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Global Stabilization of Discrete-time Chain of Integrators by Saturated Feedback ⋆

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Abstract: The global stabilization problem for discrete-time n-th order integrators system with saturated input is considered. A new class of nested type nonlinear feedback law is proposed possessing new and useful characteristics. First, this approach allows the designer to pick some parameters that facilitate the placement of the closed-loop pole set consisting of some pairs of conjugate complex numbers having negative real parts when none of the saturation elements in the control laws is saturated. Only real numbers are allowed in the other existing results. Second, there are more free parameters in this class of nonlinear feedback laws that can be further used to improve performances of the closed-loop system. Third, this class of nonlinear feedback laws possesses very simple structure and is easy to implement in practice. Some simulative experiments confirm the good behavior in term of convergence performance of the closed-loop system comparing with some other existing techniques.

1. INTRODUCTION

Practical control systems are subject to input saturation. For the special case of linear systems subject to input saturation, several important control problems have been solved. Among these problems are global stabilization (Sussmann et al. [1994]), semi-global stabilization (Lin et al. [1995]), output regulation (Santis et al. [2001]), input-output stabilization (Liu et al. [1996]) and robust stabilization (Angeli et al. [2005]). It is well-known that, a linear system subject to input saturation is globally or semi-globally stabilizable if and only if the system in the absence of the input saturation is asymptotically null controllable with bounded controls (ANCBC). It has been shown in Sussmann et al. [1991] that a simple linear system of a chain of integrators of length $n > 2$ which is ANCBC, can not be globally stabilized by saturated linear feedback. Thus, for general ANCBC linear systems, nonlinear feedback is needed for global stabilization. Still for the multiple integrators case, Teel proposed in Teel [1992] a nonlinear state feedback law of nested saturation type which not only solve the global stabilization problem but also can be used to achieve trajectory tracking for a class of bounded trajectories. This technique of using nested saturation was latterly successfully applied to achieve global stabilization of general ANCBC continuous linear systems in Sussmann et al. [1994] and discrete-time linear systems in Yang et al. [1997].

It has been shown in Rao et al. [2001] that Teel's nested saturation feedback law consisting of n (the order of the system) saturation functions exhibits good robustness, inhibits performance degradation and has excellent disturbance rejection, and is noticeably superior to some other existing feedback laws. But on the other hand, as mentioned in Marchand. [2003], for systems of larger dimensions and/or bigger initial conditions the performance of the closed-loop systems is degraded. Thus, due to the simplicity and the superiority of Teel's nested saturation feedback laws for multiple integrators, a lot of exploration and modification have been investigated by some authors. The most remarkable one comes from Marchand. [2003] where the author introduces a type of so-called state-dependent saturation functions to replace the standard saturation functions appearing in Teel's nested saturation feedback laws. This idea is then extended in Marchand et al. [2005] where another type of nonlinear feedback law is considered, and in Marchand et al. [2007] where discrete-time multiple integrators system is considered. It is found that this class of modified nonlinear feedback laws can indeed significantly improve the performance, especially the convergence performance, of the closed-loop system. Another remarkable generation comes from Johnson et al. [2003] where the author has found that Teel's nested type saturation feedback laws result in all the poles of the closed-loop system residing at -1 when none of the saturation elements in the control laws is saturated and proposed a type of modified control law that allows the eigenvalues to be in any place of the left real axis.

Two types of nested saturation feedback laws are proposed in Teel's original paper. Suppose n is the order of the system, the first one consists of n saturation functions and the other needs only $\left[\frac{n+1}{2}\right]$ saturation functions. The second type of nonlinear feedback law can greatly improve the convergence speed because it needs less saturation functions and therefore can increase the control energy significantly. The second type of nonlinear control laws also result in all the poles of the closed-loop system residing at -1 when none of the saturation elements in the control laws is saturated.

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In this paper, we consider the global stabilization of discrete-time multiple integrators by bounded controls. By adopting the basic idea given in Teel [1992], we construct a new type of nested type nonlinear feedback law that results in all the poles of the closed-loop system residing at certain locations within the unite circle when none of the saturation elements in the control laws is saturated. Moreover, we need only $\left[\frac{n+1}{2}\right]$ saturation functions to construct this class of nonlinear feedback laws. Therefore, the controller possesses a more simple structure. Also, some free parameters are introduced into the control law, which can be further used to improve the performances of the closed-loop system.

Throughout this paper, we use y_i to denote the i-th row of the state vector y. The symbol function sign is defined as $\text{sign}(y) = 1$ if $y \geq 0$ and $\text{sign}(y) = -1$ if $y < 0$. The standard saturation function can then be defined as $\text{sat}(y) = \text{sign}(y)\min\{1, |y|\}$. We use $[m]$ where $m > 0$ to denote the integer part of m and use $\sigma(A)$ to denote the eigenvalue set of matrix A.

2. MAIN RESULTS

2.1 Problem Formulation and Preliminaries

In this paper, we consider a discrete-time system by discretizing the following n-th order multiple integrators system

$$\dot{x} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \tag{1}$$

where $|u| \leq u_{\max}$ with $u_{\max}$ some known parameter representing the magnitude limitation on the control. The ZOH discretization model of (1) with sample time T is given by

$$x^+ = Gx + Hu, \tag{2}$$

where

$$G = \begin{bmatrix} 1 & T & \frac{T^2}{2!} & \cdots & \frac{T^{n-1}}{(n-1)!} \\ 0 & 1 & T & \cdots & \frac{T^{n-2}}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & T \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} \frac{T^n}{n!} \\ \frac{T^{n-1}}{(n-1)!} \\ \vdots \\ \frac{T^2}{2!} \\ T \end{bmatrix}.$$

To describe our main results, we need a new state space representation of the original system (2). Let $\widetilde{n} = \left[\frac{n+1}{2}\right]$. For a series of positive scalors $\alpha_i, i = 1, 2, \cdots, \widetilde{n}$, we define

$$A_i = \frac{4}{3+\alpha_i}\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}, i = 2, 3, \cdots, \widetilde{n}$$

$$A_1 = \begin{cases} \frac{4}{3+\alpha_1}\left[\,-1 \;\; 2\,\right], & n \,\text{is odd} \\ \frac{4}{3+\alpha_1}\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}, & n \,\text{is even} \end{cases}$$

$$A_{1i} = \begin{cases} \frac{4}{3+\alpha_i}\left[\,-1 \;\; 2\,\right] & , n \quad \text{is} \quad \text{odd} \\ A_i, i = 2, 3, \cdots, \widetilde{n}, \; n & \text{is} \quad \text{even}. \end{cases}$$

Furthermore, let

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, b_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$b_{10} = \begin{cases} 1 & , n \,\text{is odd} \\ b_0 & , n \,\text{is even} \end{cases}, A_{10} = \begin{cases} 1 & , n \,\text{is odd} \\ A_0 & , n \quad \text{is} \quad \text{even}. \end{cases}$$

We then can give the following result. Its proof is simple and can be found in Zhou et al. [2007].

Lemma 1. Let $\alpha_i, i = 1, 2, 3, \cdots, \widetilde{n}$, be a series of priori given positive numbers. Then the system (2) can be transformed to

$$y^+ = Ay + bu, \tag{3}$$

where A and b are given by

$$A = \begin{bmatrix} A_{10} & A_{12} & A_{13} & \cdots & A_{1\tilde{n}} \\ 0 & A_0 & A_3 & \cdots & A_{\widetilde{n}} \\ 0 & 0 & A_0 & \cdots & A_{\widetilde{n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_0 \end{bmatrix}, b = \begin{bmatrix} b_{10} \\ b_0 \\ b_0 \\ \vdots \\ b_0 \end{bmatrix}, \tag{4}$$

via a linear change of coordinates $y = Qx$ with Q some nonsingular matrix.

We furthermore give some notations. Let

$$k_i = \frac{4}{3+\alpha_i}\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \widetilde{y}_i = \begin{bmatrix} y_{n-1+2(i-\widetilde{n})} \\ y_{n+2(i-\widetilde{n})} \end{bmatrix},$$

for $i = 2, 3, \cdots, \widetilde{n}$, and

$$k_1^{\mathrm{T}} = \begin{cases} \frac{2}{\alpha_1 + 1}, n \,\text{is odd} \\ \frac{4}{3+\alpha_i}\left[\,-1 \;\; 2\,\right], n \,\text{is even} \end{cases}, \widetilde{y}_1^{\mathrm{T}} = \begin{cases} y_1, n \,\text{is odd} \\ \left[\, y_1 \;\; y_2 \,\right]. \end{cases}$$

For a vector $x = [x_1, x_2]^{\mathrm{T}} \in \mathbb{R}^2$, we define a quadratic function

$$V(x, \gamma) = x_1^2 + \gamma x_2^2, \tag{5}$$

where $\gamma > 1$ is independent of x. The level set of $V(x, \gamma)$ is the solid ellipsoid

$$\Omega(\rho, \gamma) = \{x \,| V(x, \gamma) \leq \rho\}.$$

The following lemma is the key technique in proving our main results of this paper. For clarity, the proof is given in Section 3.

Lemma 2. Consider the following planar nonlinear system

$$x^+ = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(x, z), \tag{6}$$

where

$$u(x, z) = -\varepsilon_2 \text{sat}\left(\frac{4(2x_2(k) - x_1(k))}{\varepsilon_2(3+\alpha)} + \frac{\varepsilon_1}{\varepsilon_2}\text{sat}(z)\right), \tag{7}$$

and $\varepsilon_2, \varepsilon_1, \alpha$ are some positive scalars. If the following inequality

$$\varepsilon_2 > \kappa(\alpha)\varepsilon_1, \tag{8}$$

in which

$$\kappa(\alpha) = \frac{3 + 2\sqrt{5}}{1 - (2 + 2\sqrt{5})\frac{|1-\alpha|}{3+\alpha}}, \tag{9}$$

is satisfied, then there exist $\rho > 0$ and $k_T > 0$ such that for arbitrary $k > k_T$, the state $x(k)$ will enter the set $\Omega(\rho, 1 + \sqrt{5})$ and remain there for ever. Moreover, for arbitrary $k > k_T$, the input $u(x, z)$ can be simplified as

$$u(x, z) = -\frac{4}{3+\alpha}(2x_2 - x_1) - \varepsilon_1 \text{sat}(z). \tag{10}$$

Remark 3. Obviously, if $\varepsilon_1 = 0$, then the system (6) is globally asymptotically stable for arbitrary $\varepsilon_2 > 0$ and $\alpha > 0$.

Remark 4. It follows from (8) that we must have $\kappa(\alpha) > 0$ which is equivalent to

$$0.465 = \frac{23 - 8\sqrt{5}}{11} < \alpha < \frac{15 + 8\sqrt{5}}{19} = 1.731. \quad (11)$$

Furthermore, $\kappa(\alpha)$ is minimized when $\alpha = 1$. In this case $\kappa(1) = 3 + 2\sqrt{5} = 7.47$.

Notice that if $u(x, z)$ is in the form of (10), then the nonlinear system (6) becomes

$$x^+ = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{1-\alpha}{3+\alpha} & \frac{2\alpha-2}{3+\alpha} \end{bmatrix}}_{A_c} x - \varepsilon_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{sat}(z),$$

which is Schur stable for arbitrary $\alpha > 0$ if $z \equiv 0$. To see this, we notice that the characteristic polynomial of A_c is

$$z^2 - \frac{2\alpha - 2}{3+\alpha} z - \frac{1-\alpha}{3+\alpha} = 0.$$

Substituting $z = \frac{w+1}{w-1}$ into the above equation and simplifying gives $w^2 + 2w + \alpha = 0$, which is Lyapunov stable for arbitrary $\alpha > 0$.

Lemma 5. Consider the following scalar discrete-time system

$$y^+ = y - \varepsilon \mathrm{sat}\left(\frac{2}{\alpha+1}\frac{y}{\varepsilon}\right), \quad (12)$$

where ε and α are positive scalars. Then the system is globally asymptotically stable.

Proof. Choose $W(y) = y^2$ as a Lyapunov function candidate. Then

$$W^+(y) - W(y) = \varepsilon \mathrm{sat}\left(\frac{2}{\alpha+1}\frac{y}{\varepsilon}\right)\left(\varepsilon \mathrm{sat}\left(\frac{2}{\alpha+1}\frac{y}{\varepsilon}\right) - 2y\right).$$

If $y > 0$, it follows from the fact $\varepsilon \mathrm{sat}\left(\frac{x}{\varepsilon}\right) \le x$, for any $\varepsilon > 0$ and $x > 0$ that

$$\varepsilon \mathrm{sat}\left(\frac{2}{\alpha+1}\frac{y}{\varepsilon}\right) - 2y \le \frac{2y}{\alpha+1} - 2y = \frac{-\alpha y}{\alpha+1} < 0.$$

Accordingly, $W^+(y) - W(y) < 0$. Similarly, if $y < 0$, it follows from the fact $\varepsilon \mathrm{sat}\left(\frac{x}{\varepsilon}\right) \le x$, for any $\varepsilon > 0$ and $x < 0$ that

$$\varepsilon \mathrm{sat}\left(\frac{2}{\alpha+1}\frac{y}{\varepsilon}\right) - 2y \ge \frac{2y}{\alpha+1} - 2y = \frac{-\alpha y}{\alpha+1} > 0,$$

and accordingly $W^+(y) - W(y) < 0$. Therefore, we have

$$W^+(y) - W(y) < 0, \forall y \ne 0,$$

which implies that the system (12) is globally asymptotically stable. ■

Lemma 6. (Marchand. [2003]) Any closed loop trajectory of any linear system can not diverge at finite time under bounded input.

2.2 The Main Result

The main result of this paper is given as follows.

Theorem 7. Let $\alpha_i, i = 1, 2, \cdots, \widetilde{n}$, be a series of priori given positive numbers in the interval $(0.465, 1.731)$ and $\varepsilon_i, i = 1, 2, \cdots, \widetilde{n}$, be some positive numbers satisfying

$$\varepsilon_i \ge \kappa(\alpha_i)\varepsilon_{i-1}, \ i = 2, 3, \cdots, \widetilde{n}, \ \varepsilon_{\widetilde{n}} \le u_{\max}. \quad (13)$$

Then the control law $u = \varepsilon_{\widetilde{n}} u_{\widetilde{n}}$ with

$$\begin{aligned} u_i &= -\mathrm{sat}\left(\frac{k_i^{\mathrm{T}}\widetilde{y}_i}{\varepsilon_i} - \frac{\varepsilon_{i-1}}{\varepsilon_i}u_{i-1}\right), \\ i &= 2, 3, \cdots, \widetilde{n}, u_1 = -\mathrm{sat}\left(\frac{k_1^{\mathrm{T}}\widetilde{y}_1}{\varepsilon_1}\right), \end{aligned} \quad (14)$$

where $y = Qx$ is given in Lemma 1, will globally asymptotically stabilize the system (2). Furthermore, under such control law, the closed-loop system will operate in linear region at finite time with a stable characteristic polynomial

$$\beta(z) = \begin{cases} \prod_{i=1}^{\widetilde{n}}\left(z^2 + \frac{2-2\alpha_i}{3+\alpha_i}z + \frac{\alpha_i - 1}{3+\alpha_i}\right) & , n \,\mathrm{is\, even} \\ \prod_{i=2}^{\widetilde{n}}\left(z^2 + \frac{2-2\alpha_i}{3+\alpha_i}z + \frac{\alpha_i - 1}{3+\alpha_i}\right)\left(z - \frac{\alpha_1 - 1}{\alpha_1 + 1}\right). \end{cases}$$

Proof. For $i = 2, 3, \cdots, \widetilde{n}$, we denote

$$A_{ci} = \begin{bmatrix} 0 & 1 \\ \frac{1-\alpha_i}{3+\alpha_i} & \frac{2\alpha_i - 2}{3+\alpha_i} \end{bmatrix}.$$

Obviously, the characteristic polynomial of A_{ci} is $\beta_i(z) = z^2 + \frac{2-2\alpha_i}{3+\alpha_i}z + \frac{\alpha_i - 1}{3+\alpha_i}$ which is stable for arbitrary $\alpha_i > 0$ as shown before. Consider the last two states of (3), *i.e.*

$$\begin{bmatrix} y_{n-1}^+ \\ y_n^+ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\varepsilon_{\widetilde{n}} u_{\widetilde{n}}. \quad (15)$$

Notice that this system is in the form of (6). Using Lemma 2, if (13) is satisfied, there exists a finite number $T_{\widetilde{n}}$, such that for $\forall k \ge T_{\widetilde{n}}$, the states $y_{n-1}(k)$ and $y_n(k)$ are linear in the control, i.e.,

$$u_{\widetilde{n}} = -\frac{8}{3+\alpha_{\widetilde{n}}}y_n + \frac{4}{3+\alpha_{\widetilde{n}}}y_{n-1} + u_{\widetilde{n}-1}. \quad (16)$$

Substituting (16) into (3) results in the following system

$$y^+ = \widetilde{A}_{\widetilde{n}-1}y + bu_{\widetilde{n}-1},$$

where

$$\widetilde{A}_{\widetilde{n}-1} = \begin{bmatrix} A_{10} & A_{12} & \cdots & A_{1\widetilde{n}-1} & 0 \\ 0 & A_0 & \cdots & A_{\widetilde{n}-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 & 0 \\ 0 & 0 & \cdots & 0 & A_{c\widetilde{n}} \end{bmatrix}.$$

Furthermore, according to Lemma 6, the other states $y_i, 1 \le i \le n-3$, will remain finite during that time.

We then consider the states y_{n-3} and y_{n-4} which are also in a difference equation having the form of (6). Do such procedure until the following system is met

$$y^+ = \widetilde{A}_2 y + bu_1, \quad (17)$$

where

$$\widetilde{A}_2 = \begin{bmatrix} A_{10} & 0 & 0 & \cdots & 0 \\ 0 & A_{c2} & 0 & \cdots & 0 \\ 0 & -A_2 & A_{c3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -A_2 & -A_3 & \cdots & A_{c\widetilde{n}} \end{bmatrix},$$

and

$$u_1 = \begin{cases} -\varepsilon_1 \mathrm{sat}\left(\frac{\frac{8}{3+\alpha_1}y_2 - \frac{4}{3+\alpha_1}y_1}{\varepsilon_1}\right) & , n \,\mathrm{is\, even} \\ -\varepsilon_1 \mathrm{sat}\left(\frac{2}{\alpha_1+1}\frac{y_1}{\varepsilon_1}\right) & , n \,\mathrm{is\, odd} \end{cases}. \quad (18)$$

When n is even, the subsystem of (17) with system matrix being A_{10} is in the form of (6) with $z = 0$. According to Remark 3, such system is globally stable and therefore there exists a finite number $k_1 > 0$ such that for arbitrary $k > k_1$, there holds $\left|\frac{8}{3+\alpha_1}y_2 - \frac{4}{3+\alpha_1}y_1\right| \leq \varepsilon_1$. When n is odd, the subsystem of (17) with system matrix being A_{10} is in the form of (12). Lemma 5 guarantees that such system is also globally stable and therefore there exists a finite number $k_1^{'} > 0$ such that for arbitrary $k > k_1^{'}$, there holds $\left|\frac{2}{\alpha_1+1}y_1\right| \leq \varepsilon_1$. In both cases, for $k > k_1\left(k_1^{'}\right)$, the closed-loop system (17) and (18) can be written as

$$y^{+} = \begin{bmatrix} A_{c1} & 0 & \cdots & 0 & 0 \\ -A_1 & A_{c2} & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ -A_1 & -A_2 & \cdots & A_{c(\widetilde{n}-1)} & 0 \\ -A_1 & -A_2 & \cdots & -A_{\widetilde{n}-1} & A_{c\tilde{n}} \end{bmatrix} y, \quad (19)$$

where

$$A_{c1} = \begin{cases} \dfrac{\alpha_1 - 1}{\alpha_1 + 1} & n \,\text{is odd} \\ \begin{bmatrix} 0 & 1 \\ \dfrac{1-\alpha_1}{3+\alpha_1} & \dfrac{2\alpha_1 - 2}{3+\alpha_1} \end{bmatrix} & n \,\text{is even} \end{cases}.$$

Clearly, the closed-loop system (19) is a linear system with characteristic polynomial $\beta(z)$. We complete the proof. ■

Remark 8. If $\alpha_i = 1$, then we have $\beta(z) = z^n$. In this case, the closed-loop system will convergence to the origin at finite step. Furthermore, $\kappa(\alpha_i)$ achieves its minimum in this case, which indicates that the control energy may achieve its maximum and the system performances can be improved.

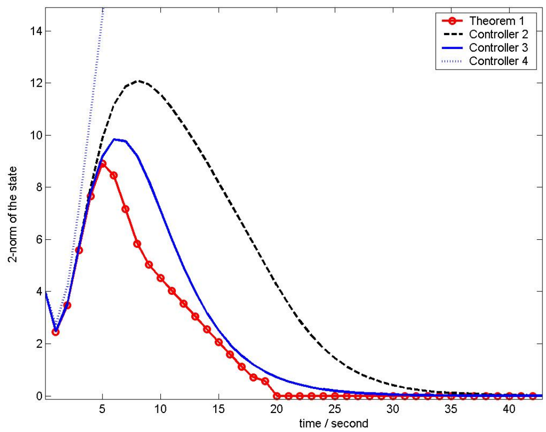

Fig. 1. Evaluation of the states under different feedback.

To illustrate the efficiency of the proposed approach, we consider a triple integrator system in the form of (2) with $|u| \leq u_{\max} = 1$ and $T = 1$ (borrowed from Marchand et al. [2007]). Fig. 1 shows the time evolution of $\|x\|$ when the control of Theorem 7 is applied for the initial condition $x_0^{\mathrm{T}} = [2, -2, 3]$. The parameters are choosing as $\alpha_1 = \alpha_2 = 1$ and $\varepsilon_2 = 7.47\varepsilon_1 = 1$. Fig. 1 also shows a comparison with some other existing results. Controller 2 and 3 are taken from Marchand et al. [2007] by using static and dynamic saturation functions respectively, while Controller 4 is the discrete-time version of the control law proposed in Sussmann et al. [1994] for continuous time integrators system. It follows that the control law given in Theorem 1 not only possesses the simplest structure, but also can improve the convergence performances of the closed-loop system. Finally we note that $\|x\|$ of the system under the control of Theorem 7 converges to origin at finite step, which coincides with Remark 8.

3. PROOF OF LEMMA 1

Consider the following region partition of the state space

$$\begin{cases} \mathbb{I}: & \left\{x \in \mathbb{R}^2 \left| \dfrac{4(2x_2 - x_1)}{3+\alpha} > \varepsilon_1 + \varepsilon_2 \right.\right\} \\ \mathbb{II}: & \left\{x \in \mathbb{R}^2 \left| \left|\dfrac{4(2x_2 - x_1)}{3+\alpha}\right| \leq \varepsilon_1 + \varepsilon_2 \right.\right\} \\ \mathbb{III}: & \left\{x \in \mathbb{R}^2 \left| \dfrac{4(2x_2 - x_1)}{3+\alpha} < -\varepsilon_1 - \varepsilon_2 \right.\right\}, \end{cases} \quad (20)$$

which means that the x_1-x_2 plane is divided into three parts by two lines $\frac{4(2x_2-x_1)}{3+\alpha} = \pm(\varepsilon_1 + \varepsilon_2)$. We will sequently prove the following two statements.

(1) Any bounded initial condition in region $\mathbb{I}$ or $\mathbb{III}$ yields a trajectory that enters the boundary of region $\mathbb{II}$ at finite time.
(2) Any state on the boundary of $\mathbb{I}$ or $\mathbb{III}$ that will enter region $\mathbb{I}$ will return to region $\mathbb{II}$ at finite time and has a lower energy level with respect to the energy function (5).

3.1 *Proof of Item 1*

Note that the system (6) in region $\mathbb{I}$ becomes

$$\begin{bmatrix} x_1^{+} \\ x_2^{+} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \varepsilon_2 \end{bmatrix}. \quad (21)$$

which is a planar linear system. Therefore, the closed form solution of the above system can be obtained as

$$\begin{aligned} x(k) &= \begin{bmatrix} 1-k & k \\ -k & 1+k \end{bmatrix} x(0) + \sum_{i=0}^{k-1} \Phi(k-i-1) Bu(i) \\ &= \begin{bmatrix} (1-k)x_1(0) + kx_2(0) - \varepsilon_2 \dfrac{k(k-1)}{2} \\ (1+k)x_2(0) - kx_1(0) - \varepsilon_2 \dfrac{k(k+1)}{2} \end{bmatrix}. \end{aligned} \quad (22)$$

We will show that the state $x(k)$ will arrive at the boundary of $\mathbb{I}$ for some k. That is to say, there is a $k > 0$ such that

$$\frac{8}{3+\alpha}x_2(k) - \frac{4}{3+\alpha}x_1(k) - (\varepsilon_2 + \varepsilon_1) = 0. \quad (23)$$

Substituting (22) into (23) gives the equation $g(k) = 0$, where $g(k)$ denotes the left side of (23). Note that

$$g(0) = (\varepsilon_2 + \varepsilon_1) - \frac{8x_2(0)}{3+\alpha} + \frac{4x_1(0)}{3+\alpha}.$$

Since the initial state $(x_1(t_0), x_2(t_0))$ is in region $\mathbb{I}$, we know that $g(0) < 0$. Because $g(k)$ is a quadratic function in k, the equation $g(k) = 0$ has a solution $k > 0$. That is to say, the states x will joint at the boundary of region $\mathbb{II}$ at finite time. The same argument holds for region $\mathbb{III}$ by symmetry.

3.2 Proof of Item 2

We now assume an initial condition locating on the boundary of region $\mathbb{III}$, i.e., an initial state $(x_1(0), x_2(0))$ such that

$$\frac{8x_2(0)}{3+\alpha} - \frac{4x_1(0)}{3+\alpha} = \varepsilon_1 + \varepsilon_2. \tag{24}$$

Note that on the boundary of $\mathbb{III}$, system (6) can be also written as (21). To enter region $\mathbb{I}$, we must have

$$\frac{8x_2^+(0)}{3+\alpha} - \frac{4x_1^+(0)}{3+\alpha} - \left(\frac{8x_2(0)}{3+\alpha} + \frac{4x_1(0)}{3+\alpha}\right) > 0,$$

Using the system equation (21), it follows from the above inequality that

$$x_2(0) < \frac{1}{4}\left((\alpha-5)\varepsilon_2 + \varepsilon_1(3+\alpha)\right). \tag{25}$$

Assume that x has been in region $\mathbb{I}$. We have shown in Item 1 that it will return to the boundary of region $\mathbb{III}$ in finite time $k_b > 0$, i.e.,

$$\frac{8x_2(0)}{3+\alpha} - \frac{4x_1(0)}{3+\alpha} = \frac{8x_2(k_b)}{3+\alpha} - \frac{4x_1(k_b)}{3+\alpha}.$$

Set $k = k_b$ in (22) and it follows from the above equation that

$$k_b = \frac{(\alpha-3)\varepsilon_2 + (3+\alpha)\varepsilon_1 - 4x_2(0)}{2\varepsilon_2}. \tag{26}$$

Using (25) and (26), we can easily obtain $k_b > 1$ which implies that the state x will return back to region $\mathbb{III}$ at least one step. Evaluating the closed form solution for x at k_b gives

$$\begin{aligned} &V^+(x(k_b),\gamma) - V(x(0),\gamma) \\ =& \frac{1}{4}\left((\alpha-3)\varepsilon_2 + (3+\alpha)\varepsilon_1 - 4x_2(0)\right) \times \\ &\left((\gamma\alpha_2 + 2\alpha_2 + 3\gamma + 6)\varepsilon_1 + (2\alpha_2 + \gamma\alpha_2 - 18 - 3\gamma)\varepsilon_2\right). \end{aligned}$$

Using inequality (25) gives

$$(\alpha-3)\varepsilon_2 + (3+\alpha)\varepsilon_1 - 4x_2(0) > 2\varepsilon_2 > 0.$$

Then $V^+(x(k_b)) - V(x(0)) < 0$ if and only if

$$\varepsilon_1(\alpha\gamma + 2\alpha + 3\gamma + 6) - \varepsilon_2(18 + 3\gamma - 2\alpha - \alpha\gamma) < 0,$$

which is equivalent to

$$\varepsilon_2 > \left(\underbrace{\frac{(2+\gamma)(3+\alpha)}{18 + 3\gamma - \alpha(2+\gamma)}}_{g(\alpha,\gamma)}\right)\varepsilon_1. \tag{27}$$

That is to say, if (27) is satisfied, any initial state leaving from the boundary of region $\mathbb{III}$ and entering to region $\mathbb{I}$ will return to the boundary of region $\mathbb{III}$ at finite time with lower energy with respect to the energy function (5). The statement also holds true for the states that leaving from the boundary of region $\mathbb{III}$ and entering to region $\mathbb{I}$ by symmetry.

3.3 Decrease of $V(\boldsymbol{x},\gamma)$ in Region $\mathbb{III}$

We need the following lemma with its proof very simple and omitted.

Lemma 9. The following inequality

$$f(t) = t\left(t - \varepsilon_2 \mathrm{sat}\left(\frac{t + \varepsilon_1 \mathrm{sat}(z)}{\varepsilon_2}\right)\right) \le |t|\,\varepsilon_1,$$

is valid if $|t| < \varepsilon_2 + \varepsilon_1$.

Consider the trajectories in region $\mathbb{III} - \Omega(\rho,\gamma)$. Denote

$$h(x,z) = \mathrm{sat}\left(\frac{4(2x_2 - x_1)}{\varepsilon_2(3+\alpha)} + \frac{\varepsilon_1}{\varepsilon_2}\mathrm{sat}(z)\right). \tag{28}$$

The time derivative of $V(x,\gamma)$ along the trajectories of system (6) is given by

$$\begin{aligned} V^+ - V &= \left(x_1^+\right)^2 + \gamma\left(x_2^+\right)^2 - \left(x_1^2 + \gamma x_2^2\right) \\ &= -\left(x_1^2 + (\gamma-1)x_2^2\right) + \gamma(2x_2 - x_1 - \varepsilon_2 h)^2 \end{aligned} \tag{29}$$

Since $\frac{4}{3+\alpha}|2x_2 - x_1| \le (\varepsilon_2 + \varepsilon_1)$, it follows that

$$(2x_2 - x_1 - \varepsilon_2 h)^2 = \left(t - \frac{4}{3+\alpha}\bar{\varepsilon}_2 \mathrm{sat}\left(\frac{t + \bar{\varepsilon}_1 \mathrm{sat}(z)}{\bar{\varepsilon}_2}\right)\right)^2,$$

where $t = 2x_2 - x_1$, $|t| < \bar{\varepsilon}_2 + \bar{\varepsilon}_1$ and

$$\frac{3+\alpha}{4}\varepsilon_2 = \bar{\varepsilon}_2,\ \frac{1}{4}(3+\alpha)\varepsilon_1 = \bar{\varepsilon}_1.$$

Using Lemma 9, we can obtain

$$\left|t - \frac{4}{3+\alpha}\bar{\varepsilon}_2 \mathrm{sat}\left(\frac{t + \bar{\varepsilon}_1 \mathrm{sat}(z)}{\bar{\varepsilon}_2}\right)\right| \le \bar{\varepsilon}_1 + \frac{|1-\alpha|}{3+\alpha}\bar{\varepsilon}_2.$$

Then it follows that

$$(2x_2 - x_1 - \varepsilon_2 h)^2 \le \left(\frac{3+\alpha}{4}\varepsilon_1 + \frac{|1-\alpha|}{4}\varepsilon_2\right)^2. \tag{30}$$

On the other hand, straightforward manipulation gives

$$\min_{x \in (\mathbb{III} - \Omega(\rho,\gamma))} \left\{x_1^2 + (\gamma-1)x_2^2\right\} = \rho^2\left(1 - \frac{1}{\gamma}\right). \tag{31}$$

By virtue of (30) and (31), inequality (29) gives

$$V^+ - V \le -\rho^2\left(1 - \frac{1}{\gamma}\right) + \gamma\left(\frac{3+\alpha}{4}\varepsilon_1 + \frac{|1-\alpha|}{4}\varepsilon_2\right)^2.$$

Then $V^+ - V < 0$ holds in $\mathbb{III} - \Omega(\rho,\gamma)$ provided

$$\rho > \frac{\gamma}{\sqrt{\gamma-1}}\left(\frac{3+\alpha}{4}\varepsilon_1 + \frac{|1-\alpha|}{4}\varepsilon_2\right). \tag{32}$$

Therefore according to Lyapunov stability theory, if (32) is satisfied, the states x will enter the region $\Omega(\rho)$ at finite time and remain there forever. On the other hand, simple calculation implies

$$\max_{\boldsymbol{x} \in \Omega(\rho)} \left\{\left|\frac{8}{3+\alpha}x_2 - \frac{4}{3+\alpha}x_1\right|\right\} = \frac{8\rho}{3+\alpha}\sqrt{\frac{4}{\gamma} + 1}.$$

Then if the following inequality

$$\frac{\frac{8\rho}{3+\alpha}\sqrt{\frac{4}{\gamma}+1}}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_2} \le 1, \tag{33}$$

is satisfied, then the inequality

$$\left|\frac{4(2x_2 - x_1)}{\varepsilon_2(3+\alpha)} + \frac{\varepsilon_1}{\varepsilon_2}\mathrm{sat}(z)\right| \le 1, \tag{34}$$

is valid for arbitrary function $z(t)$. Therefore, the nonlinear saturation function (7) is linear in its argument. Note that when (34) is satisfied, the control (7) can be simplified as (10).

To complete the proof in this part, we should show that there exists α and γ such that (32) and (33) are satisfied. Notice that (33) is equivalent to

$$\rho \le \frac{3+\alpha}{8}\sqrt{\frac{\gamma}{4+\gamma}}\,(\varepsilon_2 - \varepsilon_1). \tag{35}$$

Then inequalities (32) and (35) are equivalent to

$$\frac{\gamma}{\sqrt{\gamma-1}}\left(\frac{3+\alpha}{4}\varepsilon_1 + \frac{|1-\alpha|}{4}\varepsilon_2\right) < \frac{3+\alpha}{8}\sqrt{\frac{\gamma}{4+\gamma}}\,(\varepsilon_2 - \varepsilon_1)$$

that is to say

$$\varepsilon_2 > \left(\underbrace{\frac{\left(\frac{2\gamma}{\sqrt{\gamma-1}} + \sqrt{\frac{\gamma}{4+\gamma}}\right)}{\sqrt{\frac{\gamma}{4+\gamma}} - 2\frac{|1-\alpha|}{3+\alpha}\frac{\gamma}{\sqrt{\gamma-1}}}}_{f(\alpha,\gamma)} \right) \varepsilon_1. \tag{36}$$

3.4 Inequalities (27) and (36) can be simplified as (8)

We notice that both (27) and (36) can be written as

$$\varepsilon_2 > \kappa(\alpha,\gamma)\,\varepsilon_1, \kappa(\alpha,\gamma) = \max\{f(\alpha,\gamma), g(\alpha,\gamma)\}.$$

Denote $\kappa(\alpha) = \min_{1<\gamma<\infty}\{\kappa(\alpha,\gamma)\}$, we will show that $\kappa(\alpha)$ is in the form of (9). We first consider the function $f(\alpha,\gamma)$. Note that

$$f(\alpha,\gamma) = 1 + \left.\frac{2 + 2\frac{|1-\alpha|}{3+\alpha}}{z}\right|_{t=z+2\frac{|1-\alpha|}{3+\alpha}, t=\sqrt{\frac{(\gamma-1)}{\gamma(4+\gamma)}}}$$

We observe that $f(\alpha,\gamma)$ is a strictly increasing function of z. Therefore $f(\alpha,\gamma)$ is minimized if and only z is maximized. It follows from $t = z + 2\frac{|1-\alpha|}{3+\alpha}$ that z is maximized if and only if t is maximized. However

$$\max_{1<\gamma<\infty}\{t\} = \max_{1<\gamma<\infty}\left\{\sqrt{\frac{(\gamma-1)}{\gamma(4+\gamma)}}\right\} = \left.\frac{1}{1+\sqrt{5}}\right|_{\gamma=1+\sqrt{5}}.$$

Then we have

$$f(\alpha) = \min_{1<\gamma<\infty}\{f(\alpha,\gamma)\} = \frac{3+2\sqrt{5}}{1-\left(2+2\sqrt{5}\right)\frac{|1-\alpha|}{3+\alpha}}.$$

We next consider the function $g(\alpha,\gamma)$. Note that

$$g(\alpha,\gamma) = \left.\frac{(3+\alpha)}{(3-\alpha)} - \frac{12(3+\alpha)}{3-\alpha}\frac{1}{t}\right|_{t=18-2\alpha+(3-\alpha)\gamma}$$

Therefore $g(\alpha,\gamma)$ is strictly increasing with respect to γ. On the other hand, it is easy to know that

$$g\left(\alpha, 1+\sqrt{5}\right) = -\frac{3+a}{-12+3\sqrt{5}+a} > 0$$

and $f(\alpha) - g\left(\alpha, 1+\sqrt{5}\right) > 0$ for arbitrary α satisfying (11). Therefore, we have

$$\kappa(\alpha) = \min_{1<\gamma<\infty}\{\kappa(\alpha,\gamma)\} = \kappa\left(\alpha, 1+\sqrt{5}\right) = f(\alpha).$$

At last we notice that $\kappa(\alpha)$ is minimized if and only if $\alpha = 1$.

4. CONCLUSION

In this paper the global stabilization problem for discrete-time multiple integrators system with saturated input is considered and a new class of nested type nonlinear feedback law is proposed. This class of nonlinear feedback law inherits the advantage of the control law proposed in Teel [1992] as it needs only $\left[\frac{n+1}{2}\right]$ (n is the dimension of the system) saturation elements, and thus can significantly increase the control energy and improve the convergence performances of the closed-loop system. However, this new nonlinear control law is different from that in Teel [1992] because it can result in all the poles of the closed-loop system residing at some pairs of conjugate complex numbers having negative real parts when none of the saturation elements in the control laws is saturated. Only real numbers are allowed in the other existing results. Also, there are some free parameters ε_i and α_i in the control laws that can be well chosen by the designer to achieve better system performances. Due to these advantages, this new type of nonlinear feedback law can adequately improve the convergence performance and are superior to some other existing control laws, as illustrated by an example.

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