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Predator-Prey Dynamics Subject to a Threshold Policy with Hysteresis ⋆

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Abstract: This paper introduces a *threshold policy with hysteresis* (TPH) for the control of one-predator one-prey models. The models studied are the Lotka–Volterra (LV) and Rosenzweig–MacArthur (RM) two species density-dependent predator-prey models. The proposed policy (TPH) changes the dynamics of the system in such a way that a bounded oscillation is achieved. The policy can be designed by a suitable choice of so called virtual equilibrium points in a simple and intuitive manner.

Keywords: Nonlinear system control.

1. INTRODUCTION

This paper is concerned with the introduction of an exogenous control, called a threshold policy with hysteresis (TPH), into predator-prey models of two species. Predator-prey models play an important role in management of exploited natural resources. In order to define a TPH, the threshold policy (TP) is defined first as follows: if a certain measured variable (density or abundance) is below the single threshold no action is taken (for example, no harvesting), while above the single threshold, an action is taken (harvesting is permitted). More details about TPs can be found in Meza et al. [2005b]. Application of TPs can be seen in Collie and Spencer [1993], Costa et al. [2000], Meza et al. [2005a], Enberg [2005]. The exogenous control considered in this paper is the TPH that represents the non-ideal behavior of a practical threshold management policy which switches off at a different threshold from the one where it switches on, giving rise to an overlap region in its characteristic, known as hysteresis.

There is a substantial literature devoted to hysteresis in each of the communities of physicists, engineers and mathematicians, see, e.g. Carnevale et al. [2006], Brokate et al. [2006], Gonçalves et al. [2001], Moreno et al. [2003] and references therein.

Piecewise linear systems (PLS) are switching systems characterized by a finite number of linear dynamical models together with a set of rules for switching among these models. PLS are characterized by having both the logic in the controller and the nonlinearities in the system model (such as hysteresis) appearing as piecewise linear functions, with the system dynamics described by linear systems. The papers Gonçalves et al. [2001], Varigonda and Georgiou [2001], Moreno et al. [2003] consider linear systems subject to hysteresis as a controller or as a system nonlinearity.

In the control engineering literature, hysteresis has been shown to change the dynamics of a linear (and nonlinear) system to which it is applied in such a way that a bounded oscillation (limit cycle) can be achieved [Andronov et al., 1966, Gonçalves et al., 2001]. In the context of ecology as well, since stabilizing a system at a point is a rather unrealistic goal, this is the main motivation for the proposal in this paper of the threshold management policy (TPH), which changes the dynamics of the system in such a way that a bounded oscillation is achieved. This is exactly the objective of this paper in the specific context of two variable nonlinear predator-prey models. The policy can be designed in a simple and intuitive manner by a suitable choice of so called virtual equilibrium points, which were defined in Meza et al. [2005b].

In this paper, we study the LV and RM two species density-dependent predator-prey models.

2. MATHEMATICAL DEFINITIONS OF THRESHOLD POLICY WITH HYSTERESIS

TPs (on-off controls) for dynamical systems are strategies that switch the control inputs from one level to another whenever a certain measured variable crosses a predetermined single threshold (a line or a curve that depends on the state vector).

In the context of real systems, however, there is one important assumption that makes the TP used in Costa et al. [2000], Meza et al. [2005b] a little unrealistic: namely that as soon as the system crosses a threshold, the mode of control changes instantaneously. This allows the model to closely follow the single threshold (sliding mode), and reach a stable equilibrium (sliding equilibrium). In practice, it is likely that the threshold from the region with control towards the region without control has a

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different position than the threshold from the region without control towards the region with control. In control language, this means that we should consider hysteresis in the TP.

We propose the following hysteresis model as a relation between an input signal, $\tau(z)$, and an output signal, ϕ_{hys},

$$\phi_{hys}(\tau) = \begin{cases} \{0\}, & \text{if } \tau < -\sigma, \text{ or } \tau \leq \sigma \text{ and } \dot{\tau} > 0 \\ \{1\}, & \text{if } \tau > \sigma, \text{ or } \tau \geq -\sigma \text{ and } \dot{\tau} < 0. \end{cases} \tag{1}$$

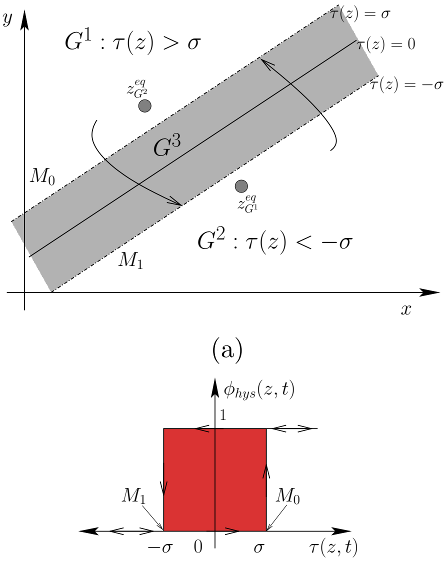
where τ is the threshold that should be chosen adequately, depending on the problem to be solved, $\sigma > 0$ is the hysteresis parameter, and $\dot{\tau}$ is the rate of change or time derivative of τ.

Remark 1. A model that is equivalent to (1) but does not use the derivative $\dot{\tau}$ explicitly can be derived from (1) by using the backward Euler approximation of $\dot{\tau}$. This requires the storage of $\tau(t)$ and $\tau(t-h)$ (value at instant $t-h$) and results in a hysteresis model similar to that proposed in Gonçalves et al. [2001].

In mathematical terms, the system subject to a threshold policy with hysteresis is as follows

$$\dot{z} = f(z,t) + u_{hys}, \tag{2}$$

where z is the state vector, $u_{hys} = u(z,t)\,\phi_{hys}(\tau)$ is the TPH, $\phi_{hys}(\tau)$ is defined as in (1), see Figure 1, and $u(z,t)$ is a continuous function. In the state space, the switching curves M_0 and M_1 (see Figure 1.(a)) of the systems are the curves where $\tau(z)$ is equal to σ and $-\sigma$, respectively. More precisely, $M_0 := \left\{z \in \mathbb{R}^2 : \tau(z) - \sigma = 0\right\}$, and $M_1 := \left\{z \in \mathbb{R}^2 : \tau(z) + \sigma = 0\right\}$.



(b)

Fig. 1. (a) Graphical representation of the TPH in a phase plane. The grey region is the hysteresis region G^3. $z^{eq}_{G^2}$ and $z^{eq}_{G^1}$ are the stable equilibrium point of the dynamics in region G^2 and G^1, respectively, and both are virtual. (b) Graphical representation of hysteresis, where z is the state vector, τ is the threshold variable and $\sigma > 0$ is the hysteresis parameter.

Definition 2. Let $z^{eq}_{G^i}$ be the stable equilibrium point of the dynamics of region G^i $\forall\ i = 1, 2$. Then $z^{eq}_{G^i}$ is called a **real equilibrium** if it belongs to G^i and a **virtual equilibrium** if it belongs to G^j, $j \neq i$.

In practice, it is likely that the threshold from G^1 to G^2 has a different position than the threshold from G^2 to G^1. The threshold of trajectories with initial conditions in G^1 moving towards G^2 is M_1, and the threshold of trajectories with initial conditions in G^2 moving towards G^1 is M_0, see Figure 1.(a).

3. ONE-PREDATOR ONE-PREY MODELS SUBJECT TO A TPH

A large class of predator-prey models can be written as the nonlinear dynamical system

$$\dot{x} = f_1(x) + f_2(x)\,y \tag{3}$$
$$\dot{y} = f_3(x)\,y \tag{4}$$

where the state variable x denotes the prey population density and the state variable y denotes the predator density; the functions f_1 and f_3 describe the prey and predator growth functions, respectively. The function f_2 describes the interaction when the predator finds the prey. These equations constitute the simplest representation of the essence of the nonlinear predator-prey interaction [Gurney and Nisbet, 1998].

We consider the introduction of an exogenous control, u^x_{hys} and u^y_{hys}, corresponding respectively to the removal of each species. If the control is applied, the model (3), (4) becomes

$$\dot{x} = f_1(x) + f_2(x)\,y - u^x_{hys} \tag{5}$$
$$\dot{y} = f_3(x)\,y - u^y_{hys}, \tag{6}$$

where u^x_{hys} and u^y_{hys} are the TPH on prey and predator, respectively.

Standard notation that will be used throughout the paper: (i) Subscripts 'fs' denotes free system (without control) and 'cs' denotes controlled system (with proportional control), (ii) $f^{G^1}(z)$ is the dynamics in region G^1, (iii) $f^{G^2}(z)$ is the dynamics in region G^2, (iv) z^{fs}_i is the stable equilibrium point of the dynamics of the free system, and (v) z^{cs}_i is the stable equilibrium point of the dynamics of the controlled system. We define the curves: (i) $V^{\phi=0}(z) = \{z \in \mathbb{R}^2_+ : \left\langle \eta, f^{G^2}(z) \right\rangle = 0\}$ and (ii) $V^{\phi=1}(z) = \{z \in \mathbb{R}^2_+ : \left\langle \eta, f^{G^1}(z) \right\rangle = 0\}$ where η is the vector normal to M_0 and M_1, and it is oriented in direction from G^2 towards G^1. We define the following regions: $G^1 := \{z \in \mathbb{R}^2_+ : \tau(z) - \sigma > 0\}$, $G^2 := \{z \in \mathbb{R}^2_+ : \tau(z) + \sigma < 0\}$, $G^3 := \{z \in \mathbb{R}^2_+ : -\sigma < \tau(z) < \sigma\}$, $\mathcal{R}^4 := \left\{z \in \mathbb{R}^2_+ : \left\langle \eta, f^{G^2}(z) \right\rangle > 0\right\}$, $\mathcal{R}^5 := \left\{z \in \mathbb{R}^2_+ : \left\langle \eta, f^{G^2}(z) \right\rangle < 0\right\}$, $\mathcal{R}^6 := \left\{z \in \mathbb{R}^2_+ : \left\langle \eta, f^{G^1}(z) \right\rangle > 0\right\}$, $\mathcal{R}^7 := \left\{z \in \mathbb{R}^2_+ : \left\langle \eta, f^{G^1}(z) \right\rangle < 0\right\}$.

We now state a general version of the main theorem of this paper (referred to as the TPH theorem), the generality permitting application to the class of predator-prey systems given in (3), (4).

Theorem 3. Consider the following general model

$$\dot{x} = f_1(x) + f_2(x)\, y - u^x_{hys} \tag{7}$$
$$\dot{y} = f_3(x)\, y - u^y_{hys} \tag{8}$$

which is the model (3), (4) subject to TPHs, and where $u^x_{hys} = u_1(z)\,\phi_{hys}$ and $u^y_{hys} = u_2(z)\,\phi_{hys}$ correspond to the removal of species x and y, where τ is a threshold that has one of the following forms

$$\tau := \alpha_1 x + \alpha_2 y - S_d, \qquad \text{slope threshold} \tag{9a}$$
$$\tau := y - y_{th}, \qquad \text{horizontal threshold} \tag{9b}$$
$$\tau := x - x_{th}, \qquad \text{vertical threshold.} \tag{9c}$$

There will exist a region that is invariant and globally attractive, if

(1) the switching lines M_0 and M_1 are defined such that the stable equilibrium points of each dynamics are virtual,
(2) the conditions $\left\langle \eta, f^{G^1}(z)\right\rangle < 0$ and $\left\langle \eta, f^{G^2}(z)\right\rangle > 0$ on the vector field are satisfied, and
(3) the intersection of the region $G^3 \bigcap \mathcal{R}^4 \bigcap \mathcal{R}^7$ is not of measure zero.

Remark 4. Condition (3) means that if $G^3 \bigcap \mathcal{R}^4 \bigcap \mathcal{R}^7$ is the segment of a curve or line then condition (3) is not satisfies.

Proof. Sketch for the LV case. For simplicity, consider the specific LV model as follows

$$f_1 = r_1\, x, \qquad f_2 = -a\, x,$$
$$f_3 = -r_2 + b\, x,$$

For this model, we demonstrate that the region IR ($A - B - C - D - A$), in Figure 2, is invariant and globally attractive.

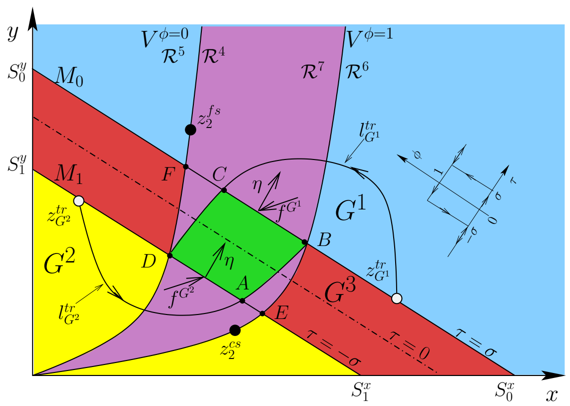


Fig. 2. Phase portrait diagram of the LV model. The globally attracting invariant region in the phase plane is the region IR ($A - B - C - D - A$). $\tau = \alpha_1 x + \alpha_2 y - Sd$, $S_0 = S_d + \sigma$ and $S_1 = S_d - \sigma$. $S_0^y = S_0/\alpha_2$, $S_1^y = S_1/\alpha_2$, $S_0^x = S_0/\alpha_1$ and $S_1^x = S_1/\alpha_1$.

Figure 2 shows the regions G^1, G^2, G^3, $\mathcal{R}^i$ for $i = 4, 5, 6, 7$, the curves $V^{\phi=0}(z)$ and $V^{\phi=1}(z)$, the switching lines M_0 and M_1 in a case where the threshold is as in equation (9a), and the stable equilibrium points z_2^{fs} and z_2^{cs}, see subsection 3.1, are virtual. The first condition of the TPH theorem is satisfied.

Conditions on the vector fields and Invariance of region $A - B - C - D - A$

The following geometrical approach is inspired by paper Boukal and Křivan [1999], in which the basic types of solution behavior along a single discontinuity line were summarized. In order to determine the behavior of trajectories along M_0 and M_1 analytically, we take a vector η normal to M_0 and M_1 and oriented in direction from G^2 towards G^1 and we examine the scalar products of this vector with $f^{G^2}(z)$ and $f^{G^1}(z)$, where $f^{G^2}(z)$ is the dynamics of the free system ($u_1 = u_2 = 0$), and $f^{G^1}(z)$ is the dynamics of the controlled system ($u_1 = \varepsilon_1\, x, u_2 = \varepsilon_2\, y$). Now, we verify direction of the vector field in switching lines M_0 and M_1,

on M_0	on M_1
$c_1^{M_0} := \left\langle \eta, f^{G^2}(z)\right\rangle > 0$	$c_1^{M_1} := \left\langle \eta, f^{G^2}(z)\right\rangle > 0$
$c_2^{M_0} := \left\langle \eta, f^{G^1}(z)\right\rangle < 0$	$c_2^{M_1} := \left\langle \eta, f^{G^1}(z)\right\rangle < 0$

From $c_1^{M_1}$, we obtain the following expression

$$\begin{aligned}\left\langle \eta,\, f^{G^2}(z)\right\rangle &= [\alpha_1 \;\; \alpha_2] \begin{bmatrix} r_1\, x - a\, x\, y \\ -r_2\, y + b\, x\, y \end{bmatrix} > 0 \\ &= \alpha_1\, r_1\, x - \alpha_2\, r_2\, y \\ &\quad + (\alpha_2\, b - \alpha_1\, a) x\, y > 0, \end{aligned} \tag{10}$$
$$\begin{aligned} V^{\phi=0}(z) &:= \alpha_1\, r_1\, x - \alpha_2\, r_2\, y \\ &\quad + (\alpha_2\, b - \alpha_1\, a) x\, y, \end{aligned} \tag{11}$$

points that satisfy (10) are on the right of $V^{\phi=0}(z)$, i.e., region $\mathcal{R}^4$; and from $c_2^{M_0}$, we obtain the following expression

$$\begin{aligned}\left\langle \eta,\, f^{G^1}(z)\right\rangle &= [\alpha_1 \;\; \alpha_2] \begin{bmatrix} r_1\, x - a\, x\, y - \varepsilon_1\, x \\ -r_2\, y + b\, x\, y - \varepsilon_2\, y \end{bmatrix} < 0 \\ &= \alpha_1 (r_1 - \varepsilon_1) x - \alpha_2 (r_2 + \varepsilon_2) y \\ &\quad + (\alpha_2\, b - \alpha_1\, a) x\, y < 0, \end{aligned} \tag{12}$$
$$\begin{aligned} V^{\phi=1}(z) &:= \alpha_1 (r_1 - \varepsilon_1) x - \alpha_2 (r_2 + \varepsilon_2) y \\ &\quad + (\alpha_2\, b - \alpha_1\, a) x\, y, \end{aligned} \tag{13}$$

points that satisfy (12) are on the left of $V^{\phi=1}(z)$, i.e., region $\mathcal{R}^7$. The points E and B are located where the curve $V^{\phi=1}(z)$ intersects the switching surfaces M_0 and M_1, respectively. The points D and F are located where the curve $V^{\phi=0}(z)$ intersects the switching surfaces M_0 and M_1, respectively. The intersection of curves $l^{tr}_{G^2}$ and M_1 is the point A, and the intersection of curves $l^{tr}_{G^1}$ and M_0 is the point C. Points that belong to $\mathcal{R}^4 \bigcap \mathcal{R}^7$ satisfy $c_1^{M_1}$ and $c_2^{M_0}$, and the intersection of regions $\mathcal{R}^4 \bigcap \mathcal{R}^7 \bigcap G^3$ is not of measure zero.

The curve labelled $l^{tr}_{G^1}$ is the trajectory that enters the region IR at the point D and remains within it thenceforth, and the curve labelled $l^{tr}_{G^2}$ is the trajectory that enters the region IR at the point B and remains within it thenceforth. The region that satisfies (10) and (12), i.e., $\mathcal{R}^4 \bigcap \mathcal{R}^7$, is the subset of points which their trajectories enter region G^3 with a transversal motion, and conditions $c_1^{M_1}$ and $c_2^{M_0}$ means that the vector fields are oriented in opposite directions, see Figures 2 and 3. Therefore, the region IR is invariant.

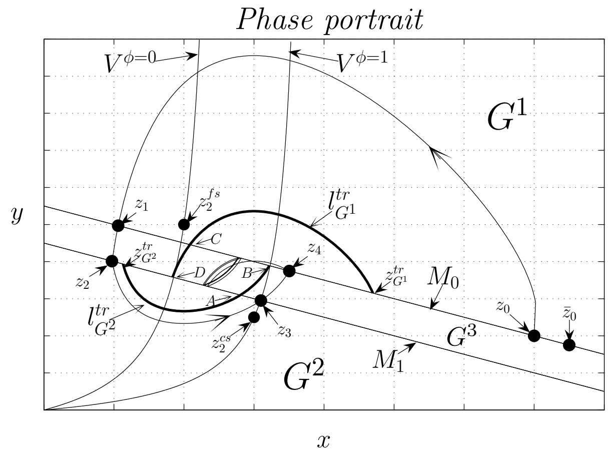


Fig. 3. Phase portrait dynamics of the the LV system with **TPH**. Parameter values: $a = 1$, $b = 1$, $r_1 = 1$, $r_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 1$, $\sigma = 0.1$, $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.5$ and $S_d = 1$. Showing the point mapping $z_0 \mapsto z_2$ on the switching line under a trajectory of the system with control, as well as the point mapping $z_2 \mapsto z_4$ under a trajectory of the system without control, for the LV system subject to the TPH, when both equilibrium points z_2^{fs} and z_2^{cs} are virtual.

Global attractivity of the region IR

Trajectories of the LV are given implicitly by the first integral. Thus the point mapping, see Figure 3, from z_0 towards z_2 must satisfy:

$$x_2^{(r_2+\varepsilon_2)} e^{-bx_2} y_2^{(r_1-\varepsilon_1)} e^{-ay_2} = x_0^{(r_2+\varepsilon_2)} e^{-bx_0} y_0^{(r_1-\varepsilon_1)} e^{-ay_0}, \tag{14}$$

and the point mapping from z_2 towards z_4 must satisfy:

$$x_4^{r_2} e^{-bx_4} y_4^{r_1} e^{-ay_4} = x_2^{r_2} e^{-bx_2} y_2^{r_1} e^{-ay_2}. \tag{15}$$

Taking logarithms of the above equations and subtracting (15) from (14) leads to:

$$\varepsilon_2 \ln\left(\frac{x_2}{x_0}\right) + \varepsilon_1 \ln\left(\frac{y_0}{y_2}\right) = r_2 \ln\left(\frac{x_0}{x_4}\right) + r_1 \ln\left(\frac{y_0}{y_4}\right) - b(x_0 - x_4) - a(y_0 - y_4) \tag{16}$$

The points z_0 and z_4 belong to line M_0 and they satisfy

$$\frac{y_0 - y_4}{x_0 - x_4} = -\frac{\alpha_1}{\alpha_2}.$$

Equation (16) can be expressed as

$$\varepsilon_2 \ln\left(\frac{x_2}{x_0}\right) + \varepsilon_1 \ln\left(\frac{y_0}{y_2}\right) = r_2 \ln\left(\frac{x_0}{x_4}\right) + r_1 \ln\left(\frac{y_0}{y_4}\right) - a(x_0 - x_4)\left(\frac{b}{a} - \frac{\alpha_1}{\alpha_2}\right) \tag{17}$$

It is reasonable to assume that $x_2 < x_0$ which implies that $y_2 > y_0$. Under this assumption the left hand side of equation (17) is negative. If $x_0 > x_4$ then the first term of right hand side of equation (17) is positive and the second term is negative. It is possible to choose the α_i (and, if necessary, the ε_i) to make the third term sufficiently negative to ensure that the left hand side and the right hand side have the same negative sign, and, in addition, satisfy the slope condition (equation (7) in Costa et al. [2000]). Therefore, the trajectory $z_0 \to z_2 \to z_4$ is contracting, in other words, if z_4 is on the left of z_0 then the mapping $z_0 \mapsto z_4$ is a contraction, see Figure 3.

In Figure 3, all trajectories that cross M_0 between points B and $z_{G^1}^{tr}$ enter region IR and remain within it thenceforth, and all trajectories that cross M_1 between points $z_{G^2}^{tr}$ and D enter region IR and remain within it thenceforth.

There exists a trajectory which maps condition $\bar{z}_0$ into $z_{tr}^{G^1}$. Any trajectory with initial condition on M_0 between the points $z_{G^1}^{tr}$ and $\bar{z}_0$ or crosses line M_0 between the points $z_{G^1}^{tr}$ and $\bar{z}_0$, entering the region IR at first, and remains within it thenceforth, as can be observed for the mapping $z_0 \mapsto z_4$ (see Figure 3). For any trajectory that crosses M_0 on the right of $\bar{z}_0$, we can demonstrate that with a second iteration of mapping, that trajectory enters the region IR and remains within it thenceforth. To economize space, this demonstration is omitted, but it follows the idea of the contraction of trajectory shown above. Therefore, the region IR is globally attractive. ■

Remark 5. The curves $V^{\phi=0}(z)$ and $V^{\phi=1}(z)$ are derived as in Boukal and Křivan [1999]. Another way to derive the curves $V^{\phi=0}(z)$ and $V^{\phi=1}(z)$ is considering a Liapunov function for each switching line M_0 and M_1, such that the time derivative of each Liapunov function must be negative. The Liapunov function for switching line M_0 is $V_{M_0}(z) = (\tau(z) - \sigma)^2/2$, and the Liapunov function for switching line M_1 is $V_{M_1}(z) = (\tau(z) + \sigma)^2/2$, where $\tau(z) = \alpha_1 x + \alpha_2 y - S_d$.

Remark 6. Trajectories that belong to the same region and have distinction never intersect. Trajectories with initial condition in region G^2 enter the region G^3 and switch when they cross the line M_0 ($\tau = \sigma$), and trajectories with initial condition in region G^1 enter to the region G^3 and switch when they cross the line M_1 ($\tau = -\sigma$), see Figure 3. Therefore, there is no possibility that the system slides in the line $\tau = 0$. Any trajectory on the left on right of $l_{G^2}^{tr}$ never intersect it, because these trajectories belong to the same region.

Figure 2 shows the regions G^1, G^2, G^3, $\mathcal{R}^i$ for $i = 4, 5, 6, 7$, the curves $V^{\phi=0}(z)$ and $V^{\phi=1}(z)$, the switching lines M_0 and M_1 in a general case where the threshold is as in (9a) and the conditions of the theorem are satisfied.

We now show the application of the TPH in the simple two-species LV and RM models.

3.1 Application of the TPH theorem to the LV model

One example of a TP, known as a weighted escapement policy (WEP), in which a threshold is built from a weighted (or linear) combination of prey and predator densities was proposed in Costa et al. [2000]. The TP was used to stabilize a LV system under simultaneous harvesting of the prey and predator. Here, we consider the LV system under simultaneous TPH on both species as follows

$$\begin{cases} \dot{x} &= r_1\, x - a\, x\, y - u_{hys}^x, \\ \dot{y} &= -r_2\, y + b\, x\, y - u_{hys}^y, \\ u_{hys}^x &= u_1(z)\, \phi_{hys}(\tau), \\ u_{hys}^y &= u_2(z)\, \phi_{hys}(\tau), \end{cases} \tag{18}$$

where the parameter r_1 is the growth rate of the prey, r_2 is the mortality rate of the predator, a, b represent the interaction coefficients between the species, $u_1 = \varepsilon_1\, x$ and $u_2 = \varepsilon_2\, y$ (proportional controls), ε_1 and ε_2 are the control effort parameters (harvesting intensities), and $\phi_{hys}(\tau)$ is defined as in equation (1); and all parameters are positive. The threshold defined in Costa et al. [2000] has the following form

$$\tau = \alpha_1 x + \alpha_2 y - S_d \tag{19}$$

where S_d is the weighted sum of species (constant), α_1 and α_2 are attributed population weights. We will use the slope threshold (19) which needs the measurement of both species densities. This is because, for the particular case of LV system, if only one species has TPH applied to it, one can satisfy conditions (1) and (2) of the TPH theorem but not condition (3).

Verification of the conditions of the TPH theorem

Note that the free LV system has the following dynamics: the origin, $z_1^{fs} = (x_1^{fs}, y_1^{fs}) = (0,0)$, is a saddle point, while $z_2^{fs} = (x_2^{fs}, y_2^{fs}) = (r_2/b, r_1/a)$ is a center point, and the controlled LV system has the following dynamics: the origin, $z_1^{cs} = (x_1^{cs}, y_1^{cs}) = (0,0)$, is a saddle point, and the point $z_2^{cs} = (x_2^{cs}, y_2^{cs}) = ((r_2+\varepsilon_2)/b, (r_1-\varepsilon_1)/a)$ is a center point. In both cases, the trajectories in the phase portrait are only closed trajectories and **not limit cycles**.

The switching lines M_0 and M_1 must be chosen so that the equilibrium points z_2^{fs} and z_2^{cs} will be virtual, this verifies the first condition of the TPH theorem. The second condition will be verified, where $\eta = [\alpha_1 \; \alpha_2]$ and f^{G^1} is the dynamics of the LV system with a proportional control, and f^{G^2} is the dynamics of the LV system without control, and the curves $V^{\phi=0}(z)$ and $V^{\phi=1}(z)$ can be calculated, and the third condition is verified, i.e., $G^3 \bigcap \mathcal{R}^4 \bigcap \mathcal{R}^7$ is not measure zero. Thus, the introduction of a TPH is responsible for new dynamic behavior, i.e., a bounded oscillation between the switching lines is achieved, see Figure 4.

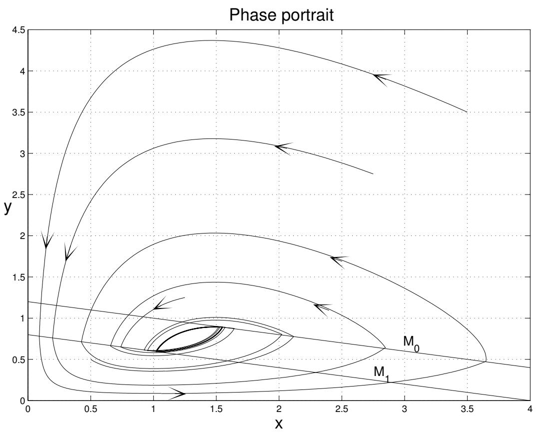


Fig. 4. Phase portrait dynamics of the LV system with **TPH** ($\varepsilon_1 = \varepsilon_2 = 0.5$). Parameter values: $a = 1$, $b = 1$, $r_1 = 1$, $r_2 = 1$, $\alpha_1 = 0.2$, $\alpha_2 = 1$, $\sigma = 0.1$, $\varepsilon_1 = 0.5$, $\varepsilon_2 = 0.5$ and $S_d = 1$.

3.2 Application of the TPH theorem to the RM model

In this section is shown that the proposed approach is successful in the control of the classical RM predator-prey model that corresponds to the choice $f_1 = r\,x\,(1 - x/K)$, $f_2 = x/(x+A)$, $f_3 = s\,A\,(x-J)/(J+A)(x+A)$ where r is the intrinsic growth rate of the prey, K is the carrying capacity of the environment, A is the half saturation constant, s is the conversion efficiency of the predator, and J is the minimum prey population for which the predator can survive. We will treat only the case $J < K$. We chose this model because it can be regarded as the simplest nontrivial paradigm that was proposed after the more classical but biologically unrealistic LV system. In this case, we consider harvesting of only the predator as follows

$$\begin{cases} \dot{x}(t) &= rx\left(1 - \dfrac{x}{K}\right) - \dfrac{xy}{x+A}, \\ \dot{y}(t) &= \dfrac{sA(x-J)}{(J+A)(x+A)}y - u_{hys}, \\ u_{hys} &= u_1(z,t)\,\phi_{hys}(\tau), \text{ and } u_1(z) = \varepsilon_2\, y \\ x(0) &= x_0 > 0, \quad y(0) = y_0 > 0, \end{cases} \tag{20}$$

where u_{hys} is the TPH, $u_1(z,t) = \varepsilon_2\, y$ is defined as a proportional control, ε_2 is the control effort parameter (harvesting intensity), and $\phi_{hys}(\tau)$ is defined as in equation (1), and τ is a threshold that has the following form

$$\tau := y - y_{th},$$

where y_{th} is the predator threshold level.

Verification of the conditions of the TPH theorem

Note that the free RM system has the following dynamics: the origin, $z_1^{fs} = (x_1^{fs}, y_1^{fs}) = (0,0)$, is a saddle point, the point $z_2^{fs} = (x_2^{fs}, y_2^{fs}) = (J, r\,(J+A)\,(K-J)/K)$ is an unstable node, and the point $z_3^{fs} = (x_3^{fs}, y_3^{fs}) = (K,0)$ is a saddle point. The behavior of the free system is a limit cycle. The controlled RM system has the following dynamics: the origin, $z_1^{cs} = (x_1^{cs}, y_1^{cs}) = (0,0)$, is a saddle point, the point $z_2^{cs} = (x_2^{cs}, y_2^{cs})$ is a stable equilibrium point, which does not belong to either to region G^1 or G^2, i.e., $z_2^{cs} \notin \mathbb{R}_+^2$, and the point $z_3^{cs} = (x_3^{fs}, y_3^{fs}) = (K,0)$ is a saddle point. The behavior of the controlled system is the extinction of the predator.

The switching lines M_0 and M_1 must be chosen so that the equilibrium points z_2^{fs} and z_2^{cs} are virtual, this verifies the first condition of the TPH theorem. The second condition is easily verified, where $\eta = [0 \; 1]$ and $f^{G^1}(z)$ is the dynamics of the RM system with a proportional control, and $f^{G^2}(z)$ is the dynamics of the RM system without control, and the curves $V^{\phi=0}(z)$ and $V^{\phi=1}(z)$ are vertical lines. The third condition is verified, i.e., $G^3 \bigcap \mathcal{R}^4 \bigcap \mathcal{R}^7$ is not measure zero. Thus, the introduction of a TPH is responsible for new dynamic behavior, i.e., a bounded oscillation between the switching lines is achieved, see Figure 5, and the amplitude of the oscillations is reduced with respect to the free system limit cycle oscillations, this being the main goal of the TPH.

4. CONCLUSION

In managing renewable resources, stabilizing a system at a point is a rather unrealistic goal, which is the main motivation for the proposal of the TPH that changes the dynamics of the system in such a way that a bounded oscillation of small amplitude is achieved, i.e., the system stabilizes in a limit cycle, as has been shown in Figures 4, 5. The TPH has been shown to be effective in the control of

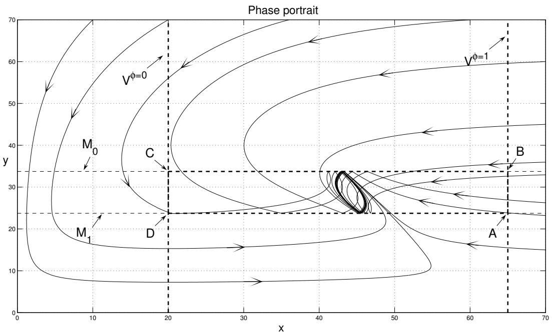


Fig. 5. Phase portrait dynamics of the the RM system with **TPH** ($\varepsilon_2 = 1/3$). Parameter values: $r = 2$, $K = 60$, $s = 1$, $A = 10$, $J = 20$, $y_{th} = 28.75$ and $\sigma = 5$.

two species (LV and RM) predator-prey model commonly used in mathematical population biology.

The TPH takes advantage of the condition of overexploitation so that the extinction of the species is avoided by switching between periods of overexploitation and no exploitation. The important novel characteristic of a TPH is that it ensures that, even though the system is subjected to a period of overexploitation it eventually stabilizes in low amplitude bounded oscillations in a desired safe region of the state space. In contrast, the commonly used proportional control cannot ensure this, and will often lead to extinction while the system is being subjected to overexploitation.

It must be stressed that when a threshold control with hysteresis effect is considered, we are implicitly considering errors and delays in the implementation of the policy, i.e. errors and delays in the measurement of the species density. The hysteresis loop around the threshold level takes explicitly into account such occurrences and therefore, stabilizes the dynamics by means of low amplitude bounded oscillations in the region delimited by the hysteresis itself. Finally, from the management standpoint, it is important to stress that the proposed strategy does not interfere directly in the harvesting intensities. As is well known, such interference usually meets several obstacles in its implementation.

Although discrete models are not studied in this paper, we expect that such systems subjected to a TPH can be analyzed with analogous techniques. This will be a topic of future research.

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