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Neo Robust Control Theory
–Beyond The Small-Gain and Passivity Paradigms–

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Abstract: The celebrated small-gain approach to robust control only makes use of the gain information of uncertainty. This results in a limitation on the achievable control bandwidth in system design. On the other hand, the alternative passivity approach has only limited applications. To relax the limitations associated with small-gain and passivity approaches, we explore the possibility of utilizing both the gain and the phase information of uncertainty in robust control design. This paper discusses the modeling of uncertainty accounting for both gain and phase, robust stability conditions and their state space characterization.

1. INTRODUCTION

The celebrated robust control theory of 1980's and 1990's is essentially based on the small-gain approach [Doyle and Stein (1981); Zhou et al. (1996)]. In this approach it is assumed implicitly that only the gain information of uncertainty can be accessed. This approach has seen a great success in the last two decades and has formed the backbone of modern robust control designs. But its limitation has also become clear in practice. Since only the gain information of uncertainty is made use of, the class of uncertainty will inevitably be enlarged in such modeling. A significant consequence is that the obtained robustness condition necessarily put a quite strong constraint on the controller gain at the low and middle frequency ranges. This limits the achievable bandwidth, i.e. fast response of the feedback systems.

Meanwhile, as a dual to the norm-bounded model of uncertainty, an uncertainty may be modeled as a positive real system when its phase is within $\pm 90°$. Then, the passivity theorem can be applied to the analysis and design of robust stable systems [Haddad and Bernstein (1991)]. In particular, for parametric uncertainty Popov theorem can be used for tighter analysis and synthesis [Haddad and Bernstein (1995)]. However, this class of uncertainty is quite limited in application.

To resolve the drawbacks of small-gain and passivity approaches, it is obviously necessary to make full use of both the gain and the phase information of uncertainty. Tits et al. (1999) made the first attempt and proposed the phase-sensitive structured singular value (PS-SSV) for robust stability analysis. In [Tits et al. (1999)] the uncertainty is modeled as a bounded class of systems with phase inside a *symmetric* interval $[-\theta(\omega), \theta(\omega)]$. This is an important step forward and effective for robust analysis. However, PS-SSV is not adequate for robust design. This is because the range of uncertainty phase is *not symmetric* in general, so an irrational transfer function has to be introduced to turn the range of uncertainty phase into a symmetric interval.

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Moreover, in the aforementioned methods it is assumed that the gain and/or phase of uncertainty are known over the whole frequency domain. However, this assumption may be excessive in some applications, as will be discussed in Sec. 2.2.

This paper tries to address these issues and puts forward several fundamental results on the modeling of uncertainty, robust stability condition in the frequency domain and its state space characterization. Several practical examples will be used to motivate and to illustrate these concepts.

Notations: $\mathbb{R}, \mathbb{C}$ denote the fields of real and complex numbers respectively. $\Re(x), \Im(x)$ are the real part and the imaginary part of complex x (either scalar or matrix) and x^* is its conjugate. The phase angle of $x \in \mathbb{C}$ is denoted by $\arg x$. $\|G\|_\infty$ is the H_∞ norm of $G(s)$, $\langle u, v\rangle$ the inner product. $\otimes$ denotes the Kronecker sum. Further $G^\sim(s) = G^T(-s)$, and the trace of square matrix $A = (a_{ij})$ is denoted by $\mathrm{Tr}(A) = \sum_{i=1}^n a_{ii}$. Finally $\mathcal{H}$ and $\mathcal{S}_+$ denote respectively the space of Hermitian matrices and the space of positive semidefinite matrices, with suitable sizes.

2. NEW WAY OF UNCERTAINTY MODELING

2.1 Motivation

Let us look at several examples first.

Example 1. The first example is simple an integrator with an uncertain gain:

$$P(s) = \frac{1+k}{s}, \quad 0 \le k \le L. \tag{1}$$

This plant can be modeled as the following class of plants

$$P(s) = \frac{1}{s}(1 + L\Delta), \quad 0 \le \Delta \le 1.$$

Suppose we use a PI compensator

$$C(s) = K\frac{s + K/(4\zeta^2)}{s}, \quad K > 0 \tag{2}$$

to compensate the plant. This 2nd order closed loop system is stable for any $k \ge 0$ as long as $K, \zeta > 0$.

In order to see what admissible uncertainty bound can be obtained by the existing robust control methods, we transform the closed loop system into a negative feedback interconnection of the uncertainty Δ and a fixed transfer function $M(s)$ given by

$$M(s) = L\frac{C(s)/s}{1 + C(s)/s} = L\frac{K(s + K/(4\zeta^2))}{s^2 + Ks + (K/2\zeta)^2}. \quad (3)$$

The Bode plot for the case $L = 1, \zeta = 0.3$ is illustrated in Fig. 1. Note that the frequency axis is normalized by K, so

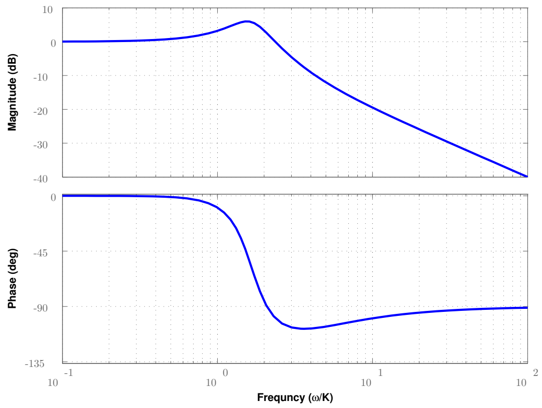

Fig. 1. Bode plot of $M(s)$ $(L = 1, \zeta = 0.3)$

the frequency bandwidth widens as the gain K increases.

If the uncertainty k is treated by the small-gain approach, then its robust stability condition is $\|M\|_\infty < 1$. Since $\|M\|_\infty = 1.99L$, the uncertainty bound is $L < \frac{1}{1.99} \approx 0.5$ which is obviously extremely conservative. This gigantic gap is caused by the modeling of the uncertainty: the phase (i.e. the sign) of gain k was ignored.

For this parametric uncertainty, many other methods such as passivity theorem, Popov theorem and real-μ analysis may be applied to estimate the stability margin. First, since $M(s)$ is not positive real (the phase of $M(s)$ is less than $-90°$ in some frequency band), the passivity approach cannot be applied directly. Meanwhile, by using Popov theorem and real-μ analysis the stability bound L obtained are approximately 0.99 and 2.52, respectively. Further, for this positive parameter uncertainty PS-SSV yields a bound better than the real-μ bound.

Example 2. The next system is a frequently encountered example arising in vibration control such as the head-positioning of HDD (hard disc drive):

$$P(s) = \frac{k}{s^2} + \frac{\kappa_1}{s^2 + 2\zeta_1\omega_1 s + \omega_1^2} + \frac{\kappa_2}{s^2 + 2\zeta_2\omega_2 s + \omega_2^2}. \quad (4)$$

Here the signs of κ_1, κ_2 change according to the mechanical design. It is a well-known fact in vibration control that the system is much harder to control when κ_1, κ_2 take opposite signs (called out-of-phase) [Ono and Teramoto (1992)]. Meanwhile, when κ_1, κ_2 take the same sign (in-phase) a wider control bandwidth is achievable. Clearly, this difference of achievable performance stems from the difference in the phases of the last 2 terms.

Usually the 3rd term is regarded as an uncertainty because its parameters are uncertain, as is the case in HDD control [Atsumi (2006)]. However, in the small-gain approach the phase of uncertainty is ignored and the uncertainty is modeled only by its gain. This inevitably leads to the extremely conservative result that the best achievable performance is no better than that of the out-of-phase uncertainty case.

Further, it is known that the passivity theorem is not applicable in the out-of-phase case [Atsumi (2006)].

Example 3. As the last example, let us consider the time-delay systems given in Morari and Zafiriou (1989) which arises in process systems

$$P(s) = \frac{ke^{-\theta s}}{\tau s + 1} \quad (5)$$
$$0.8 \le k \le 1.2,\ 0.8 \le \theta \le 1.2,\ 0.7 \le \tau \le 1.3.$$

When the nominal parameters are chosen as their respective mean values and the uncertainty modeled as a multiplicative one

$$1 + W(s)\Delta(s) = \frac{P(s)}{P_0(s)} = \frac{k}{k_0}\frac{\tau_0 s + 1}{\tau s + 1}e^{-(\theta - \theta_0)s},\ \|\Delta\|_\infty \le 1,$$

then with the following IMC controller

$$K(s) = \frac{Q(s)}{1 - P_0(s)Q(s)} = \frac{\tau_0 s + 1}{\epsilon s + 1 - e^{-\theta_0 s}}\frac{1}{k_0} \quad (6)$$

the lower bound of ϵ (which is inversely proportional to the bandwidth) obtained is 0.21 [Morari and Zafiriou (1989)].

However, for this sort of structured uncertainty, mean value of parameter is in general not a good choice for the nominal value. It will be shown later that, by minimizing both the gain and the phase of uncertainty via suitable selection of nominal parameters, smaller lower bound of ϵ can be obtained and the bandwidth of the closed loop system is widened. This sheds new light on the selection of suitable nominal value rather than the mean value.

2.2 Modeling of uncertainty

Assumption 4. $|\Delta(j\omega)|$ is even and $\arg\Delta(j\omega)$ is odd in the frequency ω. □

When the uncertainty is an LTI system, this assumption is satisfied trivially. A class of LTI delay systems is also included in this class of uncertainties. Owing to this property, we can restrict the discussions in the positive frequency domain.

A key observation from the preceding examples is that the range of uncertainty phase may be obtained to some extent. So it is reasonable to assume that the uncertainty satisfies

$$|\Delta(j\omega)| \le |W(j\omega)|,\ \theta_L(\omega) \le \arg\Delta(j\omega) \le \theta_H(\omega). \quad (7)$$

for all frequency ω.

However, the phase information at high frequency domain may not be so useful for the following reasons:

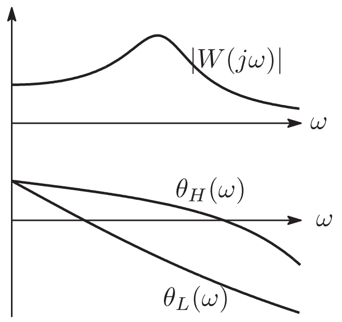

Fig. 2. Bode plot of uncertainty

(1) The phase of uncertainty may be too messy to use. For example, in Example 2 higher order vibration modes come in at high frequency domain and the phase bound becomes too wide to be useful in feedback design.
(2) The main objective of using uncertainty phase is to widen the control bandwidth. Since the bandwidth in general can not get so high due to input saturation, the loop gain inevitably needs to be significantly rolled-off at high frequency domain so that the phase becomes irrelevant to stability and performance.

Therefore, a more sensible uncertainty model is

$$|\Delta(j\omega)| \le |W(j\omega)|,\ \forall\omega \in [0,\infty)$$
$$\theta_L(\omega) \le \arg\Delta(j\omega) \le \theta_H(\omega),\ \omega \in [0,\omega_B]. \quad (8)$$

Here ω_B may be either the required bandwidth or the frequency band in which the phase information is reliable. This type of uncertainty is named as **Type 1 uncertainty**.

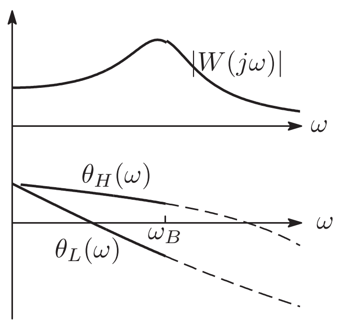

Fig. 3. Bode plot of type 1 uncertainty

Further, one may even be tempted to ignore the gain information of uncertainty below ω_B (**Type 2 uncertainty**):

$$\theta_L(\omega) \le \arg\Delta(j\omega) \le \theta_H(\omega),\ \omega \in [0,\omega_B]$$
$$|\Delta(j\omega)| \le |W(j\omega)|,\ \forall\omega \in [\omega_B,\infty). \quad (9)$$

The payoff for such simplification is that the analysis and design can be significantly simplified.

3. ROBUST STABILITY CONDITIONS

The robust stability of the feedback system in Fig.5 is considered throughout this paper.

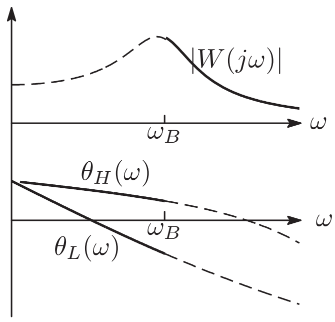

Fig. 4. Bode plot of type 2 uncertainty

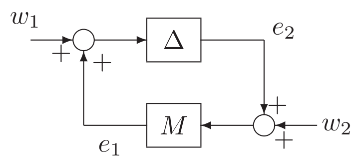

Fig. 5. Uncertain system

3.1 Type 1 Uncertainty

Theorem 5. Assume that both $M(s)$ and $\Delta(s)$ are stable transfer functions and $\Delta(s)$ belongs to Type 1 uncertainty. Define a set of frequencies as (c.f. Fig. 6)

$$\Omega = \{\omega \in [0,\omega_B] : -\theta_H(\omega) \le \arg M(j\omega) \le -\theta_L(\omega)\} \quad (10)$$

Then

(1) The closed loop system is robustly stable if
$$|M(j\omega)W(j\omega)| < 1\ \forall\omega \in \Omega \cup (\omega_B,\infty). \quad (11)$$
(2) Further, when $\Omega = \emptyset$ the preceding condition is necessary and sufficient for the robust stability.

Proof. The closed loop system is stable iff the return difference $R(s)$ satisfies

$$R(s) = 1 - M(s)\Delta(s) \ne 0$$

on the closed right half plane. When $\Delta(s) = 0$, $R(s) = 1$. So due to the continuity of $R(s)$, this stability condition fails iff

$$1 - M(j\omega)\Delta(j\omega) = 0 \quad (12)$$

holds for some uncertainty $\Delta(s)$ and at some frequency ω. This condition is in turn equivalent to

$$|M(j\omega)\Delta(j\omega)| = 1,\ \arg M(j\omega) + \arg\Delta(j\omega) = 0. \quad (13)$$

The second equation may be true only when $\omega \in \Omega$ or $\omega > \omega_B$. But the first equation does not hold in this frequency range due to (11). So Statement (1) is obvious.

To show the necessity of Statement (2), one needs to construct a destabilizing uncertainty in Type 1 uncertainty when the given condition fails. This can be done in the same way as the proof of small-gain theorem [Zhou et al. (1996)]. Suppose $|M(j\omega_0)W(j\omega_0)| = 1$ at some $\omega_0 > \omega_B$. Note that the phase of uncertainty is arbitrary in this frequency domain. We define an angle as $\phi = \arg M(j\omega_0) + \arg W(j\omega_0)$ and construct an uncertainty

$$\Delta(s) = W(s)\delta(s)$$

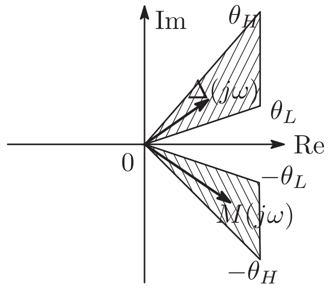

Fig. 6. Phase relation for $\Omega \neq \emptyset$

in which

$$\delta(s) = \begin{cases} \dfrac{s-a}{s+a} \ (a = \dfrac{\omega_0}{\tan(\phi/2)} > 0), \text{ if } \phi \neq 0 \\ 1, \text{ if } \phi = 0. \end{cases}$$

Obviously $|\Delta(j\omega_0)| = |W(j\omega_0)|$ and $\arg \delta(j\omega_0) = -\phi$. So

$$\arg \Delta(j\omega_0) = \arg W(j\omega_0) - \phi = -\arg M(j\omega_0)$$

and (12) holds at ω_0. This means that the closed loop system has an unstable at $j\omega_0$.

Statement (2) provides a robust stability condition which puts, separately, phase requirement ($\Omega = \emptyset$) on $M(s)$ only at low frequency domain and gain requirement only at high frequency domain. In particular, at low frequency the gain of $M(s)$ is not constrained so that there is a high potential for enhancing the control bandwidth in control design.

Example 6. Let us look at Example 1 once again. Clearly, $\arg k = 0°$. In the feedback system of Fig. 5

$$M(s) = -L\frac{K(s + K/(4\zeta^2))}{s^2 + Ks + (K/2\zeta)^2},$$

so its phase angle is 180° behind that of Fig. 1 and is contained in $[-180°, -300°)$, or equivalently $(60°, 180°]$. Since 0° does not intersect this interval, $\Omega_0 = \emptyset$ for any ω_B including the infinity. Therefore, according to Theorem 5 the robust stability is guaranteed for any gain uncertainty $k \geq 0$ and any controller gain $K > 0$ which is the best stability margin compared with the estimates of other methods shown in Example 1. Further the bandwidth is proportional to $K\sqrt{1+k} \geq K$, so any convergence rate is achievable by increasing the controller gain.

3.2 Type 2 Uncertainty

Theorem 7. Assume that both $M(s)$ and $\Delta(s)$ are stable transfer functions and $\Delta(s)$ belongs to Type 2 uncertainty. Then the closed loop system is robustly stable iff

$$\Omega = \emptyset, \ |M(j\omega)W(j\omega)| < 1 \ \forall\omega \in (\omega_B, \infty) \qquad (14)$$

in which the set Ω is as defined in Theorem 5.

Proof. From the proof of Theorem 5, robust stability is equivalent to that there is no uncertainty $\Delta(s)$ satisfying

$$|M(j\omega)\Delta(j\omega)| = 1, \ \arg M(j\omega) + \arg \Delta(j\omega) = 0 \quad (15)$$

for any frequency ω. Under the given condition (14), the second equation in (15) does not hold for $\omega \leq \omega_B$ and the first equation fails for $\omega > \omega_B$. So the sufficiency is obvious.

Conversely, if the second condition in (14) fails, then one can always construct an uncertainty in Type 2 uncertainty such that the closed loop system has a pole at $j\omega_0$ ($\omega_0 > \omega_B$), just as in the proof of Theorem 5. If the first condition fails, then there exists a stable $\Delta(s)$ satisfying $\arg M(j\omega_1) + \arg \Delta(j\omega_1) = 0$ at a frequency $\omega_1 \in [0, \omega_B]$ and $|M(j\omega_1)| > 0$ in general. Since the gain of uncertainty $\Delta(s)$ is unconstrained in this frequency band, this $\Delta(s)$ can always be constructed in such a way that $|M(j\omega_1)\Delta(j\omega_1)| = 1$ also holds. This shows the necessity of the robust stability condition.

Theorem 7 means that the phase robustness condition $\Omega = \emptyset$ is not so conservative when arbitrary gain uncertainty is allowed at the low frequency domain.

Example 8. Consider Example 3 once again. Let the nominal parameters corresponding to the nominal plant $P_0(s)$ be denoted by (k_0, τ_0, θ_0). Define the multiplicative uncertainty as

$$\Delta(s) = \frac{P(s)}{P_0(s)} - 1 = \frac{k}{k_0}\frac{1+\tau_0 s}{1+\tau s}e^{(\theta_0-\theta)s} - 1.$$

When the same IMC controller is used, the corresponding $M(s)$ becomes

$$M(s) = -\frac{K(s)P_0(s)}{1+K(s)P_0(s)} = -\frac{e^{-\theta_0 s}}{\epsilon s + 1}.$$

It is easy to get

$$|\Delta(j\omega)| = \sqrt{r^2 - 2r\cos\phi + 1}$$
$$\arg \Delta(j\omega) = \arctan\frac{\sin\phi}{\cos\phi - 1/r}$$
$$|M(j\omega)| = \frac{1}{\sqrt{(\epsilon\omega)^2 + 1}}$$
$$\arg M(j\omega) = -\pi - \theta_0\omega - \arctan(\epsilon\omega)$$

where

$$r = \frac{k}{k_0}\sqrt{\frac{(\tau_0\omega)^2+1}{(\tau\omega)^2+1}}$$
$$\phi = \arctan(\tau_0\omega) - \arctan(\tau\omega) + (\theta_0 - \theta)\omega.$$

We have the following observations:

(1) If $r < 1$ for all $\omega \neq 0$, then $\cos\phi - 1/r$ is dominated by $-1/r$ and both the value and the variation range of $\arg \Delta(j\omega)$ will be reduced. So to minimize the range of $\arg \Delta(j\omega)$, we should minimize r first. This leads to the selection of $k_0 = k_{\max}, \tau_0 = \tau_{\min}$. Subject to this choice, the bound of $|\Delta(j\omega)|$ is also reduced.
(2) To minimize the variation range of ϕ due to uncertain θ. a good option is $\theta_0 = \frac{\theta_{\max}+\theta_{\min}}{2}$. This contributes to a further reduction of the variation range of $\arg \Delta(j\omega)$.

Through numerical computation, it is found that $\epsilon_{\min} \leq 0.14$ for $k_0 = k_{\max}, \theta_0 = \frac{\theta_{\max}+\theta_{\min}}{2}, \tau_0 = \tau_{\min}$. This minimal ϵ (inverse of the bandwidth of $M(s)$) is much lower than the value $\epsilon = 0.21$ obtained based on the small-gain condition [Morari and Zafiriou (1989)].

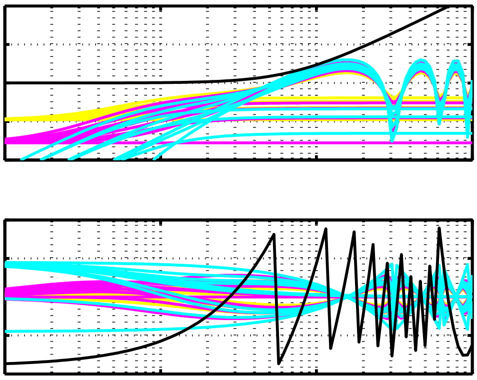

Fig. 7. $\epsilon = 0.14$: Bode plots of $\Delta(s)$ and $1/M(s)$ (bold)

3.3 Characterization of Phase Condition in Frequency Domain

This subsection discusses how to characterize the phase robustness condition $\Omega = \emptyset$ in terms of transfer function $M(s)$ in the frequency domain. To this end, it is assumed that

Assumption 9. There exist rational transfer functions $W_L(s)$, $W_H(s)$ satisfying

$$\arg W_L(j\omega) = \theta_L(\omega), \quad \arg W_H(j\omega) = \theta_H(\omega) \qquad (16)$$

for all $\omega \in [0, \omega_B]$.

Since the phase angle of a rational function is an odd function, $\arg W_L^*(j\omega) = \arg W_L(-j\omega) = -\theta_L(\omega)$ as well as $\arg W_H^*(j\omega) = -\theta_H(\omega)$ hold.

Assumption 10. For all $\omega \in [0, \omega_B]$, the phase uncertainty satisfies

$$\theta_H(\omega) - \theta_L(\omega) \in [0, \pi). \qquad (17)$$

This is assumed because otherwise the range of phase uncertainty will be too wide to be useful in robust control.

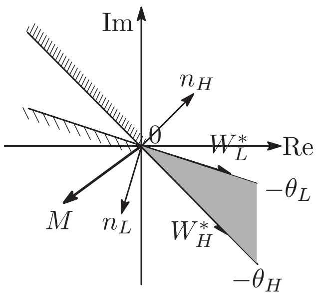

Fig. 8. Phase condition

Each point ζ in the complex plane corresponds to a 2-dimensional real vector $[\Re(\zeta),\ \Im(\zeta)]^T$. So W_L^* and W_H^* correspond to $[\Re(W_L),\ -\Im(W_L)]^T$ and $[\Re(W_H),\ -\Im(W_H)]^T$ respectively. Their normals are given by (c.f. Fig. 8)

$$n_L(\omega) = -\begin{bmatrix} \Im(W_L(j\omega)) \\ \Re(W_L(j\omega)) \end{bmatrix},\ n_H(\omega) = \begin{bmatrix} \Im(W_H(j\omega)) \\ \Re(W_H(j\omega)) \end{bmatrix}.$$

(Note that the difference of signs in the normals stems from the their directions) Then the half plane above the line indicated by $-\theta_H$ is characterized by

$$H_H(\omega) := \{x : x \in \mathbb{R}^2,\ \langle x, n_H(\omega)\rangle \geq 0\} \qquad (18)$$

and the half plane below the line indicated by $-\theta_L$ is characterized by

$$H_L(\omega) := \{x : x \in \mathbb{R}^2,\ \langle x, n_L(\omega)\rangle \geq 0\}. \qquad (19)$$

Under Assumption 10, the intersection of these two half planes forms a convex cone (illustrated by the gray area in Fig. 8)

$$\mathrm{Co}\{W_H^*(j\omega), W_L^*(j\omega)\} := H_H(\omega) \cap H_L(\omega) \qquad (20)$$

when ω is fixed. Here an abuse of notation is made for simplicity. A complex number $\zeta \in \mathrm{Co}\{W_H^*(j\omega), W_L^*(j\omega)\}$ means $[\Re(\zeta),\ \Im(\zeta)]^T \in H_H(\omega) \cap H_L(\omega)$. This convex cone rotates with respect to the origin when ω varies. So it will be called a moving convex cone.

This convex cone is characterized as follows.

Lemma 11. Let ζ be a complex number. The following statements are equivalent.

(1) $\zeta \in \mathrm{Co}\{W_H^*(j\omega), W_L^*(j\omega)\}$.
(2) $\Im(\zeta W_H(j\omega)) \geq 0$ and $\Im(\zeta W_L(j\omega)) \leq 0$.
(3) $j\zeta W_H(j\omega) + (j\zeta W_H(j\omega))^* \leq 0$ and $j\zeta W_L(j\omega) + (j\zeta W_L(j\omega))^* \geq 0$.

Proof. This follows from $[\Re(\zeta),\ \Im(\zeta)]^T \in H_H(\omega) \cap H_L(\omega)$ and simple computations such as

$$\begin{bmatrix} \Im(W_H) \\ \Re(W_H) \end{bmatrix}^T \begin{bmatrix} \Re(\zeta) \\ \Im(\zeta) \end{bmatrix} = \Im(W_H)\Re(\zeta) + \Re(W_H)\Im(\zeta)$$
$$= \Im(\zeta W_H) = -\Re(j\zeta W_H) = \frac{-1}{2}[j\zeta W_H + (j\zeta W_H)^*].$$

From Fig. 8, it is clear that $\Omega = \emptyset$ iff $M(j\omega)$ is outside the gray moving convex cone, i.e. $M(j\omega) \notin \mathrm{Co}\{W_H^*(j\omega), W_L^*(j\omega)\}$ for all $\omega \in [0, \omega_B]$. So the phase robustness condition can be characterized as follows.

Corollary 12. $\Omega = \emptyset$ iff $M(j\omega) \notin \mathrm{Co}\{W_H^*(j\omega), W_L^*(j\omega)\}$ for all $\omega \in [0, \omega_B]$.

4. STATE SPACE CHARACTERIZATION

The robust stability conditions derived up to now are given as phase and gain constraints in finite frequency domains. For synthesis purpose, it is convenient to characterize this kind of finite frequency domain property in the state space. This may be achieved based on the generalized KYP lemma developed by Iwasaki and Hara (2005).

4.1 Preliminaries

A brief review of the generalized KYP lemma will be provided for the sake of readers.

First of all, an Hermitian function

$$\sigma(f, \Gamma) := \begin{bmatrix} f \\ 1 \end{bmatrix}^* \Gamma \begin{bmatrix} f \\ 1 \end{bmatrix},\ f \in \mathbb{C}^m,\ \Gamma^* = \Gamma \qquad (21)$$

is defined. Then, it is easy to see that

$$s = j\omega \ \Leftrightarrow\ \sigma(s, J) = 0,\ J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (22)$$

Further, $|\omega| \in [\omega_B, \infty)$ is equivalent to

$$\begin{bmatrix} j\omega \\ 1 \end{bmatrix}^* \Psi_1 \begin{bmatrix} j\omega \\ 1 \end{bmatrix} \geq 0, \; \Psi_1 = \begin{bmatrix} -1 & 0 \\ 0 & -\omega_B^2 \end{bmatrix} \tag{23}$$

and $\omega \in [0, \omega_B]$ is equivalent to

$$\begin{bmatrix} j\omega \\ 1 \end{bmatrix}^* \Psi_2 \begin{bmatrix} j\omega \\ 1 \end{bmatrix} \geq 0, \; \Psi_2 = \begin{bmatrix} -1 & j\dfrac{\omega_B}{2} \\ -j\dfrac{\omega_B}{2} & 0 \end{bmatrix}. \tag{24}$$

That is, a finite frequency domain can be characterized by

$$\sigma(s, J) = 0, \quad \sigma(s, \Psi) \geq 0 \tag{25}$$

in which J is given in (22) and Ψ is a matrix such as Ψ_1, Ψ_2 above.

The following generalized KYP lemma provides a state space characterization for a class of finite frequency properties of transfer function, which is fundamental for system synthesis.

Lemma 13. (Theorem 2 of Iwasaki and Hara (2005)). Let (A, B, C, D) be a (real) state space realization of $G(s)$. $G(s)$ satisfies

$$\begin{bmatrix} G \\ 1 \end{bmatrix}^* \Pi \begin{bmatrix} G \\ 1 \end{bmatrix} < 0, \; \Pi^* = \Pi \tag{26}$$

for all frequency in the domain given by (25) iff there exist Hermitian matrices P, Q such that $Q > 0$ and

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T (J \otimes P + \Psi \otimes Q) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^T \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0. \tag{27}$$

When all parameter matrices are real, both P and Q can be taken as real symmetric.

Further, this equivalence still holds even when the strict inequalities in (26) and (27) are both replaced by non-strict inequalities, if (A, B) is controllable. In this case, $Q \geq 0$. □

Moreover, the lemma below gives the infeasibility condition for an LMI, which could be proved along the same line as Meinsma et al. (1997), based on the separating hyperplane argument.

Lemma 14. (Boyd et al. (1994); Meinsma et al. (1997)). Let $\mathcal{X}$ be a convex cone of matrices and $F^*(X) = F(X)$ be affine in X. Then, there is no $X \in \mathcal{X}$ satisfying $F(X) \leq 0$ iff there exists a positive semidefinite Hermitian matrix $W^* = W \geq 0$ such that

$$\text{Tr}(F(X)W) > 0, \; \forall X \in \mathcal{X}. \tag{28}$$

4.2 Robust stability condition in state space

The robust stability conditions derived in the preceding section will be characterized in the state space in this subsection.

First, the gain condition $|M(j\omega)W(j\omega)| < 1$ for all $\omega \in [\omega_B, \infty)$ is equivalent to that the inequality holds for all $|\omega| \in [\omega_B, \infty)$ since the gain is an even function of ω. This, in turn, can be written as

$$\begin{bmatrix} G_1 \\ 1 \end{bmatrix}^* E \begin{bmatrix} G_1 \\ 1 \end{bmatrix} < 0 \tag{29}$$

$$G_1(s) = M(s)W(s), \quad E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for all frequencies ω in this domain.

The next issue is how to characterize the phase condition $\Omega = \emptyset$ in terms of transfer function $M(s)$ in the frequency domain. This is achieved based on Lemma 11 and Corollary 12. Due to Lemma 11, $M(j\omega) \in \text{Co}\{W_H^*(j\omega), W_L^*(j\omega)\}$ iff

$$jM(j\omega)W_H + (jM(j\omega)W_H(j\omega))^* \leq 0$$
$$-jM(j\omega)W_L(j\omega) - (jM(j\omega)W_L(j\omega))^* \leq 0$$

hold for all $\omega \in [0, \omega_B]$. This, in turn, is equivalent to

$$\begin{bmatrix} G_2(j\omega) \\ I_2 \end{bmatrix}^* \Pi_2 \begin{bmatrix} G_2(j\omega) \\ I_2 \end{bmatrix} \leq 0, \; \forall \omega \in [0, \omega_B] \tag{30}$$

$$G_2(s) = \begin{bmatrix} M(s)W_H(s) & 0 \\ 0 & M(s)W_L(s) \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} 0 & -jE \\ jE & 0 \end{bmatrix}. \tag{31}$$

Then, application of Lemmas 13 and 14 to these frequency domain conditions yields the state space characterization for robust stability.

Theorem 15. Suppose that both $M(s)$ and $\Delta(s)$ are stable transfer functions, $\Delta(s)$ belongs to Type 1 or 2 uncertainty and Assumptions 4, 9 and 10 hold. Let $G_1(s) = (A_1, B_1, C_1, D_1)$, $G_2(s) = (A_2, B_2, C_2, D_2)$ and (A_2, B_2) be controllable. Then the closed loop system is robustly stable if the following statements hold.

(1) There exist real symmetric matrices P_1, Q_1 such that $Q_1 > 0$ and

$$\begin{bmatrix} A_1 & B_1 \\ I & 0 \end{bmatrix}^T (J \otimes P_1 + \Psi_1 \otimes Q_1) \begin{bmatrix} A_1 & B_1 \\ I & 0 \end{bmatrix} + \begin{bmatrix} C_1 & D_1 \\ 0 & 1 \end{bmatrix}^T E \begin{bmatrix} C_1 & D_1 \\ 0 & 1 \end{bmatrix} < 0. \tag{32}$$

(2) There exist real symmetric matrix $P_2 \geq 0$ and real skew-symmetric matrix Q_2 satisfying

$$[A_2 \;\; B_2]P_2 \begin{bmatrix} I \\ 0 \end{bmatrix} + [I \;\; 0]P_2 \begin{bmatrix} A_2^T \\ B_2^T \end{bmatrix} = 0 \tag{33}$$

$$[A_2 \;\; B_2]Q_2 \begin{bmatrix} I \\ 0 \end{bmatrix} + [I \;\; 0]Q_2 \begin{bmatrix} A_2^T \\ B_2^T \end{bmatrix} = 0 \tag{34}$$

$$[A_2 \;\; B_2]P_2 \begin{bmatrix} A_2^T \\ B_2^T \end{bmatrix} - \omega_B [A_2 \;\; B_2]Q_2 \begin{bmatrix} I \\ 0 \end{bmatrix} \leq 0 \tag{35}$$

$$\text{Tr}\left(E \cdot [C_2 \;\; D_2]Q_2 \begin{bmatrix} 0 \\ I \end{bmatrix} \right) < 0. \tag{36}$$

Proof. We prove that the gain condition and phase condition are guaranteed by Conditions (1) and (2) respectively.

(1) The small-gain condition in Theorems 1, 2 is equivalent to that (29) holds in the domain of (23), which in turn is equivalent to that (32) has real symmetric solutions P_1 and $Q_1 (> 0)$ according to Lemma 13.

(2) It suffices to guarantee that $\Omega = \emptyset$ holds, which is equivalent to $M(j\omega) \notin \text{Co}\{W_H^*(j\omega), W_L^*(j\omega)\}$ for all $\omega \in [0, \omega_B]$ (Corollary 12).

Due to the previous argument, $M(j\omega) \in \text{Co}\{W_H^*(j\omega), W_L^*(j\omega)\}$ for all $\omega \in [0, \omega_B]$ iff there exist Hermitian matrices U, V such that $V \geq 0$ and $F(U, V) \leq 0$ where

$$F(U,V) = \begin{bmatrix} A_2 & B_2 \\ I & 0 \end{bmatrix}^T (J \otimes U + \Psi_2 \otimes V) \begin{bmatrix} A_2 & B_2 \\ I & 0 \end{bmatrix} + \begin{bmatrix} C_2 & D_2 \\ 0 & I \end{bmatrix}^T \Pi_2 \begin{bmatrix} C_2 & D_2 \\ 0 & I \end{bmatrix}. \quad (37)$$

Then $\Omega = \emptyset$ is equivalent to that $F(U, V) \leq 0$ is infeasible on the space $\mathcal{H} \otimes \mathcal{S}_+$. So by Lemma 14, this is equivalent to the existence of an Hermitian matrix

$$W = P_2 + jQ_2 \geq 0$$

(note $P_2^T = P_2, Q_2^T = -Q_2$) such that

$$\text{Tr}(F(U,V)W) > 0 \quad \forall U \in \mathcal{H},\ V \in \mathcal{S}_+.$$

By using the property $\text{Tr}(AB) = \text{Tr}(BA)$, the following equivalent condition is obtained easily:

$$\text{Tr}(X \cdot J \otimes U) + \text{Tr}(X \cdot \Psi_2 \otimes V) + \text{Tr}(Y \cdot \Pi_2) > 0 \quad (38)$$

holds for all $(U, V) \in \mathcal{H} \otimes \mathcal{S}_+$, where

$$X = \begin{bmatrix} A_2 & B_2 \\ I & 0 \end{bmatrix} W \begin{bmatrix} A_2 & B_2 \\ I & 0 \end{bmatrix}^T, Y = \begin{bmatrix} C_2 & D_2 \\ 0 & I \end{bmatrix} W \begin{bmatrix} C_2 & D_2 \\ 0 & I \end{bmatrix}^T.$$

Let X_{ij}, Y_{ij} denote the (i, j) block of 2×2 partitions of X and Y respectively, each is compatible with the matrix multiplications in (38). Since X, Y are Hermitian, we have $X_{21} = X_{12}^*$ and $Y_{21} = Y_{12}^*$. Then it is easy to get

$$\text{Tr}(X \cdot J \otimes U) = \text{Tr}((X_{12} + X_{12}^*)U)$$
$$\text{Tr}(X \cdot \Psi_2 \otimes V) = -\text{Tr}(X_{11}V + j\frac{\omega_B}{2}(X_{12} - X_{12}^*)V)$$
$$\text{Tr}(Y \cdot \Pi_2) = \text{Tr}(j(Y_{12} - Y_{12}^*)E) = j\text{Tr}(E(Y_{12} - Y_{12}^*)).$$

As $\text{Tr}((X_{12} + X_{12}^*)U)$ is an affine function on $\mathcal{H}$, it is bounded from below iff

$$X_{12} + X_{12}^* = 0 \quad (39)$$

which is equivalent to (33) and (34). Then

$$\text{Tr}(X \cdot \Psi_2 \otimes V) = -\text{Tr}((X_{11} + j\omega_B X_{12})V). \quad (40)$$

Second, since V may be arbitrarily close to 0, (38) holds for all $(U, V) \in \mathcal{H} \otimes \mathcal{S}_+$ only if

$$\text{Tr}(Y \cdot \Pi_2) = j\text{Tr}(E(Y_{12} - Y_{12}^*)) > 0. \quad (41)$$

Due to the symmetry of E, $\text{Tr}(EZ^T) = \text{Tr}(ZE) = \text{Tr}(EZ)$ holds for any compatible real matrix Z. Then

$$\begin{aligned}
&\text{Tr}(E(Y_{12} - Y_{12}^*)) \\
&= \text{Tr}(E[\Re(Y_{12}) - \Re(Y_{12})^T]) + j\text{Tr}(E[\Im(Y_{12}) + \Im(Y_{12})^T]) \\
&= \text{Tr}(E\Re(Y_{12})) - \text{Tr}(E\Re(Y_{12})^T) \\
&\quad + j[\text{Tr}(E\Im(Y_{12})) + \text{Tr}(E\Im(Y_{12})^T)] \\
&= 2j\text{Tr}(E\Im(Y_{12})).
\end{aligned}$$

So (41) is equivalent to $\text{Tr}(E\Im(Y_{12})) < 0$, which is exactly (36).

Third, since $X_{11} + j\omega_B X_{12}$ is Hermitian and $\text{Tr}((X_{11} + j\omega_B X_{12})V) = \text{Tr}(V^{\frac{1}{2}}(X_{11} + j\omega_B X_{12})V^{\frac{1}{2}})$, (38) holds for arbitrary $V \in \mathcal{S}_+$ only if

$$X_{11} + j\omega_B X_{12} \leq 0. \quad (42)$$

Taking into account the skew-symmetry of $\Im(X_{11})$ and $\Re(X_{12})$ (due to (39)), this condition is equivalent to

$$\Re(X_{11}) - \omega_B \Im(X_{12}) \leq 0$$

which, in turn, is equivalent to (35).

Finally, it is obvious that (38) holds when all these conditions are satisfied. This ends the proof.

5. CONCLUDING REMARK

This paper has explored the possibility of making full use of both the gain and the phase information of uncertainty, so as to overcome the limitation of small-gain and passivity approaches and to achieve the highest possible system performance in robust control design. To this end, we have proposed new model of uncertainty accounting for both gain and phase, robust stability conditions in the frequency domain and their state space characterization, for SISO systems. This is a very promising field for robust control research.

The extension to MIMO systems remains to be an open problem which is even harder.

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