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On Statistical Control of Stochastic Servo-Systems: Performance-Measure Statistics and State-Feedback Paradigm ⋆

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Abstract: This paper provides a concise and up-to-date analysis of the foundations of performance robustness of a linear-quadratic class of servo-systems with respect to variability in a stochastic environment. The dynamics of servo-systems are corrupted by a standard stationary Wiener process and include input functions that are controlled by controllers. Basic assumptions will be that controllers have access to the current value of the states of the systems and would like to learn about performance uncertainty of the systems that is now affected by other non-cooperative learners, i.e. model deviations and environmental disturbances named Nature. The controller considered here optimizes a multi-objective criterion over time where optimization takes place with high regard for possible random sample realizations by Nature who may more likely not be acting in concert. It is found that the optimal servo in the finite horizon case is a novel two-degrees-of-freedom controller with: one, a feedback controller with state measurements that is robust against performance uncertainty; two, a model-following controller that minimizes the difference between the reference model and the system outputs.

Keywords: Servo-systems, linear-quadratic structure, performance-measure statistics, statistical control, Mayer problems, dynamic programming

1. INTRODUCTION

In recent works Pham (2000) through Pham (2007), the statistical control theory for a class of optimal stochastic regulator problems has been developed for the task of returning systems to either zero or pre-specified states in a complete statistical description of Chi-squared random measures of performance. In fact, this regulator problem class is a special case of a wider class of problems where it is required that the outputs of a system follow a reference signal which in turn belongs to a known class of signals. The present research investigation now examines a possible extension of this generalized stochastic regulator theory developed so far in such a way that the resulting servo controller consists of a feedback controller together with a model-following controller involving processing of the desired reference signals to ensure the outputs of a linear stochastic system follow as closely as possible the outputs of a reference system in accordance of a given target probability density function of Chi-squared random measure of performance. To the best knowledge of the author, the theoretical development in the sequel appears to be the first of its kind and its novel solution concepts are related well with the extensive literature on tracking and feedforward control design problems such as the earlier works of Davison (1973, 1976) and Yuksel (2006); just to name a few. Most of these works only considered the traditional measure of average performance using dynamic programming approach.

The results here will effectively address two key issues that have not been dealt with so far in the stochastic control literature; namely, how to quantify higher-order characteristics of performance uncertainty with respect to all sample realizations of the underlying stochastic process (i.e. Nature) and how to design new model-following strategies that directly influence the performance distribution and robustness. Therefore, the enabling solution included here will bring one step closer the realization of optimal tracking of stochastic systems with multi-attribute performance guarantees. Future work will answer the question of applications as may be seen in longitudinal acceleration command tracking, speed capture systems, and etc.

2. PERFORMANCE-MEASURE STATISTICS

Precisely stated, both state and output dynamics of a stochastic linear system modeled on $[t_0, t_f]$ are given by

$$dx(t) = (A(t)x(t) + B(t)u(t))\,dt + G(t)dw(t),\; x(t_0) \quad (1)$$
$$z(t) = C(t)x(t) \quad (2)$$

where the deterministic coefficients $A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times n})$, $B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times m})$, $C \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r\times n})$, and $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times p})$. Nature, $w(t) \triangleq w(t,\omega) : [t_0, t_f] \times \Omega \mapsto \mathbb{R}^p$ is the p-dimensional Wiener process starting from t_0 with action space of Ω, which is independent of the known $x(t_0) \triangleq x_0$. $\{\mathcal{F}_{t\geq t_0>0}\}$ is its filtration on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t\geq t_0>0}\}, \mathcal{P})$ over $[t_0, t_f]$ with the correlation of independent increments

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$$E\left\{[w(\tau)-w(\xi)][w(\tau)-w(\xi)]^T\right\} = W|\tau-\xi|\,, \quad W>0\,.$$

The admissible control $u \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0,t_f];\mathbb{R}^m))$ defined by a subset of Hilbert space of $\mathbb{R}^m$-valued square-integrable processes on $[t_0,t_f]$ that are adapted to the σ-field $\mathcal{F}_t$ generated by $w(t)$ with $E\left\{\int_{t_0}^{t_f} u^T(\tau)u(\tau)d\tau\right\} < \infty$ is robust against Nature's actions ω so that the resulting system outputs $z \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0,t_f];\mathbb{R}^r))$ effectively follows the desired transient responses $z_d \in L^2([t_0,t_f];\mathbb{R}^r)$ of the given linear reference model

$$dx_d(t) = A_d(t)x_d(t)dt\,, \quad x_d(t_0) = x_{d0} \tag{3}$$

$$z_d(t) = C_d(t)x_d(t) \tag{4}$$

where the deterministic coefficients $A_d \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n_d\times n_d})$ and $C_d \in \mathcal{C}([t_0,t_f];\mathbb{R}^{r\times n_d})$.

In addition, the stochastic system (1) in the absence of process noises is supposed to be uniformly exponentially stable. That is, there exist positive constants η_1 and η_2 such that the pointwise matrix norm of the closed-loop state transition matrix satisfies the inequality

$$||\Phi(t,\tau)|| \le \eta_1 e^{-\eta_2(t-\tau)} \qquad \forall\, t\ge\tau\ge t_0\ .$$

The pair $(A(t),B(t))$ is stabilizable if there exists a bounded matrix-valued function $K(t)$ such that $dx(t) = (A(t)+B(t)K(t))\,x(t)dt$ is uniformly exponentially stable. Similarly, the pair $(C_d(t),A_d(t))$ is assumed detectable. Hence, there must exist a bounded matrix-valued function $L_d(t)$ such that $dx_d(t) = (A_d(t)-L_d(t)C_d(t))\,x_d(t)dt$ is also uniformly exponentially stable. Associated with the admissible 3-tuple $(x_(\cdot), x_{d0}(\cdot); u(\cdot))$ is a finite-horizon integral-quadratic performance-measure $J: \mathbb{R}^n \times \mathbb{R}^{n_d} \times L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m)) \mapsto \mathbb{R}^+$

$$J(x_0,x_{d0};u(\cdot)) = [z(t_f)-z_d(t_f)]^T Q_f\,[z(t_f)-z_d(t_f)] \tag{5}$$
$$+\int_{t_0}^{t_f}\Big\{[z-z_d]^T(\tau)Q(\tau)\,[z-z_d]\,(\tau) + u^T(\tau)R(\tau)u(\tau)\Big\}d\tau$$

where design parameters $Q_f \in \mathbb{R}^{r\times r}$, $Q \in \mathcal{C}([t_0,t_f];\mathbb{R}^{r\times r})$, and invertible $R \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m\times m})$ are deterministic, symmetric and positive semi-definite relative weighting of the terminal state, state trajectory, and control input.

In view of the linear system (1)-(2) and the quadratic performance-measure (5), admissible control laws are therefore, restricted to being generated by the mapping $\gamma: [t_0,t_f]\times L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^n)) \times L^2(\mathcal{C}([t_0,t_f];\mathbb{R}^{n_d})) \mapsto L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m))$ with the rule of actions

$$u(t) = \gamma(t,x(t),x_d(t)) \triangleq K_x(t)x(t) + K_{x_d}(t)x_d(t) \tag{6}$$

where the deterministic matrix-valued functions $K_x \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m\times n})$ and $K_{x_d} \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m\times n_d})$ are, respectively, the admissible feedback gain and the feedforward gain on the reference model states from restraint sets which are yet to be defined. To convert the stochastic servo problem to a stochastic regulator problem, it requires to define augmented state variables and system parameters

$$x_a \triangleq \begin{bmatrix} x \\ x_d \end{bmatrix};\ A_a \triangleq \begin{bmatrix} A+BK_x & BK_{x_d} \\ 0 & A_d \end{bmatrix};\ G_a \triangleq \begin{bmatrix} G \\ 0 \end{bmatrix}$$

together with the state and terminal penalty weightings

$$N_a \triangleq \begin{bmatrix} C^TQC + K_x^TRK_x & -C^TQC_d + K_x^TRK_{x_d} \\ -C_d^TQC + K_{x_d}^TRK_x & C_d^TQC_d + K_{x_d}^TRK_{x_d} \end{bmatrix};$$

$$Q_{fa} \triangleq \begin{bmatrix} C^T(t_f)Q_fC(t_f) & -C^T(t_f)Q_fC_d(t_f) \\ -C_d^T(t_f)Q_fC(t_f) & C_d^T(t_f)Q_fC_d(t_f) \end{bmatrix}$$

which lead to the equivalent performance-measure of (5)

$$J(x_{a0};K_x(\cdot),K_{x_d}(\cdot)) = x_a^T(t_f)Q_{fa}(t_f)x_a(t_f) + \int_{t_0}^{t_f} x_a^T(\tau)N_a(\tau)x_a(\tau)d\tau \tag{7}$$

subject to the augmented system of (1)-(4)

$$dx_a(t) = A_a(t)x_a(t)dt + G_a(t)dw(t),\ x_a(t_0) = x_{a0} \tag{8}$$

with initial value condition

$$x_{a0} = \begin{bmatrix} x_0 \\ x_{d0} \end{bmatrix}.$$

As previously established by Pham (2000)-Pham (2005), the following theorem contains an efficient and tractable procedure for calculating all the performance-measure statistics of any order that completely capture the performance uncertainty of the augmented stochastic system (8) and (7).

Theorem 1. Performance-Measure Statistics.
Suppose that (A,B) is uniformly stabilizable and (C_d,A_d) is uniformly detectable. The kth-cumulant or equivalently kth-order statistic of the Chi-squared performance-measure (7) is given by

$$\kappa_k(t_0,x_{a0}) = x_{a0}^TH_a(t_0,k)x_{a0} + D_a(t_0,k) \tag{9}$$

where the cumulant-generating components $H_a(\alpha,k)$ and $D_a(\alpha,k)$ evaluated at $\alpha = t_0$ satisfy the cumulant-generating equations (with the dependence of $H_a(\alpha,k)$ and $D_a(\alpha,k)$ upon K_x and K_{x_d} suppressed)

$$\frac{d}{d\alpha}H_a(\alpha,1) = -A_a^T(\alpha)H_a(\alpha,1) - H_a(\alpha,1)A_a(\alpha) - N_a(\alpha)\,, \tag{10}$$

$$\frac{d}{d\alpha}H_a(\alpha,i) = -A_a^T(\alpha)H_a(\alpha,i) - H_a(\alpha,i)A_a(\alpha) - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H_a(\alpha,j)G_a(\alpha)WG_a^T(\alpha)H_a(\alpha,i-j)\,, \tag{11}$$

$$\frac{d}{d\alpha}D_a(\alpha,i) = -\mathrm{Tr}\left\{H_a(\alpha,i)G_a(\alpha)WG_a^T(\alpha)\right\} \tag{12}$$

with the terminal-value conditions $H_a(t_f,1) = Q_{fa}(t_f)$, $H_a(t_f,i) = 0$ for $2\le i\le k$ and $D_a(t_f,i) = 0$ for $1\le i\le k$.

Note that this computational procedure now allows the incorporation of classes of linear-feedback strategies in the statistical control problems. Moreover, these performance-measure statistics or cumulants (9) are then interpreted in terms of variables and system parameters of the original servo as follows

$$\kappa_k(t_0,x_0,z_0) = x_0^TH_{11}^k(t_0)x_0 + 2x_0^TH_{12}^k(t_0)z_0 + z_0^TH_{22}^k(t_0)z_0 + D^k(t_0)\,, \tag{13}$$

provided that the matrix partition is of the form

$$H_a(\alpha,i) = \begin{bmatrix} H_{11}^i(\alpha) & H_{12}^i(\alpha) \\ H_{12}^{iT}(\alpha) & H_{22}^i(\alpha) \end{bmatrix},\quad 1\le i\le k \tag{14}$$

from which the components $\left\{H_{11}^i(\alpha)\right\}_{i=1}^k$, $\left\{H_{12}^i(\alpha)\right\}_{i=1}^k$, and $\left\{H_{22}^i(\alpha)\right\}_{i=1}^k$ implicitly depend on K_x and K_{x_d} and satisfy the cumulant-supporting equations

$$\frac{d}{d\alpha}H_{11}^1(\alpha) = -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T H_{11}^1(\alpha) - H_{11}^1(\alpha)\left[A(\alpha)+B(\alpha)K_x(\alpha)\right] - K_x^T(\alpha)R(\alpha)K_x(\alpha) - C^T(\alpha)Q(\alpha)C(\alpha) \tag{15}$$

$$\frac{d}{d\alpha}H_{11}^i(\alpha) = -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T H_{11}^i(\alpha) - H_{11}^i(\alpha)\left[A(\alpha)+B(\alpha)K_x(\alpha)\right] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H_{11}^j(\alpha)G(\alpha)WG^T(\alpha)H_{11}^{i-j}(\alpha) \quad (16)$$

$$\frac{d}{d\alpha}H_{12}^1(\alpha) = -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T H_{12}^1(\alpha) - H_{11}^1(\alpha)B(\alpha)K_{x_d}(\alpha) - H_{12}^1(\alpha)A_d(\alpha) - K_x^T(\alpha)R(\alpha)K_{x_d}(\alpha) + C^T(\alpha)Q(\alpha)C_d(\alpha) \quad (17)$$

$$\frac{d}{d\alpha}H_{12}^i(\alpha) = -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T H_{12}^i(\alpha) - H_{11}^i(\alpha)B(\alpha)K_{x_d}(\alpha) - H_{12}^i(\alpha)A_d(\alpha) - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H_{11}^j(\alpha)G(\alpha)WG^T(\alpha)H_{12}^{i-j}(\alpha) \quad (18)$$

$$\frac{d}{d\alpha}H_{22}^1(\alpha) = -A_d^T(\alpha)H_{22}^1(\alpha) - H_{22}^1(\alpha)A_d(\alpha) - K_{x_d}^T(\alpha)B^T(\alpha)H_{12}^1(\alpha) - H_{12}^{1T}(\alpha)B(\alpha)K_{x_d}(\alpha) - K_{x_d}^T(\alpha)R(\alpha)K_{x_d}(\alpha) - C_d^T(\alpha)Q(\alpha)C_d(\alpha) \quad (19)$$

$$\frac{d}{d\alpha}H_{22}^i(\alpha) = -A_d^T(\alpha)H_{22}^i(\alpha) - H_{22}^i(\alpha)A_d(\alpha) - K_{x_d}^T(\alpha)B^T(\alpha)H_{12}^i(\alpha) - H_{12}^{iT}(\alpha)B(\alpha)K_{x_d}(\alpha) - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H_{12}^{jT}(\alpha)G(\alpha)WG^T(\alpha)H_{12}^{i-j}(\alpha) \quad (20)$$

$$\frac{d}{d\alpha}D^i(\alpha) = -\mathrm{Tr}\left\{H_{11}^i(\alpha)G(\alpha)WG^T(\alpha)\right\} \quad (21)$$

with terminal-value conditions $H_{11}^1(t_f) = C^T(t_f)Q_fC(t_f)$, $H_{11}^i(t_f) = 0$ for $2 \le i \le k$; $H_{12}^1(t_f) = -C^T(t_f)Q_fC_d(t_f)$, $H_{12}^i(t_f) = 0$ for $2 \le i \le k$; $H_{22}^1(t_f) = C_d^T(t_f)Q_fC_d(t_f)$, $H_{22}^i(t_f) = 0$ for $2 \le i \le k$; and $D^i(t_f) = 0$ for $1 \le i \le k$.

3. PROBLEM STATEMENTS

Although different states $x(t)$ will result in different values for the "performance-to-come" wherein (7) is redefined with the lower integration limit t_0 being replaced by the running variable α, the cumulant values are however, functions of time-backward evolutions of the cumulant-generating variables $H_{11}^i(\alpha)$, $H_{12}^i(\alpha)$, $H_{22}^i(\alpha)$ and $D^i(\alpha)$ and do not take into account of all the intermediate values $x(t)$. This fact therefore makes the new optimization problem as being considered in statistical control particularly unique as compared with the more traditional dynamic programming class of investigations. In other words, the time-backward trajectories (15)-(21) should be considered as the "new" dynamical equations for statistical control from which the resulting Mayer optimization and associated value function in the framework of dynamic programming Fleming (1975) therefore depend on these "new" state variables $H_{11}^i(\alpha)$, $H_{12}^i(\alpha)$, $H_{22}^i(\alpha)$ and $D^i(\alpha)$, not the classical states $x(t)$ as the people may often expect.

For notational simplicity and compact formulation, it is required to introduce the convenient mappings to denote the right members of (15)-(21)

$$\mathcal{F}_{11}^i : [t_0,t_f] \times (\mathbb{R}^{n\times n})^k \times \mathbb{R}^{m\times n} \mapsto \mathbb{R}^{n\times n}$$
$$\mathcal{F}_{12}^i : [t_0,t_f] \times (\mathbb{R}^{n\times n_d})^k \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n_d} \mapsto \mathbb{R}^{n\times n_d}$$
$$\mathcal{F}_{22}^i : [t_0,t_f] \times (\mathbb{R}^{n_d\times n_d})^k \times (\mathbb{R}^{n\times n_d})^k \times \mathbb{R}^{m\times n_d} \mapsto \mathbb{R}^{n_d\times n_d}$$
$$\mathcal{G}^i : [t_0,t_f] \times (\mathbb{R}^{n})^k \mapsto \mathbb{R}$$

with the rules of action

$$\mathcal{F}_{11}^1(\alpha,\mathcal{H}_{11},K_x) \triangleq -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T \mathcal{H}_{11}^1(\alpha) - \mathcal{H}_{11}^1(\alpha)\left[A(\alpha)+B(\alpha)K_x(\alpha)\right] - K_x^T(\alpha)R(\alpha)K_x(\alpha) - C^T(\alpha)Q(\alpha)C(\alpha)\,,$$

$$\mathcal{F}_{11}^i(\alpha,\mathcal{H}_{11},K_x) \triangleq -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T \mathcal{H}_{11}^i(\alpha) - \mathcal{H}_{11}^i(\alpha)\left[A(\alpha)+B(\alpha)K_x(\alpha)\right] - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{11}^j(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{11}^{i-j}(\alpha)\,,$$

$$\mathcal{F}_{12}^1(\alpha,\mathcal{H}_{12},K_x,K_{x_d}) \triangleq -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T \mathcal{H}_{12}^1(\alpha) - \mathcal{H}_{11}^1(\alpha)B(\alpha)K_{x_d}(\alpha) - \mathcal{H}_{12}^1(\alpha)A_d(\alpha) - K_x^T(\alpha)R(\alpha)K_{x_d}(\alpha) + C^T(\alpha)Q(\alpha)C_d(\alpha)\,,$$

$$\mathcal{F}_{12}^i(\alpha,\mathcal{H}_{12},K_x,K_{x_d}) \triangleq -\left[A(\alpha)+B(\alpha)K_x(\alpha)\right]^T \mathcal{H}_{12}^i(\alpha) - \mathcal{H}_{11}^i(\alpha)B(\alpha)K_{x_d}(\alpha) - \mathcal{H}_{12}^i(\alpha)A_d(\alpha) - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{11}^j(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{12}^{i-j}(\alpha)\,,$$

$$\mathcal{F}_{22}^1(\alpha,\mathcal{H}_{22},\mathcal{H}_{12},K_{x_d}) \triangleq -A_d^T(\alpha)\mathcal{H}_{22}^1(\alpha) - \mathcal{H}_{22}^1(\alpha)A_d(\alpha) - K_{x_d}^T(\alpha)B^T(\alpha)\mathcal{H}_{12}^1(\alpha) - \mathcal{H}_{12}^{1T}(\alpha)B(\alpha)K_{x_d}(\alpha) - K_{x_d}^T(\alpha)R(\alpha)K_{x_d}(\alpha) - C_d^T(\alpha)Q(\alpha)C_d(\alpha)\,,$$

$$\mathcal{F}_{22}^i(\alpha,\mathcal{H}_{22},\mathcal{H}_{12},K_{x_d}) \triangleq -A_d^T(\alpha)\mathcal{H}_{22}^i(\alpha) - \mathcal{H}_{22}^i(\alpha)A_d(\alpha) - K_{x_d}^T(\alpha)B^T(\alpha)\mathcal{H}_{12}^i(\alpha) - \mathcal{H}_{12}^{iT}(\alpha)B(\alpha)K_{x_d}(\alpha) - \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_{12}^{jT}(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{12}^{i-j}(\alpha)\,,$$

$$\mathcal{G}^i(\alpha,\mathcal{H}_{11}) \triangleq -\mathrm{Tr}\left\{\mathcal{H}_{11}^i(\alpha)G(\alpha)WG^T(\alpha)\right\}\,,$$

where the components of k-tuple variables $\mathcal{H}_{11}$, $\mathcal{H}_{12}$, $\mathcal{H}_{22}$ and $\mathcal{D}$ defined by $\mathcal{H}_{11}(\cdot) \triangleq \left(\mathcal{H}_{11}^1(\cdot),\ldots,\mathcal{H}_{11}^k(\cdot)\right)$, $\mathcal{H}_{12}(\cdot) \triangleq \left(\mathcal{H}_{12}^1(\cdot),\ldots,\mathcal{H}_{12}^k(\cdot)\right)$, $\mathcal{H}_{22}(\cdot) \triangleq \left(\mathcal{H}_{22}^1(\cdot),\ldots,\mathcal{H}_{22}^k(\cdot)\right)$, and $\mathcal{D}(\cdot) \triangleq \left(\mathcal{D}^1(\cdot),\ldots,\mathcal{D}^k(\cdot)\right)$ provided that each element $\mathcal{H}_{11}^i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{n\times n})$, $\mathcal{H}_{12}^i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{n\times n_d})$, $\mathcal{H}_{22}^i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{n_d\times n_d})$, and $\mathcal{D}^i \in \mathcal{C}^1([t_0,t_f];\mathbb{R})$ have the representations $\mathcal{H}_{11}^i(\cdot) \triangleq H_{11}^i(\cdot)$, $\mathcal{H}_{12}^i(\cdot) \triangleq H_{12}^i(\cdot)$, $\mathcal{H}_{22}^i(\cdot) \triangleq H_{22}^i(\cdot)$, and $\mathcal{D}^i(\cdot) \triangleq D^i(\cdot)$. Thus, the product mappings of the equations (15)-(21)

$$\mathcal{F}_{11} : [t_0,t_f] \times (\mathbb{R}^{n\times n})^k \times \mathbb{R}^{m\times n} \mapsto (\mathbb{R}^{n\times n})^k$$
$$\mathcal{F}_{12} : [t_0,t_f] \times (\mathbb{R}^{n\times n_d})^k \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n_d} \mapsto (\mathbb{R}^{n\times n_d})^k$$
$$\mathcal{F}_{22} : [t_0,t_f] \times (\mathbb{R}^{n_d\times n_d})^k \times (\mathbb{R}^{n\times n_d})^k \times \mathbb{R}^{m\times n_d} \mapsto (\mathbb{R}^{n_d\times n_d})^k$$
$$\mathcal{G} : [t_0,t_f] \times (\mathbb{R}^{n\times n})^k \mapsto \mathbb{R}^k$$

in statistical control for servo problems have the rules of action given by

$$\frac{d}{d\alpha}\mathcal{H}_{11}(\alpha) = \mathcal{F}_{11}(\alpha, \mathcal{H}_{11}(\alpha), K_x(\alpha))\,, \tag{22}$$

$$\frac{d}{d\alpha}\mathcal{H}_{12}(\alpha) = \mathcal{F}_{12}(\alpha, \mathcal{H}_{12}(\alpha), K_x(\alpha), K_{x_d}(\alpha))\,, \tag{23}$$

$$\frac{d}{d\alpha}\mathcal{H}_{22}(\alpha) = \mathcal{F}_{22}(\alpha, \mathcal{H}_{22}(\alpha), \mathcal{H}_{12}(\alpha), K_x(\alpha))\,, \tag{24}$$

$$\frac{d}{d\alpha}\mathcal{D}(\alpha) = G(\alpha, \mathcal{H}_{11}(\alpha))\,, \tag{25}$$

under the obvious definitions $\mathcal{F}_{11} \triangleq \mathcal{F}_{11}^1 \times \cdots \times \mathcal{F}_{11}^k$, $\mathcal{F}_{12} \triangleq \mathcal{F}_{12}^1 \times \cdots \times \mathcal{F}_{12}^k$, $\mathcal{F}_{22} \triangleq \mathcal{F}_{22}^1 \times \cdots \times \mathcal{F}_{22}^k$, and $\mathcal{G} \triangleq \mathcal{G}^1 \times \cdots \times \mathcal{G}^k$ together with the terminal values $\mathcal{H}_{11}(t_f) \triangleq \mathcal{H}_{11}^f = \big(C^T(t_f)Q_fC(t_f), 0, \ldots, 0\big)$, $\mathcal{H}_{12}(t_f) \triangleq \mathcal{H}_{12}^f = \big(-C^T(t_f)Q_fC_d(t_f), 0, \ldots, 0\big)$, $\mathcal{H}_{22}(t_f) \triangleq \mathcal{H}_{22}^f = \big(C_d^T(t_f)Q_fC_d(t_f), 0, \ldots, 0\big)$, and $\mathcal{D}(t_f) \triangleq \mathcal{D}^f = (0, \ldots, 0)$.

Recall that the product system uniquely determines $\mathcal{H}_{11}$, $\mathcal{H}_{12}$, $\mathcal{H}_{22}$, and $\mathcal{D}$ once admissible feedback and feedforward gains K_x and K_{x_d} are specified. Therefore, it should be considered $\mathcal{H}_{11} \equiv \mathcal{H}_{11}(\cdot, K_x, K_{x_d})$, $\mathcal{H}_{12} \equiv \mathcal{H}_{12}(\cdot, K_x, K_{x_d})$, $\mathcal{H}_{22} \equiv \mathcal{H}_{22}(\cdot, K_x, K_{x_d})$, and $\mathcal{D} \equiv \mathcal{D}(\cdot, K_x, K_{x_d})$. The performance index in the statistical control problem can now be formulated in K_x and K_{x_d}.

Definition 1. Performance Index.
Fix $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$. Then for the given 3-tuple (t_0, x_0, x_{d0}), the performance index in statistical control for explicit model-following problems is given by
$\phi_s : \{t_0\} \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^{n\times n_d})^k \times (\mathbb{R}^{n_d\times n_d})^k \times \mathbb{R}^k \mapsto \mathbb{R}^+$
with the rule of action

$$\begin{aligned}&\phi_s\left(t_0, \mathcal{H}_{11}(t_0), \mathcal{H}_{12}(t_0), \mathcal{H}_{22}(t_0), \mathcal{D}(t_0)\right)\\ &\triangleq \sum_{i=1}^k \mu_i \kappa_i(t_0, x_0, x_{d0}; K_x, K_{x_d}) = \sum_{i=1}^k \mu_i \Big[x_0^T \mathcal{H}_{11}^i(t_0) x_0\\ &\quad + 2x_0^T \mathcal{H}_{12}^i(t_0) x_{d0} + x_{d0}^T \mathcal{H}_{22}^i(t_0) x_{d0} + \mathcal{D}^i(t_0)\Big]\end{aligned} \tag{26}$$

where the scalar, real constants μ_i represent parametric design freedom and the unique solutions $\left\{\mathcal{H}_{11}^i(\alpha)\right\}_{i=1}^k$, $\left\{\mathcal{H}_{12}^i(\alpha)\right\}_{i=1}^k$, $\left\{\mathcal{H}_{22}^i(\alpha)\right\}_{i=1}^k$, and $\{\mathcal{D}^i(\alpha)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the dynamic equations of motion (22)-(25).

It is worth to observe the performance index (26) adopts a new and comprehensive optimization criterion which introduces additional parametric design freedom in the class of feedback control laws that will then result in a broad class of problem solutions as one can directly derive from these solutions to other related results in LQG and Risk-Sensitive control problems. More importantly, the ultimate objective here is to introduce a competition among performance-measure statistics as they directly influence on the performance distribution of (5).

Definition 2. Admissible Feedback & Feedforward Gains.
For given terminal data $(t_f, \mathcal{H}_{11}^f, \mathcal{H}_{12}^f, \mathcal{H}_{22}^f, \mathcal{D}^f)$, the classes of admissible feedback and feedforward gains are defined as follows. Let compact subsets $\overline{K}_x \subset \mathbb{R}^{m\times n}$ and $\overline{K}_{x_d} \subset \mathbb{R}^{m\times n_d}$ be the sets of allowable gain values. For the given $k \in \mathbb{Z}^+$ and sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the sets of admissible $\mathcal{K}^x_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu}$ and $\mathcal{K}^{x_d}_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu}$ are respectively assumed to be the classes of $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m\times n})$ and $\mathcal{C}([t_0, t_f]; \mathbb{R}^{m\times n_d})$ with values $K_x(\cdot) \in \overline{K}_x$ and $K_{x_d}(\cdot) \in \overline{K}_{x_d}$ for which solutions to the dynamic equations of motion (22)-(25) exist on the interval of optimization $[t_0, t_f]$.

Definition 3. Optimization Problem.
Suppose that $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then, the optimization problem over $[t_0, t_f]$ is defined as the minimization of the performance index (26) with respect to (K_x, K_{x_d}) over $\mathcal{K}^x_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu} \times \mathcal{K}^{x_d}_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu}$ and subject to the equations of motion (22)-(25).

Definition 4. Reachable Set.
Let the reachable set $\mathcal{Q} \triangleq \Big\{(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z}) \in [t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^{n\times n_d})^k \times (\mathbb{R}^{n_d\times n_d})^k \times \mathbb{R}^k$ such that $\mathcal{K}^x_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu} \neq 0$ and $\mathcal{K}^{x_d}_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu} \neq 0\Big\}$.

By adapting to the initial cost problem and the terminologies present in statistical control, the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function is derived from the excellent treatment in Fleming (1975).

Theorem 2. HJB Equation-Mayer Problem.
Let $(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ be any interior point of the reachable set $\mathcal{Q}$ at which the value function $\mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ is differentiable. If there exist optimal gains $K_x^* \in \mathcal{K}^x_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu}$ and $K_{x_d}^* \in \mathcal{K}^{x_d}_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu}$, then the partial differential equation of dynamic programming

$$\begin{aligned}0 = \min_{K_x\in\overline{K}_x,\, K_{x_d}\in\overline{K}_z} &\Bigg\{\frac{\partial}{\partial\varepsilon}\mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})\\ &+ \frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Y}_{11})}\mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})\,\mathrm{vec}\left(\mathcal{F}_{11}(\varepsilon, \mathcal{Y}_{11}, K_x)\right)\\ &+ \frac{\partial}{\partial\mathrm{vec}(\mathcal{Y}_{12})}\mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})\mathrm{vec}(\mathcal{F}_{12}(\varepsilon, \mathcal{Y}_{12}, K_x, K_{x_d}))\\ &+ \frac{\partial}{\partial\mathrm{vec}(\mathcal{Y}_{22})}\mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})\mathrm{vec}(\mathcal{F}_{22}(\varepsilon, \mathcal{Y}_{22}, \mathcal{Y}_{12}, K_{x_d}))\\ &+ \frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Z})}\mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})\,\mathrm{vec}\left(\mathcal{G}(\varepsilon, \mathcal{Y}_{11})\right)\Bigg\}\end{aligned} \tag{27}$$

is satisfied. The boundary condition for (27) is given by $\mathcal{V}\left(t_0, \mathcal{H}_{11}^0, \mathcal{H}_{12}^0, \mathcal{H}_{22}^0, \mathcal{D}^0\right) = \phi_s\left(t_0, \mathcal{H}_{11}^0, \mathcal{H}_{12}^0, \mathcal{H}_{22}^0, \mathcal{D}^0\right)$.

Theorem 3. Verification Theorem.
Fix $k \in \mathbb{Z}^+$ and let $\mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ be a continuously differentiable solution of (27) which satisfies the boundary condition $\mathcal{W}(t_0, \mathcal{H}_{11}^0, \mathcal{H}_{12}^0, \mathcal{H}_{22}^0, \mathcal{D}^0) = \phi_s\left(t_0, \mathcal{H}_{11}^0, \mathcal{H}_{12}^0, \mathcal{H}_{22}^0, \mathcal{D}^0\right)$. Let $(t_f, \mathcal{H}_{11}^f, \mathcal{H}_{12}^f, \mathcal{H}_{22}^f, \mathcal{D}^f)$ be in $\mathcal{Q}$; (K_x, K_{x_d}) a pair of gains in $\mathcal{K}^x_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu} \times \mathcal{K}^{x_d}_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu}$; $\mathcal{H}_{11}$, $\mathcal{H}_{12}$, $\mathcal{H}_{22}$ and $\mathcal{D}$ the solutions of (22)-(25). Then, $\mathcal{W}(\alpha, \mathcal{H}_{11}(\alpha), \mathcal{H}_{12}(\alpha), \mathcal{H}_{22}(\alpha), \mathcal{D}(\alpha))$ is a time-backward increasing function of α. If $(K_x^*, K_{x_d}^*)$ is a gain set in $\mathcal{K}^x_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu} \times \mathcal{K}^{x_d}_{t_f,\mathcal{H}_{11}^f,\mathcal{H}_{12}^f,\mathcal{H}_{22}^f,\mathcal{D}^f;\mu}$ defined on $[t_0, t_f]$ with corresponding solutions, $\mathcal{H}_{11}^*$, $\mathcal{H}_{12}^*$, $\mathcal{H}_{22}^*$ and $\mathcal{D}^*$ of (22)-(25) such that, for $\alpha \in [t_0, t_f]$

$$\begin{aligned}0 &= \frac{\partial}{\partial\varepsilon}\mathcal{W}(\alpha, \mathcal{H}_{11}^*(\alpha), \mathcal{H}_{12}^*(\alpha), \mathcal{H}_{22}^*(\alpha), \mathcal{D}^*(\alpha))\\ &\quad + \frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Y}_{11})}\mathcal{W}(\alpha, \mathcal{H}_{11}^*(\alpha), \mathcal{H}_{12}^*(\alpha), \mathcal{H}_{22}^*(\alpha), \mathcal{D}^*(\alpha))\\ &\qquad\qquad \cdot \mathrm{vec}(\mathcal{F}_{11}(\alpha, \mathcal{H}_{11}^*(\alpha), K_x^*(\alpha)))\end{aligned}$$

$$+ \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y}_{12})} \mathcal{W}(\alpha, \mathcal{H}_{11}^*(\alpha), \mathcal{H}_{12}^*(\alpha), \mathcal{H}_{22}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \operatorname{vec}(\mathcal{F}_{12}(\alpha, \mathcal{H}_{12}^*(\alpha), K_x^*(\alpha), K_{x_d}^*(\alpha))) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Y}_{22})} \mathcal{W}(\alpha, \mathcal{H}_{11}^*(\alpha), \mathcal{H}_{12}^*(\alpha), \mathcal{H}_{22}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \operatorname{vec}(\mathcal{F}_{22}(\alpha, \mathcal{H}_{22}^*(\alpha), \mathcal{H}_{12}^*(\alpha), K_{x_d}^*(\alpha))) + \frac{\partial}{\partial \operatorname{vec}(\mathcal{Z})} \mathcal{W}(\alpha, \mathcal{H}_{11}^*(\alpha), \mathcal{H}_{12}^*(\alpha), \mathcal{H}_{22}^*(\alpha), \mathcal{D}^*(\alpha)) \cdot \operatorname{vec}(\mathcal{G}(\alpha, \mathcal{H}_{11}^*(\alpha))), \quad (28)$$

then K_x^* and $K_{x_d}^*$ are optimal feedback and feedforward gains. Moreover

$$\mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z}) = \mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z}) \quad (29)$$

where $\mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ is the value function.

4. STATISTICAL CONTROL SOLUTION

Recall that the optimization problem being considered herein is in "Mayer form" and can be solved by applying an adaptation of the Mayer form verification theorem of dynamic programming given in Fleming (1975). In the framework of dynamic programming, it is often required to denote the terminal time and states of a family of optimization problems by $(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ rather than $\left(t_f, \mathcal{H}_{11}^f, \mathcal{H}_{12}^f, \mathcal{H}_{22}^f, \mathcal{D}^f\right)$. Thus, the value of these optimization problems depends on their terminal conditions. In particular, for any $\varepsilon \in [t_0, t_f]$, the states of the equations (22)-(25) are denoted by $\mathcal{H}_{11}(\varepsilon) = \mathcal{Y}_{11}$, $\mathcal{H}_{12}(\varepsilon) = \mathcal{Y}_{12}$, $\mathcal{H}_{22}(\varepsilon) = \mathcal{Y}_{22}$, and $\mathcal{D}(\varepsilon) = \mathcal{Z}$. Then, the quadratic-affine nature of (26) implies that a solution to the HJB equation (27) is suggested of the form as follows.

Theorem 4. Fix $k \in \mathbb{Z}^+$ and let $(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ be any interior point of the reachable set $\mathcal{Q}$ at which the real-valued function $\mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ described by

$$\mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z}) = x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{Y}_{11}^i + \frac{d}{d\varepsilon} \mathcal{E}_{11}^i(\varepsilon) \right) x_0 + 2x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{Y}_{12}^i + \frac{d}{d\varepsilon} \mathcal{E}_{12}^i(\varepsilon) \right) x_{d0} + \sum_{i=1}^k \mu_i \left(\mathcal{Z}^i + \frac{d}{d\varepsilon} \mathcal{T}^i(\varepsilon) \right) + x_{d0}^T \sum_{i=1}^k \mu_i \left(\mathcal{Y}_{22}^i + \frac{d}{d\varepsilon} \mathcal{E}_{22}^i(\varepsilon) \right) x_{d0} \quad (30)$$

is differentiable. Parameterizations $\mathcal{E}_{11}^i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{n \times n})$, $\mathcal{E}_{12}^i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{n \times n_d})$, $\mathcal{E}_{22}^i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R}^{n_d \times n_d})$, and $\mathcal{T}^i \in \mathcal{C}^1([t_0, t_f]; \mathbb{R})$ are yet to be determined. Then, the derivative of $\mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ with respect to ε is

$$\frac{d}{d\varepsilon} \mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z}) = x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_{11}^i(\varepsilon, \mathcal{Y}_{11}, K_x) + \frac{d}{d\varepsilon} \mathcal{E}_{11}^i(\varepsilon) \right) x_0 + 2x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_{12}^i(\varepsilon, \mathcal{Y}_{12}, K_x, K_{x_d}) + \frac{d}{d\varepsilon} \mathcal{E}_{12}^i(\varepsilon) \right) x_{d0} + x_{d0}^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_{22}^i(\varepsilon, \mathcal{Y}_{22}, \mathcal{Y}_{12}, K_{x_d}) + \frac{d}{d\varepsilon} \mathcal{E}_{22}^i(\varepsilon) \right) x_{d0} + \sum_{i=1}^k \mu_i \left(\mathcal{G}^i(\varepsilon, \mathcal{Y}_{11}) + \frac{d}{d\varepsilon} \mathcal{T}^i(\varepsilon) \right), \quad (31)$$

provided $(K_x, K_{x_d}) \in \overline{K}_x \times \overline{K}_{x_d}$.

Replacing the guess solution (30) into the HJB equation (27), one obtains

$$\min_{(K_x, K_{x_d}) \in \overline{K}_x \times \overline{K}_{x_d}} \left\{ x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_{11}^i(\varepsilon, \mathcal{Y}_{11}, K_x) + \frac{d}{d\varepsilon} \mathcal{E}_{11}^i(\varepsilon) \right) x_0 + 2x_0^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_{12}^i(\varepsilon, \mathcal{Y}_{12}, K_x, K_{x_d}) + \frac{d}{d\varepsilon} \mathcal{E}_{12}^i(\varepsilon) \right) x_{d0} + x_{d0}^T \sum_{i=1}^k \mu_i \left(\mathcal{F}_{22}^i(\varepsilon, \mathcal{Y}_{22}, \mathcal{Y}_{12}, K_{x_d}) + \frac{d}{d\varepsilon} \mathcal{E}_{22}^i(\varepsilon) \right) x_{d0} + \sum_{i=1}^k \mu_i \left(\mathcal{G}^i(\varepsilon, \mathcal{Y}_{11}) + \frac{d}{d\varepsilon} \mathcal{T}^i(\varepsilon) \right) \right\} \equiv 0. \quad (32)$$

Differentiating the expression within the bracket (32) with respect to K_x and K_{x_d} yields the necessary conditions for an interior extremum of the performance index (26) on the finite horizon $[t_0, t_f]$

$$0 = -2B^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_{11}^i M_{11}^0 - 2\mu_1 R(\varepsilon) K_x M_{11}^0 - 2B^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_{12}^i M_{12}^0 - 2\mu_1 R(\varepsilon) K_{x_d} M_{12}^0 \quad (33)$$

$$0 = -2B^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_{12}^i M_{22}^0 - 2\mu_1 R(\varepsilon) K_{x_d} M_{22}^0 - 2B^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_{11}^i M_{12}^0 - 2\mu_1 R(\varepsilon) K_x M_{12}^0 \quad (34)$$

where matrices $M_{11}^0 \triangleq x_0 x_0^T$, $M_{12}^0 \triangleq x_{d0} x_0^T$, and $M_{22}^0 \triangleq x_{d0} x_{d0}^T$. Because these matrices are arbitrary and have rank one, the extremizing K_x and K_{x_d} must be

$$K_x(\varepsilon, \mathcal{Y}_{11}) = -R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_{11}^r, \quad (35)$$

$$K_{x_d}(\varepsilon, \mathcal{Y}_{12}) = -R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_{12}^r, \quad (36)$$

in which $\widehat{\mu}_r = \mu_i / \mu_1$. In view of (35) and (36), the value of the expression inside of the bracket of (32) becomes

$$x_0^T \left[\sum_{i=1}^k \mu_i \frac{d}{d\varepsilon} \mathcal{E}_{11}^i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_{11}^i - \sum_{i=1}^k \mu_i \mathcal{Y}_{11}^i A(\varepsilon) + \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_{11}^r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{i=1}^k \mu_i \mathcal{Y}_{11}^i + \sum_{i=1}^k \mu_i \mathcal{Y}_{11}^i B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^k \widehat{\mu}_r \mathcal{Y}_{11}^r - \mu_1 C^T(\varepsilon) Q(\varepsilon) C(\varepsilon) \right.$$

$$- \mu_1 \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_{11}^r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{Y}_{11}^s \Bigg] x_0$$

$$+ 2x_0^T \Bigg[\sum_{i=1}^{k} \mu_i \frac{d}{d\varepsilon} \mathcal{E}_{12}^i(\varepsilon) - A^T(\varepsilon) \sum_{i=1}^{k} \mu_i \mathcal{Y}_{12}^i - \sum_{i=1}^{k} \mu_i \mathcal{Y}_{12}^i A_d(\varepsilon)$$

$$+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_{11}^r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{i=1}^{k} \mu_i \mathcal{Y}_{12}^i$$

$$+ \sum_{i=1}^{k} \mu_i \mathcal{Y}_{11}^i B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_{12}^r + \mu_1 C^T(\varepsilon) Q(\varepsilon) C_d(\varepsilon)$$

$$- \mu_1 \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_{11}^r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{Y}_{12}^s$$

$$- \sum_{i=2}^{k} \mu_i \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{Y}_{11}^j G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{12}^{i-j} \Bigg] x_{d0}$$

$$+ x_{d0}^T \Bigg[\sum_{i=1}^{k} \mu_i \frac{d}{d\varepsilon} \mathcal{E}_{22}^i(\varepsilon) - A_d^T(\varepsilon) \sum_{i=1}^{k} \mu_i \mathcal{Y}_{22}^i - \sum_{i=1}^{k} \mu_i \mathcal{Y}_{22}^i A_d(\varepsilon)$$

$$+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_{12}^r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{i=1}^{k} \mu_i \mathcal{Y}_{12}^i$$

$$+ \sum_{i=1}^{k} \mu_i \mathcal{Y}_{12}^{iT} B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_{12}^r - \mu_1 C_d^T(\varepsilon) Q(\varepsilon) C_d(\varepsilon)$$

$$- \mu_1 \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{Y}_{12}^r B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{Y}_{12}^s$$

$$- \sum_{i=2}^{k} \mu_i \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{Y}_{12}^{jT} G(\varepsilon) W G^T(\varepsilon) \mathcal{Y}_{12}^{i-j} \Bigg] x_{d0}$$

$$+ \sum_{i=1}^{k} \mu_i \frac{d}{d\varepsilon} \mathcal{T}^i(\varepsilon) - \sum_{i=1}^{k} \mu_i \operatorname{Tr} \left\{ \mathcal{Y}_{11}^i G(\varepsilon) W G^T(\varepsilon) \right\} . \quad (37)$$

The minimum of (37) equal to zero for any $\varepsilon \in [t_0, t_f]$ when $\mathcal{Y}_{11}^i$, $\mathcal{Y}_{12}^i$, $\mathcal{Y}_{22}^i$ and $\mathcal{Z}^i$ evaluated at (22)-(25) requires

$$\frac{d}{d\varepsilon} \mathcal{E}_{11}^1(\varepsilon) = A^T(\varepsilon) \mathcal{H}_{11}^1(\varepsilon) + \mathcal{H}_{11}^1(\varepsilon) A(\varepsilon) + C^T(\varepsilon) Q(\varepsilon) C(\varepsilon)$$

$$- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_{11}^1(\varepsilon)$$

$$- \mathcal{H}_{11}^1(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon)$$

$$+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_{11}^s(\varepsilon) \quad (38)$$

$$\frac{d}{d\varepsilon} \mathcal{E}_{11}^i(\varepsilon) = A^T(\varepsilon) \mathcal{H}_{11}^i(\varepsilon) + \mathcal{H}_{11}^i(\varepsilon) A(\varepsilon)$$

$$- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_{11}^i(\varepsilon)$$

$$- \mathcal{H}_{11}^i(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon)$$

$$+ \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_{11}^j(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{11}^{i-j}(\varepsilon) \quad (39)$$

$$\frac{d}{d\varepsilon} \mathcal{E}_{12}^1(\varepsilon) = A^T(\varepsilon) \mathcal{H}_{12}^1(\varepsilon) + \mathcal{H}_{12}^1(\varepsilon) A_d(\varepsilon) - C^T(\varepsilon) Q(\varepsilon) C_d(\varepsilon)$$

$$- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_{12}^1(\varepsilon)$$

$$- \mathcal{H}_{11}^1(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{12}^r(\varepsilon)$$

$$+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_{12}^s(\varepsilon) \quad (40)$$

$$\frac{d}{d\varepsilon} \mathcal{E}_{12}^i(\varepsilon) = A^T(\varepsilon) \mathcal{H}_{12}^i(\varepsilon) + \mathcal{H}_{12}^i(\varepsilon) A_d(\varepsilon)$$

$$- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{11}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_{12}^i(\varepsilon)$$

$$- \mathcal{H}_{11}^i(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{12}^r(\varepsilon)$$

$$+ \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_{11}^j(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{12}^{i-j}(\varepsilon) \quad (41)$$

$$\frac{d}{d\varepsilon} \mathcal{E}_{22}^1(\varepsilon) = A_d^T(\varepsilon) \mathcal{H}_{22}^1(\varepsilon) + \mathcal{H}_{22}^1(\varepsilon) A_d(\varepsilon) + C_d^T(\varepsilon) Q(\varepsilon) C_d(\varepsilon)$$

$$- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{12}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_{12}^1(\varepsilon)$$

$$- \mathcal{H}_{12}^{1T}(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{12}^r(\varepsilon)$$

$$+ \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{12}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{s=1}^{k} \widehat{\mu}_s \mathcal{H}_{12}^s(\varepsilon) \quad (42)$$

$$\frac{d}{d\varepsilon} \mathcal{E}_{22}^i(\varepsilon) = A_d^T(\varepsilon) \mathcal{H}_{22}^i(\varepsilon) + \mathcal{H}_{22}^i(\varepsilon) A_d(\varepsilon)$$

$$- \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{12}^r(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \mathcal{H}_{12}^i(\varepsilon)$$

$$- \mathcal{H}_{12}^{iT}(\varepsilon) B(\varepsilon) R^{-1}(\varepsilon) B^T(\varepsilon) \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_{12}^r(\varepsilon)$$

$$+ \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} \mathcal{H}_{12}^{jT}(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \mathcal{H}_{12}^{i-j}(\varepsilon) \quad (43)$$

$$\frac{d}{d\varepsilon} \mathcal{T}^i(\varepsilon) = \operatorname{Tr} \left\{ \mathcal{H}_{11}^i(\varepsilon) G(\varepsilon) W G^T(\varepsilon) \right\} . \quad (44)$$

The boundary condition of $\mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$ implies that the initial conditions $\mathcal{E}_{11}^i(t_0) = 0$, $\mathcal{E}_{12}^i(t_0) = 0$, $\mathcal{E}_{22}^i(t_0) = 0$, and $\mathcal{T}^i(t_0) = 0$ for the equations (38)-(44) and yields a value function

$$\mathcal{W}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z}) = \mathcal{V}(\varepsilon, \mathcal{Y}_{11}, \mathcal{Y}_{12}, \mathcal{Y}_{22}, \mathcal{Z})$$

$$= x_0^T \sum_{i=1}^{k} \mu_i \mathcal{H}_{11}^i x_0 + 2x_0^T \sum_{i=1}^{k} \mu_i \mathcal{H}_{12}^i x_{d0}$$

$$+ x_{d0}^T \sum_{i=1}^{k} \mu_i \mathcal{H}_{22}^i x_{d0} + \sum_{i=1}^{k} \mu_i \mathcal{D}^i \quad (45)$$

for which the sufficient condition (28) of the verification theorem is satisfied so that the extremizing gains (35) and (36) become optimal

$$K_x^*(\varepsilon) = -R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_{11}^{*r}(\varepsilon)\,,$$

$$K_{x_d}^*(\varepsilon) = -R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_{12}^{*r}(\varepsilon)\,.$$

Theorem 5. Multi-Cumulant Solution for Servo Problems. Under the assumptions of (A, B) uniformly stabilizable and (C_d, A_d) uniformly detectable, the servo dynamics governed by (1)-(2) attempt to follow the desired trajectory of the system (3)-(4) via the Chi-squared measure of performance (5). Suppose $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^{k}$ with $\mu_1 > 0$ are fixed. Then, the statistical control solution for the servo problem over a finite horizon is a two-degrees-of-freedom controller with time-varying gains

$$u^*(t) = K_x^*(t)x^*(t) + K_{x_d}^*(t)x_d(t)\,, \tag{46}$$

$$K_x^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_{11}^{r*}(\alpha)\,, \tag{47}$$

$$K_{x_d}^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_{12}^{r*}(\alpha)\,, \tag{48}$$

where $\widehat{\mu}_r = \mu_i/\mu_1$ represent different levels of influence as they deem important to the performance distribution.

There is the addition of a feedforward part, which is the state x_d of the reference model (3)-(4). The feedback part of the optimal servo is dependent of A_d, C_d, and $x_d(t_0)$. Finally, $\{\mathcal{H}_{11}^{r*}(\alpha)\}_{r=1}^{k}$, and $\{\mathcal{H}_{12}^{r*}(\alpha)\}_{r=1}^{k}$ are the solutions of the time-backward matrix differential equations

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}_{11}^{1*}(\alpha) = &-\left[A(\alpha)+B(\alpha)K_x^*(\alpha)\right]^T\mathcal{H}_{11}^{1*}(\alpha)\\ &-\mathcal{H}_{11}^{1*}(\alpha)\left[A(\alpha)+B(\alpha)K_x^*(\alpha)\right]\\ &-K_x^{*T}(\alpha)R(\alpha)K_x^*(\alpha)-C^T(\alpha)Q(\alpha)C(\alpha)\,,\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}_{11}^{r*}(\alpha) = &-\left[A(\alpha)+B(\alpha)K_x^*(\alpha)\right]^T\mathcal{H}_{11}^{r*}(\alpha)\\ &-\mathcal{H}_{11}^{r*}(\alpha)\left[A(\alpha)+B(\alpha)K_x^*(\alpha)\right]\\ -\sum_{s=1}^{r-1}&\frac{2r!}{s!(r-s)!}H_{11}^{s*}(\alpha)G(\alpha)WG^T(\alpha)H_{11}^{(r-s)*}(\alpha)\,,\end{aligned} \tag{50}$$

together with

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}_{12}^{1*}(\alpha) = &-\left[A(\alpha)+B(\alpha)K_x^*(\alpha)\right]^T\mathcal{H}_{12}^{1*}(\alpha)\\ &-\mathcal{H}_{11}^{1*}(\alpha)B(\alpha)K_{x_d}^*(\alpha)-\mathcal{H}_{12}^{1*}(\alpha)A_d(\alpha)\\ &-K_x^{*T}(\alpha)R(\alpha)K_{x_d}^*(\alpha)+C^T(\alpha)Q(\alpha)C_d(\alpha)\,,\end{aligned} \tag{51}$$

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}_{12}^{r*}(\alpha) = &-\left[A(\alpha)+B(\alpha)K_x^*(\alpha)\right]^T\mathcal{H}_{12}^{r*}(\alpha)\\ &-\mathcal{H}_{11}^{r*}(\alpha)B(\alpha)K_{x_d}^*(\alpha)-H_{12}^{r*}(\alpha)A_d(\alpha)\\ -\sum_{s=1}^{r-1}&\frac{2r!}{s!(r-s)!}H_{11}^{s*}(\alpha)G(\alpha)WG^T(\alpha)H_{12}^{(r-s)*}(\alpha)\end{aligned} \tag{52}$$

where terminal-value conditions $\mathcal{H}_{11}^{1*}(t_f) = C^T(t_f)Q_fC(t_f)$, $\mathcal{H}_{11}^{r*}(t_f) = 0$ for $2 \leq r \leq k$ together with $\mathcal{H}_{12}^{1*}(t_f) = -C^T(t_f)Q_fC_d(t_f)$, $\mathcal{H}_{12}^{r*}(t_f) = 0$ for $2 \leq r \leq k$.

5. CONCLUSIONS

The results here demonstrate a successful combination of the compactness offered by logic from the state-space model description (7)-(8) and the quantitativity from a-priori probabilistic knowledge of adverse environmental disturbances so that the uncertainty of performance-measure (7) can now be represented in a compact and robust way. Moreover, there is a feature of interactive learning in the context of performance uncertainty where the servo controller not only optimizes criteria for evaluating for its performance but also interacts with the external environment. To be specific, the model-reference system (8) consists of a statistical controller plus a special agent called Nature. It is assumed that the statistical controller has a finite set of performance-measure statistics, while Nature has a finite set of sample path realizations. The optimal statistical controller composed of cumulant-based feedback (47) and feedforward (48) gains that operates dynamically on the time-backward histories of the cumulant-supporting equations (49)-(50) and (51)-(52) from the final to the current time. Finally, the present framework emphasizes the amount of information in performance-measure statistics (which actually are functions of noise process characteristics) needed to implement a risk-averse learning rule that effectively shapes the closed-loop performance distribution beyond the long-run average performance. The results also stress on the limits of what can be achieved: in such robust control designs there exists a kind of statistical controllers (46) that trades the property of certainty equivalence principle as would be inherited from the special case of classical linear-quadratic tracking, for the adaptability to deal with uncertain environments.

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