following set valued maps,

$$L_V^0(x) = \{u \in R^m \mid V_x[f(x)+g(x)u] < -\sigma(x)\} \tag{A-1}$$

$$K_V^0(x) = U(x) \cap L_V^0(x) \tag{A-2}$$

Suppose Ж is an open set in R^m, thus, $x_0 \in (K_V^0)^{-1}$(Ж) implies $U(x_0) \cap L_V^0(x_0) \cap$Ж $\neq 0$. For every $y_0 \in K_V^0(x_0) \cap$Ж, from the openness of $L_V^0(x_0) \cap$Ж, there always exists a positive number η, such that $B(y_0, 2\eta) \subset L_V^0(x_0) \cap$Ж. Simultaneously, for every x, if $V_x g(x) \neq 0$, $V_x f(x) + V_x g(x) u = -\sigma(x)$ is an super-plane in the m dimensional Euclidean space. And, we can compute the Euclidean distance between point y_0 and the super-plane,

$$d_{x,y_0} = \frac{|V_x[f(x)+g(x)y_0]+\sigma(x)|}{V_x g(x) g^T(x) V_x^T} \tag{A-3}$$

First, if $V_{x0}g(x_0) \neq 0$, $B(y_0, 2\eta) \subset L_V^0(x_0) \cap$Ж implies $d_{x0,y0} > 2\eta$. Meanwhile, from the continuity of $V_x(x)$, $f(x)$, $g(x)$, and $\sigma(x)$, there exists a neighbourhood Д$_1$ of x_0 such that for every $x' \in$ Д$_1$, $d_{x',y0} > \eta$, i.e., $B(y_0, \eta) \subset L_V^0(x') \cap$Ж. Second, if $V_{x0}g(x_0) = 0$, and $x \neq 0$, then $L_V^0(x_0) = IR^m$, and $V_x f(x) + \sigma(x) < 0$ implies $d_{x,y0}$ is positive infinite. Thus, it is obvious that a neighbourhood Д$_1$ of x_0 can be found such that $B(y_0, \eta) \subset L_V^0(x') \cap$Ж.

$U(x)$ is lsc means we can find a neighbourhood Д$_2$ of x_0 such that $U(x) \cap B(y_0, \eta) \neq \Phi$. Let Д := Д$_1 \cap$Д$_2$, thus, for every $x' \in$ Д, we have $K_V^0(x') \cap$Ж $\neq \phi$, i.e., $x_0 \in$ Д $\subset (K_V^0)^{-1}$(Ж). That is, $(K_V^0)^{-1}$ (Ж) is an open set. And from theorem 1.3.6 of (Li, 2003), $K_V^0(x)$ is lsc on $\Omega_{cm,W} \backslash \{0\}$. $K_V(x)$ is the closure of $K_V^0(x)$, from proposition 1.3.10 of (Li, 2003), $K_V(x)$ is lsc too.

From the continuity of $V_x(x)$, $f(x)$, and $g(x)$, the function $V_x(x)[f(x)+g(x)u]$ is continuous with respect to x and u. Thus, if $V_x(x)[f(x_1)+g(x_1)u] > \sigma(x_1)$, there exist two open neighborhoods Д$_3$ and Д$_4$ of x_1 and u_1, such that for every $x \in$ Д$_3$ and $u \in$ Д$_4$, $D(x_1, u_1) > \sigma(x_1)$, that means Graph($L_V$) is closed. Thus, that Graph(U) is closed implies Graph(K_V) is closed. ■

Theorem A-1,

Let T be a set valued map from a metric space X to a Banach space Y that is strictly convex and reflexive. If $\zeta(x)$: $X \rightarrow Y$ is a single continuous function. Then the newly defined set valued map, R: $X \rightarrow Y$ ($R(x) = T(x) - \zeta(x)$) such that

(1) If T is a closed and convex valued, so does R;
(2) If T is lsc, so does R;
(3) If Graph(T) is closed, so does graph(R). ■

Proof of Theorem A-1:(1) can be directly obtained by the definition of set valued map in (aubin, 1990).

If T is lsc, i.e., for every sequence $\{x_i\} \in X$ converging to x and every $y \in R(x)$ there exists a sequence $\{y_i\} \in Y$ converging to $y + \zeta(x)$ and $N \geq 1$ such that $y_i \in T(x_i)$ for all $i \geq N$. On the one hand, from the definition of $R(x)$, $y_i - \zeta(x_i) \in T(x_i) - \zeta(x_i) = R(x_i)$, and from the continuity of $\zeta(\ )$, $\zeta(x_i)$ converges to $\zeta(x)$, thus, we conclude that the sequence $\{\bar{y}_i = y_i - \zeta(x_i)\} \in Y$ converges to y, and when $i \geq N$, $\bar{y}_i \in R(x_i)$. Thus, we completed the proof of (2).

Now we will prove (3). For every $(t, l) \in X \times Y$-Graph(R), we have $(t, l+\zeta(t)) \in X \times Y$-Graph($T$), from the closing of graph T, $X \times Y$-Graph(T) is open, thus there exists a positive number ε such that $B((t, l+\zeta(t)), \varepsilon) \subset X \times Y$-Graph($T$). From the definition of R, $B((t, l), \varepsilon) \subset X \times Y$-Graph($R$), thus, $X \times Y$-Graph(R) is open in the product space $X \times Y$, i.e., Graph(R) is closed. ■

Theorem A-2:

If the following conditions are satisfied,
i) $\Theta(x,u,\omega)$ and $\sigma(x)$ are continuous vector-value function;
ii) For each x and ω, the mapping $u \rightarrow \Theta(x,u,\omega)$ is affine;
iii) $W(x)$ is a continuous with nonempty compact values;
iv) $U(x)$ is lsc, closed valued and convex valued set valued map, and Graph(U) is closed;
v) $K(x) \cap U(x) \neq \Phi$.
Then the following results can be obtained,
i) $K(x) \cap U(x)$ is lsc on $\Omega_{c2} \backslash \Omega_{c1}$;
ii) $K(x) \cap U(x)$ is closed valued and convex valued;
iii) Graph($K(x) \cap U(x)$) is closed.
where $K(x)$ is defined as follows,

$$D(x,u) \triangleq \max_{\omega \in W(x)} \Theta(x,u,\omega)$$

$$K(x) = \{u \in IR^m : D(x,u) \leq -\sigma(x)\}$$ ■

Proof of Theorem A-2: We first define the following new set valued map,

$$K^0(x) = \{u \in IR^m : D(x,u) \leq -\sigma(x)\}$$

from conditions i) and iii), Theorem 1.4.16 (Aubin, 1990) ensures the upper semi-continuity of $D(x,u)$. Thus, for every x_0 and u_0 such that $D(x_0, u_0) < -\sigma(x_0)$, we have,

$$\limsup_{x \to x_0, u \to u_0} D(x,u) + \sigma(x) \leq D(x_0, u_0) + \sigma(x_0) < 0$$

that means, there exists open neighbourhoods of x_0 and u_0 Ж$_1$ and Ж$_2$ such that for every $x \in$ Ж$_1$ and $u \in$ Ж$_2$, the following inequality is satisfied,

$$D(x,u) + \sigma(x) < 0$$

That means, Graph($K^0(x)$) is open.

For every point $x_0 \in IR^n$ and every open set $N \subset IR^m$ such that $U(x_0) \cap K^0(x_0) \cap N \neq \Phi$, the lower semi-continuity of $U(x)$ implies that there always exists an open neighbourhood Ж$_3$ of x_0 such that $U(\xi) \cap N \neq \Phi$ for all $\xi \in$ Ж$_3$. On the other hand, since Graph($K^0(x)$) is open, from Proposition 1.10.1 and Prop 1.10.3 (Aubin, 1984), $K^0(x)$ is lsc, thus there exists an open neighbourhood Ж$_4$ of x_0 such that $K^0(\xi) \cap N \neq \Phi$, for all $\xi \in$ Ж$_4$. Select Ж$_5$ = Ж$_3 \cap$ Ж$_4$, we have, $U(\xi) \cap K^0(\xi) \cap N \neq \Phi$ for all $\xi \in$ Ж$_5$, i.e., $K^0(x) \cap U(x)$ is lsc.It is not difficult to prove that $\overline{K^0(x) \cap U(x)} = K(x) \cap U(x)$, and from Prop1.3.10 (Li, 2003), we can conclude that $K(x) \cap U(x)$ is also lsc set valued map. Thus, we complete the proof of i).

Secondly, we will prove the convexness of $K(x) \cap U(x)$. First, from condition ii), for any $0 \leq k \leq 1$, we have

$$\begin{aligned} & D(x, ku_1 + (1-k)u_2) \\ &= \max_{\omega \in W(x)} [k\Theta(x,u_1,\omega) + (1-k)\Theta(x,u_2,\omega)] \\ &\leq k \max_{\omega \in W(x)} \Theta(x,u_1,\omega) + (1-k) \max_{\omega \in W(x)} \Theta(x,u_2,\omega) \\ &= kD(x,u_1) + (1-k)D(x,u_2) \end{aligned}$$

i.e., $D(x,u)$ is convex. Further, if $u_1 \in K(x)$ and $u_2 \in K(x)$, for any $0 \leq k \leq 1$, we have

$$\begin{aligned} & D(x, ku_1 + (1-k)u_2) + \sigma(x) \\ &\leq kD(x,u_1) + (1-k)D(x,u_2) + \sigma(x) \\ &\leq kD(x,u_1) + k\sigma(x) + (1-k)D(x,u_2) + (1-k)\sigma(x) \\ &\leq 0 \end{aligned}$$

i.e., $K(x)$ is convex. Thus, the convexness of $K(x) \cap U(x)$ can be easily obtained by the convexness of $U(x)$. And the closeness of $K(x) \cap U(x)$ is clear from the closeness of $K(x)$ and $U(x)$. Thus, we have proved ii). If $W(x)$ is continuous, from Theorem 1.4.16 (Aubin, 1990), $D(x,u)$ is continuous, thus Graph($K(x)$) is closed, and then, Graph($K(x) \cap U(x)$)= Graph($K(x)$)$\cap$Graph($U(x)$) is closed too. Up to now, we have completed the proof of i)-iii) of the theorem. ■

Theorem A-3

If all the conditions of Theorem A-2 are satisfied. Furthermore, $c_1 = 0$, and the following property is satisfied,

For every $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < \|x\| < \delta$ implies

$$\inf\{D(x,u) : u \in U(x), \|u\| < \varepsilon\} < 0$$

Then, $K(x) \cap \rho(x)D(x)$ is lsc with closed and convex values on Ω_{c2}, where $\rho : \overline{IR^-} \to [0,1]$ is monotonically increasing function such that $\rho(t) = 0$ if and only if $t = 0$. ■

Proof of Theorem A-3: First, we show that $\rho(\|x\|)U(x)$ is lsc. For every sequence {xi} converging to x_0 (not zero), and every $y \in \rho(\|x_0\|)U(x_0)$, we have $y/\rho(\|x_0\|) \in U(x_0)$. And from the lower semi-continuity of $U(x)$, there always exists a sequence $\{y_i\}$ converging to $y/\rho(\|x_0\|)$ and $N \geq 1$ such that $y_i \in U(x_i)$ for all $i \geq N$. And from the continuity of $\rho(.)$, sequence $\{\rho(\|x_i\|)y_i\}$ converges to y; $\rho(\|x_i\|)y_i \in \rho(\|x_i\|)U(x_i)$ for all $i \geq N$, i.e., $\rho(\|x\|)U(x)$ is lsc at x_0. That means $\rho(\|x\|)U(x)$ is lsc. If $x_0 = 0$, for every sequence $\{x_i\}$ converging to 0, we have $0 \in U(x_0)$, from the lower semi-continuity of $U(x)$, there always exists a sequence $\{y_i\}$ converging to 0 and $N \geq 1$ such that $y_i \in U(x_i)$ for all $i \geq N$. And from the continuity of $\rho(.)$, sequence $\{\rho(\|x_i\|)y_i\}$ converges to 0, and $\rho(\|x_i\|)y_i \in 0$ for all $i \geq N$, i.e., $\rho(\|x\|)U(x)$ is lsc at x=0. Thus, $\rho(\|x\|)U(x)$ is lsc. The convexness and closeness of $\rho(\|x\|)U(x)$ can be easily proved from the definition of them.

Thus, we have proved that $\rho(\|x\|)U(x)$ is closed valued, convex valued and lsc on Ω_{c2}, from theorem A-2, $K(x) \cap D(x)$ is lsc on $\Omega_{c2}/\{0\}$. If x=0, we can suppose an open neighbourhood of zero state Ж, thus, by the given property, there exists an open neighbourhood Ж' of zero in IR^n such that for every $x' \in$ Ж', we have $K(x') \cap \rho(\|x\|)U(x) \cap$ Ж $\neq \Phi$, i.e., $K(x') \cap \rho(\|x\|)U(x)$ is lsc at x=0. ■