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Rejection of Persistent Bounded Disturbance for a Class of Time-Delay Systems ⋆

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Abstract: This paper considers the problem of persistent bounded disturbance rejection for a class of time-delay systems by Lyapunov function and positively invariant set analysis method. Sufficient conditions for internal stability and L_1-performance analysis are given in terms of linear matrix inequalities (LMIs). Based on which, a dynamic output-feedback controller is then designed. All the obtained results are delay-dependent, and therefore, are less conservative. A numerical example is included to illustrate the proposed method.

1. INTRODUCTION

Many control problems involve designing a controller capable of stabilizing a given system while minimizing the worst-case response to some exogenous disturbances, see Vidyasagar (1986) and the references therein. When the disturbances involved are persistent bounded with size measured in terms of peak time-domain values, it leads to the problem of peak-to-peak gain minimization, i.e., the L_1 or induced L_∞ problem, formulated by Vidyasagar (1986), Vidyasagar (1991). It has recently attached much attention because of incorporating time domain specifications directly, see, e.g., Bobillo et al. (1992), Abedor et al. (1996), Hao et al. (2003), Blanchini et al. (1995), Tang et al. (2004), Lin et al. (2003). Among these works, analysis and control synthesis problems of persistent bounded disturbance rejection are considered for systems without time delay. On the other hand, time delay is, in many cases, a source of instability. The problem of rejection of persistent bounded disturbance for time-delay systems is, therefore, of theoretical and practical importance. However, there have been very few works concerning the same problem for time delay systems so far, which motivates the present paper.

In this paper, we consider the problem of persistent bounded disturbance rejection for time-delay systems. By Lyapunov function and invariant-set analysis method, we obtain some sufficient conditions on a robust ellipsoidal attractor that ensure the internal stability and the desired L_1 performance of the systems. In particular, the conditions are delay-dependent, therefore, they are less conservative. Based on which, a dynamic output-feedback controller is designed for time-delay systems.

For simplification, we use the following notation. R is the set of all real numbers. R^n is the set of all $n-$tuples of real numbers. The symbol $Sym\{\cdot\}$ denotes $Sym\{X\} \overset{\text{def}}{=} X + X^T$, and the symbol $*$ denotes the symmetric part.

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2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following time-delay system:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_d x(t-d) + B_1 w(t)\\ z(t) &= Cx(t) + Dw(t)\\ x(t) &= x(0) = \phi(t), \forall t \in [-d, 0]\end{aligned} \quad (1)$$

where $x(t) \in R^n$ is the state, $z(t) \in R^r$ is the controlled output signal, $w(t) \in R^l$ is the external disturbance signal, $\phi(t)$ is the initial condition, and A, Ad, B_1, C, D are known constant matrices of appropriate dimensions. The time delay $d > 0$ is assumed to be known. Assume that the admissible disturbance set is $W := \{w : R \to R^{n_w}, \|w\|_\infty \le 1\}$. The L_∞ norm is defined by $\|w\|_\infty =: sup_t\|w(t)\|_2$.

We firstly present the following definition and lemma which will be used in the development of our main results.

Definition 1. A set Ξ is said to be positively invariant for dynamical system if the trajectory $x(t)$ of the system remains in Ξ for all $t > 0$ whenever $x(0) \in \Xi$.

Definition 2. The origin reachable set $(R_\infty(0))$ is said to be the set that the state of the system can reach from the origin. It is the minimal closed positively invariant set containing the origin.

Definition 3. A set Ω is said to be a robust attractor of a system with respect to $w \in W$, if all the state trajectories initiating from the exterior of Ω eventually enter and remain in Ω for all $w \in W$. Obviously, a robust attractor is also a positively invariant set.

Definition 4. For a given scalar $\rho > 0$, we say that system (1) with initial state has ρ performance if $\|z\|_\infty \le \rho$ for all $w \in W$. Define the performance set as follows:

$$\Omega(\rho) = \{x : \|z\|_\infty = \|Cx + Dw\|_\infty \le \rho, \forall w \in W\}$$

Thus, if $R_\infty(0) \subset \Omega(\rho)$, then the system has ρ performance.

Lemma 1. Let P be an $n \times n$ matrix, then, for any scalar $\varepsilon > 0$, it follows that

$$2x^T PB_1 w \le \varepsilon^{-1} x^T PB_1 B_1^T Px + \varepsilon w^T w$$

Lemma 2. Given any real matrices Σ_1, Σ_2, Σ_3 with appropriate dimensions, where $\Sigma_3 > 0$ is a symmetric and positive-definite matrix. Then, the following inequality holds:

$$\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 \le \Sigma_1^T \Sigma_3 \Sigma_1 + \Sigma_2^T \Sigma_3^{-1} \Sigma_2$$

3. MAIN RESULTS

3.1 L_1 Performance Analysis

For a symmetric and positive-definite matrix $P > 0$, let the ellipsoid $\Omega_P = \{x(t) : x^T(t)Px(t) \le 1\}$.

Theorem 1. For prescribed positive scalars $\rho > 0$, $d > 0$, $\alpha > 0$, $\sigma > 0$, if there exist symmetric and positive-definite matrices $P > 0, Q > 0$ such that

$$\begin{bmatrix} (1,1) & -PA_d & 0 & d(A+A_d)^T Q & PB_1 \\ * & -Q & 0 & -dA_d^T Q & 0 \\ * & * & -I & dB_1^T Q & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -\alpha I \end{bmatrix} < 0 \qquad (2)$$

$$\begin{bmatrix} \sigma P & 0 & C^T \\ * & (\rho^2 - \sigma)I & D^T \\ * & * & I \end{bmatrix} \ge 0 \qquad (3)$$

where $(1,1) = Sym\{P(A + A_d)\} + (\alpha + 1)P$. Then system (1) is stable and Ω_P is a robust attractor of it with respect to $w \in W$. Moreover, $\Omega_P \subset \Omega(\rho)$, and hence, the system has ρ performance.

Proof: Consider a Lyapunov-Krasovskii functional candidate $V(t)$ as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

where

$$V_1(t) = x^T(t)Px(t)$$
$$V_2(t) = \int_{t-d}^{t} \left[\int_{s}^{t} \dot{x}^T(\theta)d\theta\right] Q \left[\int_{s}^{t} \dot{x}(\theta)d\theta\right] ds$$
$$V_3(t) = \int_{0}^{d} ds \int_{t-s}^{t} (\theta - t + s)\dot{x}^T(\theta)Q\dot{x}(\theta)d\theta$$

If the time derivative of $V(t)$ along the trajectory of the system (1) is negative for any $x \notin \Omega_P$, then Ω_P is a robust attractor of (1) with respect to $w \in W$. Note the identity (Leibniz-Newton): $\int_a^b \dot{v}(t)dt = v(b) - v(a)$.

By Lemma 1, we have the following equation for any scalar $\alpha > 0$

$$\begin{aligned} \dot{V}_1(t) &= 2x^T(t)P[(A + A_d)x(t) - A_d \int_{t-d}^{t} \dot{x}(\theta)d\theta + B_1 w(t)] \\ &\le 2x^T(t)P(A + A_d)x(t) - 2x^T(t)PA_d \int_{t-d}^{t} \dot{x}(\theta)d\theta \\ &\quad +\alpha^{-1}x^T(t)PB_1B_1^TPx(t) + \alpha w^T(t)w(t) \\ &= \xi^T \Xi_1 \xi - (\alpha + 1)[x^T(t)Px(t) - w^T(t)w(t)] \end{aligned}$$

where

$$\xi = \begin{bmatrix} x^T(t) & \int_{t-d}^{t} \dot{x}^T(\theta)d\theta & w^T(t) \end{bmatrix}^T$$

$$\Xi_1 = \begin{bmatrix} (1,1) & -PA_d & 0 \\ * & 0 & 0 \\ * & * & -I \end{bmatrix}$$

$$(1,1) = Sym\{P(A + A_d)\} + \alpha^{-1}PB_1B_1^TP + (\alpha + 1)P$$

$$\begin{aligned} \dot{V}_2(t) &= 2\int_{t-d}^{t} (\theta - t + d)\dot{x}^T(t)Q\dot{x}(\theta)d\theta \\ &\quad - \left[\int_{t-d}^{t} \dot{x}^T(\theta)d\theta\right] Q \left[\int_{t-d}^{t} \dot{x}(\theta)d\theta\right] \end{aligned}$$

$$\dot{V}_3(t) = \frac{1}{2}d^2\dot{x}^T(t)Q\dot{x}(t) - \int_{t-d}^{t} (\theta - t + d)\dot{x}^T(\theta)Q\dot{x}(\theta)d\theta$$

By lemma 1, it can be shown that

$$2\dot{x}^T(t)Q\dot{x}(\theta) \le \dot{x}^T(t)Q\dot{x}(t) + \dot{x}^T(\theta)Q\dot{x}(\theta)$$

Therefore

$$\begin{aligned} \dot{V}_2(t) &\le \frac{1}{2}d^2\dot{x}^T(t)Q\dot{x}(t) + \int_{t-d}^{t} (\theta - t + d)\dot{x}^T(\theta)Q\dot{x}(\theta)d\theta \\ &\quad - \left[\int_{t-d}^{t} \dot{x}^T(\theta)d\theta\right] Q \left[\int_{t-d}^{t} \dot{x}(\theta)d\theta\right] \end{aligned}$$

Then, we can get

$$\begin{aligned} \dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) \\ &\le -(\alpha + 1)[x^T(t)Px(t) - w^T(t)w(t)] \\ &\quad +\xi^T(\Xi_1 + \Xi_2 + \Xi_3)\xi \end{aligned}$$

where

$$\Xi_2 = d^2 \begin{bmatrix} (A + A_d)^T \\ -A_d^T \\ B_1^T \end{bmatrix} Q \begin{bmatrix} A + A_d & -A_d & B_1 \end{bmatrix}$$

$$\Xi_3 = diag\{0, -Q, 0\}$$

By the Schur complement formula, (2) is equivalent to $\Xi_1 + \Xi_2 + \Xi_3 < 0$. Therefore, we have $\dot{V}(t) < -(\alpha + 1)x^T(t)Px(t) < 0$ whenever $w = 0$. Furthermore, since

$x^T(t)Px(t) > 1$ for $x \notin \Omega_P$ and $w^T(t)w(t) \le 1$ for $w \in W$, we have $\dot{V}(t) < \xi^T(\Xi_1 + \Xi_2 + \Xi_3)\xi < 0$ for any $w \in W$. Therefore, system (1) is stable and Ω_P is a robust attractor of it with respect to $w \in W$.

Again by the Schur complement formula, (3) is equivalent to the following inequality

$$\begin{bmatrix} \sigma P - C^TC & C^TD \\ * & (\rho^2 - \sigma)I - D^TD \end{bmatrix} \ge 0$$

It follows that

$$\sigma x^T(t)Px(t) - x^T(t)C^TCx(t) - 2x^T(t)C^TDw(t) + (\rho^2 - \sigma)w^T(t)w(t) - w^T(t)D^TDw(t) \ge 0$$

Then, we can get

$$(\rho^2 - \sigma)w^T(t)w(t) - \|Cx(t) + Dw(t)\|^2\sigma + x^T(t)Px(t) \ge 0$$

From this, it is clear that if $x(t) \in \Omega_P$ and $w(t) \in W$, $\|z(t)\|^2 = \|Cx(t) + Dw(t)\|^2 \le \sigma + (\rho^2 - \sigma) = \rho^2$. This shows that $\Omega_P \subset \Omega(\rho)$ and hence $R_\infty(0) \subset \Omega_P \subset \Omega(\rho)$. Therefore, system (1) has ρ performance. This completes the proof.

For given positive scalars $\alpha, \sigma > 0$, (2) and (3) are LMIs. And it is clear that the results are delay-dependent.

3.2 Output-Feedback Control

In this section, the problem of designing an output-feedback controller for a time-delay system is investigated. Consider the following time-delay system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_dx(t-d) + B_1w(t) + B_2u(t) \\ z(t) &= C_1x(t) + D_1w(t) \\ y(t) &= C_2x(t) + D_2w(t) \end{aligned} \tag{4}$$

where $u(t) \in R^{n_u}$ is the control input , $y(t) \in R^{n_y}$ is the measured output signal and the other signals are the same with system (1).

The goal is to design a dynamic output-feedback controller

$$\begin{aligned} \dot{\hat{x}}(t) &= A_K\hat{x}(t) + B_Ky(t) \\ u(t) &= C_K\hat{x}(t) \end{aligned} \tag{5}$$

where $\hat{x}(t) \in R^n$ is the controller state. Applying the controller (5) to system (4) will result in the closed loop system:

$$\begin{aligned} \dot{x}_c(t) &= A_cx(t) + A_{dc}x(t-d) + B_{1c}w(t) \\ z_c(t) &= C_cx(t) + D_{1c}w(t) \end{aligned} \tag{6}$$

where

$$\begin{aligned} x_c &= \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, A_c = \begin{bmatrix} A & B_2C_K \\ B_KC_2 & A_K \end{bmatrix}, A_{dc} = \begin{bmatrix} A_d & 0 \\ 0 & 0 \end{bmatrix} \\ B_{1c} &= \begin{bmatrix} B_1 \\ B_KD_2 \end{bmatrix}, C_c = [\,C_1 \ 0\,], D_{1c} = D_1 \end{aligned} \tag{7}$$

Then we present a sufficient condition under which there exists an output-feedback controller with form (5) for system (6).

Theorem 2. For prescribed positive scalars $\rho > 0, d > 0$, there exists a dynamical output-feedback controller such that the closed loop system (6) is asymptotically stable and has $\rho-$performance if there exist positive scalars $\alpha > 0, \beta > 0, \sigma > 0$, symmetric and positive-definite matrices $X > 0, Y > 0$, $R > 0$ and matrices $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ such that

$$\begin{bmatrix} (1,1) & (1,2) & 0 & (1,4) & (1,5) & 0 \\ * & -\begin{bmatrix} \beta I & 0 \\ * & R \end{bmatrix} & 0 & (2,4) & 0 & \begin{bmatrix} 0 \\ \beta Y \end{bmatrix} \\ * & * & -I & (3,4) & 0 & 0 \\ * & * & * & (4,4) & 0 & 0 \\ * & * & * & * & -\alpha I & 0 \\ * & * & * & * & * & -\beta I \end{bmatrix} < 0 \tag{8}$$

$$\begin{bmatrix} \sigma X & \sigma I & 0 & XC_1^T \\ * & \sigma Y & 0 & C_1^T \\ * & * & (\rho^2 - \sigma)I & D_1^T \\ * & * & * & I \end{bmatrix} \ge 0, \ \begin{bmatrix} X & I \\ * & Y \end{bmatrix} > 0 \tag{9}$$

where

$$(1,1) = \begin{bmatrix} \Xi_1 & \mathcal{L}_1^T + A + A_d + (\alpha+1)I \\ * & Sym\{Y(A + A_d) + \mathcal{L}_2C_2\} + (\alpha+1)Y \end{bmatrix}$$

$$\Xi_1 = Sym\{(A + A_d)X + B_2\mathcal{L}_3\} + (\alpha+1)X$$

$$(1,2) = -\beta \begin{bmatrix} A_d & 0 \\ YA_d & 0 \end{bmatrix}$$

$$(1,4) = d\begin{bmatrix} X(A + A_d)^T + \mathcal{L}_3^TB_2^T & \mathcal{L}_1^T \\ (A + A_d)^T & (A + A_d)^TY + C_2^T\mathcal{L}_2^T \end{bmatrix}$$

$$(1,5) = \begin{bmatrix} B_1 \\ YB_1 + \mathcal{L}_2D_2 \end{bmatrix}, (2,4) = -d\begin{bmatrix} A_d^T & A_d^TY \\ 0 & 0 \end{bmatrix}$$

$$(3,4) = d\,[\,B_1^T \ \ B_1^TY + D_2^T\mathcal{L}_2^T\,], \ \ (4,4) = -\begin{bmatrix} \beta I & \beta Y \\ * & R \end{bmatrix}$$

Proof: Applying Theorem 1 to the closed loop system (6), we can get

$$\begin{bmatrix} (1,1) & -PA_{dc} & 0 & d(A_c + A_{dc})^TQ & PB_{1c} \\ * & -Q & 0 & -dA_{dc}^TQ & 0 \\ * & * & -I & dB_{1c}^TQ & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -\alpha I \end{bmatrix} < 0 \tag{10}$$

where $(1,1) = Sym\{P(A_c + A_{dc})\} + (\alpha+1)P$. Partition P and its inverse P^{-1} as

$$P = \begin{bmatrix} Y & N \\ * & W \end{bmatrix}, P^{-1} = \begin{bmatrix} X & M \\ * & Z \end{bmatrix} \tag{11}$$

where $X, Y \in R^{n\times n}$ are symmetric and positive-definite matrices. From $P^{-1}P = I$, it follows that

$$MN^T = I - XY \tag{12}$$

Define

$$F_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}, F_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}, V = \begin{bmatrix} I & 0 \\ 0 & N^T \end{bmatrix} \tag{13}$$

Then, we have

$$F_1^TPF_1 = \begin{bmatrix} X & I \\ * & Y \end{bmatrix} > 0$$

Pre- and post-multiply the equation (10) by $diag\{F_1^T, V^TQ^{-1}, I, F_1^TPQ^{-1}, I\}$ and its inverse, respectively. Substitute (7), (11), (13) into the obtained equation. And define

$$\begin{aligned} \mathcal{L}_1 &= Y(A+A_d)X + NB_K C_2 X + YB_2 C_K M^T \\ &\quad + NA_K M^T \\ \mathcal{L}_2 &= NB_K,\ \mathcal{L}_3 = C_K M^T,\ Q^{-1} = diag\{\beta I, S\} \\ R &= N^T SN + \beta YY \end{aligned} \tag{14}$$

The equation (8) can be obtained immediately. Similarly to the above process, the equation (9) can be get. This completes the proof.

Remark 1. By solving the LMIs (8) and (9) in Theorem 2, we can obtain a controller stabilizing system (6) and achieving the desired performance level of persistent bounded disturbance rejection in the closed loop system. Given any solution of the LMIs in theorem 2, a corresponding controller with the form (5) will be constructed as follows: 1.Compute a factorization MN^T of $I - XY$ and deduce the invertible matrices M and N. 2.Solve the equation (14) for B_K, C_K and A_K(in that order) using the matrices M and N obtained above.

4. ILLUSTRATIVE EXAMPLES

Consider the system (4) with the following parameters:

$$A = \begin{bmatrix} -1 & 0 \\ 0.2 & 0.1 \end{bmatrix}, A_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.01 \end{bmatrix}, B_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.3 & 1.8 \end{bmatrix}, D_1 = 0, C_2 = [\,1\ \ 6\,]$$

$$D_2 = [\,0.4\ \ 0.7\,]$$

Notice that the system matrix A is unstable, and the pair (A, B_2) is not controllable. Here, we use Theorem 2 to find a linear output feedback controller to stabilize the system and guarantee the closed loop system to have the ρ performance with $\rho = 0.9$.

Assume $d = 1, \alpha = 0.1, \beta = 8, \sigma = 0.8$, we can solve the LMIs in Theorem 2 and obtain

$$A_k = \begin{bmatrix} -6.9498 & -8.0003 \\ -37.6226 & -50.2073 \end{bmatrix}, B_k = \begin{bmatrix} -1.2895 \\ -6.7149 \end{bmatrix}$$

$$C_k = [\,1.0261\ \ 1.2994\,]$$

Furthermore, let the external bounded disturbance $w = \left[\,(1/\sqrt{2})\sin(\pi t + 1)\ \ (1/\sqrt{2})\cos(2\pi t + 1)\,\right]^T$. The numerical simulation of the state response of the system without disturbances is shown in Fig. 1, and that of the system involving the disturbance effects in Fig. 2. The initial states are both chosen to be (0.6, -0.1).

5. CONCLUSIONS

This paper considers the problem of persistent bounded disturbance rejection for a class of linear systems with a constant time-delay. Delay dependent sufficient conditions are derived that ensure the internal stability and desired level of persistent bounded disturbances. Then the dynamic output feedback disturbance rejection controller is designed. A numerical example shows the effectiveness of the method. Further results on persistent bounded disturbance rejection for uncertain time-delay systems will be presented elsewhere.

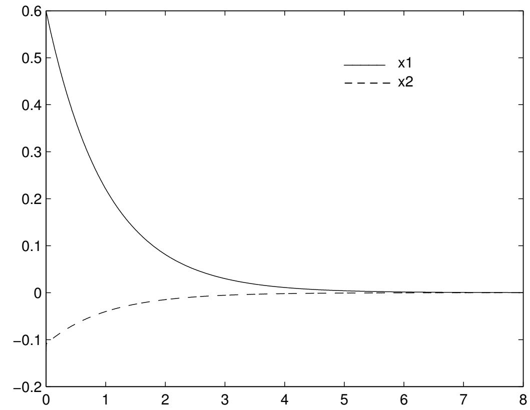

Fig. 1. State response of the system without disturbances

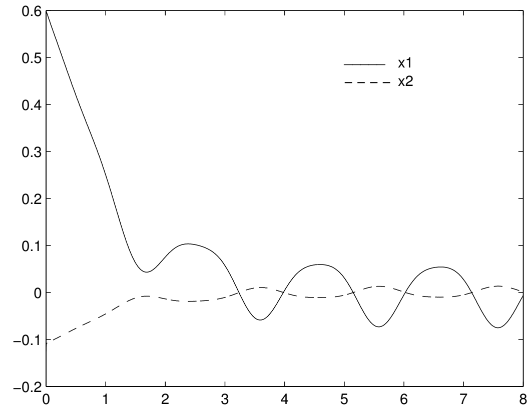

Fig. 2. State response of the system with external disturbances

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