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Trajectory stabilization of wheeled system

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Abstract: The control problem for wheeled systems similar to a mobile robot, a vehicle, a wheeled tractor, etc. is studied. These systems belong to the class of nonholonomic mechanical systems. Analysis in this paper is limited to kinematic models, and the dynamics of control drives is taken into account. Control that stabilizes the motion of a wheeled system along a given trajectory (planar smooth curve) is constructed. The stabilizability is substantiated in large with respect to basic variables of the system. The similar fact is confirmed when perturbations are taken into account.

1. INTRODUCTION

Kinematic models of nonholonomic mechanical systems with rolling are the subject of many studies [*Kolmanovsky et al., 1995*] since they describe the motion of real wheeled systems (WS) sufficiently adequately. At present, these models are used in solving a large number of control problems. Among them are the problems of stabilization of a vehicle's motion on a road, automatic parking and traffic planning problems, controllability problems, etc. [*Kolmanovsky et al., 1995*]. There is considerable interest in solving many applied WS control problems [*Ackermann et al., 1995*, *Cordesses et al., 2000*].

In this work, the problem of stabilization of WS motion along a given trajectory is solved. Such control problems are widespread. For example, ensuring the motion along a given trajectory is necessary for realization of construction and other technological operations (cabling, trenching, etc.). In agro-industrial complex the whole range of such operations is carried out (ploughing, planting, weeding, etc. [*Cordesses et al., 2000*]).

The solution of this control problem was obtained under natural conditions. The given trajectory of WS motion has the shape of a planar smooth curve. Only one control that is the control of the WS front axle is used in the system. It is assumed that the velocity of the system motion along the curve is given (by the driver with the help of manual control means - gas, transmission) and only "lateral stability" is provided. The specific feature of this work is that the dynamics of the front axle drive (actuator) is taken into account in a general form, and that the stability of WS motion in large is achieved. The similar fact is confirmed when perturbations are taken into account. The perturbations are related to the state measurement errors. The perturbations are also related to side slip of wheels (along their axes). It is assumed that slipping can occur due to a slope of the surface along which the wheeled system moves.

It should be noted that the WS trajectory control problem being considered is the problem of stabilization of the WS state space manifold. The solution to the general control problem (for example, point stabilization) encounters known difficulties. For example, smooth stationary control laws do not provide a possibility of ensuring exponential stability of motion of a nonholonomic system in the general case [*Kolmanovsky et al., 1995*]. The smoothness of the control law is the significant condition, since it is required to take into account the dynamics of the system drives. The exponential stability is necessary for compensation for different perturbations.

The WS dynamics is described in Section 2. The WS control problem and the work problem are formulated in Section 3. A control law stabilizing the WS motion in the absence of perturbations is constructed in Section 4. In Section 5, it is shown that this control law admits small state system measurement errors and the deviation of the WS motion from the given trajectory is small. In Section 6, it is also shown that the control law admits small WS wheels slip. The deviation tends to zero if the slip value is directly taken into account in control law. The simulation results are presented.

2. MECHANICAL WHEELED SYSTEM AS A CONTROL OBJECT

The general schematic diagram of the wheeled system under study is shown in Fig. 1. The wheeled system contains a body, a driving back axle, and a controlled front axle. The state of the back axle is characterized by the angle a and the coordinates x,y in the system $\{X,Y\}$ of some point p of the back axle. The state of the front axle is given by the controlled angle b. By taking into account these notations, the WS motion is described by the system of equations

$$\dot{x} = v\cos a, \quad \dot{y} = v\sin a, \quad \dot{a} = v\,\mathrm{tg}(b)/L, \quad \dot{b} = F(b,u,t). \quad (2.1)$$

The first three equations of (2.1) represent the translational and angular WS motion. The last equation describes the dynamics of the controlled front axle drive, u is the control, v,L=const>0. The correlation $p\in S$ reflects the WS control objective, where S is the given curve (Fig. 1).

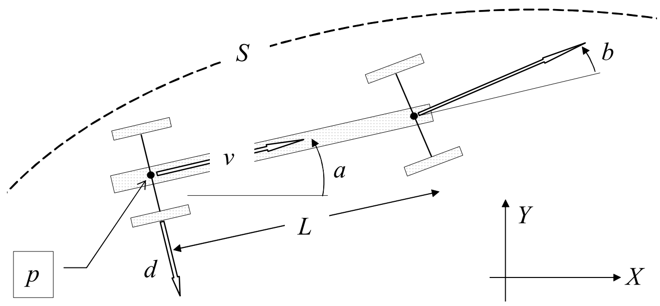

Fig. 1. General view of a wheeled system

The relations (2.1) contain the description of the mechanical constraints imposed on the system. The first two relations describe the first mechanical constraint. Its meaning is that the back WS wheels do not slip along the wheels axes. The similar second constraint for the front wheels provides possibility of construction of the third equation of (2.1).

Dynamic WS model (2.1) and its generalizations are being studied intensively [*Kolmanovsky et al., 1995*]. For example, in some studies, the inertial properties of the mechanical wheeled system were taken into account additionally. This provides a possibility of detailed investigation of the effect of external forces, for example, wind or inertia forces, on WS [*Ackermann et al., 1995*]. Note that the systems with rolling are the classical object of investigation in the framework of analytical mechanics of nonholonomic systems [*Neimark et al., 1967*].

Note as a rule, that the WS drive dynamics has not been taken into account directly [*Kolmanovsky et al., 1995*]. In this case, the variable b was considered as a control parameter, for example, from the class of continuous or smooth functions of time [*Micaeli et al., 1994*]. In this work, the drive dynamics is taken into account by the last equation of (2.1). Usually, a hydraulic drive is used as the front axle drive of a real WS. In the framework of (2.1), only its most general properties are taken into account. In particular, the description of the WS drive dynamics contains undefined parameters; therefore, the form of the function F in (2.1) may be unknown. However, it is known that the function F is bounded and satisfies the relations

$$F(b,h,t) \geq H, \quad F(b,-h,t) \leq -H \qquad (2.2)$$

$\forall b, \forall t, |u| \leq h$, $H, h = \text{const}>0$. This means that the angular velocity $\dot{b}$ of the controlled WS front axle is limited, and the control u, in essence, may change only the sign of the velocity $\dot{b}$. Property given in (2.2), in one form or another expresses the necessary property of any real control unit. Thus, the dynamics of WS actuator is taken into account in the general form.

3. STATEMENT OF THE CONTROL PROBLEM

The original control objective $p \in S$ of the wheeled system (2.1) is formalized by the relations

$$\rho=0, \qquad \rho^2 = \min_{(x^1,y^1)\in S} \left\langle (x-x^1)^2 + (y-y^1)^2 \right\rangle, \qquad (3.1)$$

where (x,y) are the coordinates of the point p. The solution to problem (3.1) is denoted by (x^*,y^*). The point p^* with the coordinates (x^*,y^*) lies on the curve S and is the closest to the point p (Figs. 1, 2). Thus, according to (3.1), the WS control objective is achieved if $x=x^*$, $y=y^*$.

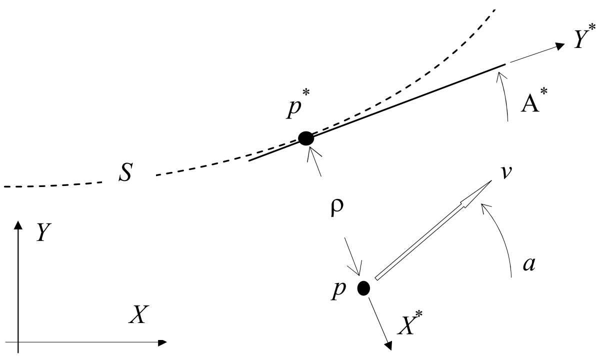

Fig. 2. Distance ρ from the point p to the given curve S.

The smooth curve S in the plane X,Y is considered as trajectory of the WS motion (Figs. 1, 2). Here, it is assumed that the curve S is defined parametrically, $x=\Phi_x(s), y=\Phi_y(s)$, $(d\Phi_x/ds)^2+(d\Phi_y/ds)^2 \neq 0$. The parameter s determines the coordinates x and y of points of the curve S in the system $\{X,Y\}$. Quantity $A(s)$ is the tangent angle to the curve S at the point s. $A^*=A(s^*)$, where s^* is the value of s for the point $(x^*,y^*)\in S$ that is the closest to the point (x,y). The curvature $A'(s)$ of the curve S and its derivative are limited

$$|A'| \leq \bar{A}', \quad |A''| \leq \bar{A}'', \quad \bar{A}', \bar{A}'' = \text{const} \geq 0, \quad A' = dA(s)/ds. \; (3.2)$$

Note that along with control objective (3.1), other control objectives for wheeled systems are considered [*Kolmanovsky et al., 1995*]. For example, the description of the control objective different from (3.1) may contain requirements for the velocity of motion along the trajectory S. In practical problems it is necessary to introduce a number of additional limiting conditions (for example, a fixed range of variation of the angle b of the WS front axle, WS "lateral" acceleration [*Ackermann et al., 1995*], and so on). In automatic parking problems, the trajectory S is not given. The trajectory is constructed (planned) for the initial and final WS positions. The orientation of the system's body, its velocity, and the surrounding obstacles, including movable objects can be taken into account.

The WS control problem consists of constructing a control law $u=U(x,y,a,b)$ which provides stability of motion (3.1) of the closed-loop system (2.1). Note that the relation (3.1) ρ=0 determines the set of points on the curve S in the plane X,Y. In other words, (3.1) forms a manifold of the state space $\{x,y,a,b\}$ of the wheeled system (2.1). Therefore, in this work, the stability of manifold (3.1) of the closed-loop system is considered [*Matyukhin, 2001*]. It is assumed that the state x,y,a,b of the system is available with the system parameters v,L, Eqs. (2.1), and the description of the curve S. Note that in practice, the description of the curve S is contained in the WS on-board computer. It can be created automatically, for example, in the storage mode. For this purpose, the operator leads, e.g., an agricultural tractor along the trajectory satisfying the technological operations of field preparation [*Cordesses et al., 2000*]. At this stage, satellite sensors of tractor position are usually used (GPS sensors [*Rapoport, 2004*]). They can be applied further for on-line tractor control in an automatic mode.

The goal of this work is to investigate the WS control problem formulated above. The control law providing stability of the WS motion of form (3.1) under these conditions is constructed.

4. WHEELED SYSTEM CONTROL LAW

Let us write the WS control objective (3.1) in the form

$$r = 0. \tag{4.1}$$

Here r denotes the abscissa of the point p in the system $\{X^*,Y^*\}$ (Fig. 2). Obviously, $\rho=|r|$, and the original control objective $\rho=0$ in the form (3.1) follows from the relation (4.1). The case in which r in (4.1) is uniquely determined is considered. Namely, problem (3.1) is solved at some time instant before the initial moment $t=0$, when the motion is started. If this problem has several solutions, one of them is to be chosen (e.g., arbitrary). For the solution x^*,y^* chosen, the quantity s is denoted by s^{*0}. Then, for $t>0$, problem (3.1) is reduced to the following local problem:

$$s^* = \underset{|s-s^{*1}|\le d}{\arg\min}\left\langle \left(x-\Phi_x(s)\right)^2+\left(y-\Phi_y(s)\right)^2\right\rangle,\ s^{*1}(t)=s^*(t\text{-}0). \tag{4.2}$$

With account of proposition (3.2), this problem has a unique solution, since the number $d>0$ is chosen sufficiently small, $d<1/\overline{A}'$. Therefore, for $t\ge0$, the coordinate system $\{X^*,Y^*\}$ and r in relation (4.1) are uniquely determined.

Lemma 1. For system (2.1) and the curve S satisfying conditions (3.2), we have

$$\dot r=-v\sin(a-A^*),\quad \dot s^*(1+r\,A'^*)=v\cos(a-A^*). \tag{4.3}$$

Here, $\dot s^*$ is the value of $\dot s=ds/dt$ for the point $p^*\in S$ with the coordinates x^*,y^* closest to the point p with the coordinates x,y, $A^*=A(s^*)$. The idea of the proof of the lemma is related to the analysis of the vector equality $\mathbf{P}^*+r\mathbf{n}=\mathbf{P}$, where $\mathbf{P}$ and $\mathbf{P}^*$ are the radius vectors of the points p and p^* in the system $\{X,Y\}$ (Fig. 2), $\mathbf{n}$ and $\boldsymbol{\tau}$ are the orts of the coordinate system $\{X^*,Y^*\}$.

By taking into account lemma 1, we construct the control law of the wheeled system (2.1) in the form

$$u=-h\,\mathrm{sign}\left(b-b_z\right), \tag{4.4}$$

$$v\,\mathrm{tg}(b_z)/L\overset{\text{def}}{=}\dot a_z+\varphi(a-a_z),\qquad \sin\left(a_z-A^*\right)\overset{\text{def}}{=}-f\left(r\right).$$

Here, the quantity a_z is determined by the last relation, and the second relation determines the quantity b_z. Expressions (4.4) allow us to construct the equalities

$$\dot e=\varphi(e)+v\left(\mathrm{tg}b-\mathrm{tg}b_z\right)/L,\quad e=a-a_z, \tag{4.5}$$

$$\dot r=v\,f(r)+v\sin(a_z-A^*)-v\sin(a-A^*),$$

where propositions (4.3) of lemma 1 and Eqs. (2.1) should be taken into account. The idea is to realize the sliding mode of the form $b=b_z$ in system (2.1) using discontinuous control (4.4). In this case, the first relation in (4.5) takes the form $\dot e=\varphi(e)$, $e=a\text{-}a_z$. The function $\varphi(e)$ is chosen so that the motion $e=0$ of this system is exponentially stable, i.e., $a\approx a_z$. Then, the second correlation (4.5) takes the form $\dot r\approx vf(r)$, where the function $f(r)$ is similar to the function $\varphi(e)$. This implies the stability of the motion $r=0$ of (2.1).

Theorem 1. Assume that for system (2.1), the arbitrary numbers x^0,y^0,a^0,b^0 ($0<b^0<\pi/2$) are given, and the following inequalities are valid:

$$|x(0)|\le x^0,\quad |y(0)|\le y^0,\quad |a(0)|\le a^0,\quad |b(0)|\le b^0. \tag{4.6}$$

Then there exist numbers $\overline{A}',\overline{A}''$ from proposition (3.2) and a control of the form (4.4) such that the stability of the motion $r=0$ along the trajectory S is ensured.

The proof scheme of the theorem is given in the Appendix and Ref. [*Matyukhin, 2006*]. The meaning of the theorem is that, first of all, that there exists a solution to the WS control problem in the general nonlinear formulation. According to the theorem, the constraints on the initial values $(x(0),y(0),a(0),b(0))$ are almost absent, and the WS moves along the given trajectory S for any initial values. Constraint (4.6), where $0<b^0<\pi/2$, is not restrictive, in particular, from a practical point of view. Indeed, the state of the real wheeled system required in (4.6) can be realized relatively simply. Note also that control law (4.4) depends on x^0,y^0,a^0,b^0 in the general case. Therefore, in theorem 1, stabilizability of the system in large with respect to basic variables x,y,a of WS state is established. Constraint (3.2) of the theorem for the curvature of the curve S seems to be natural. It is related to the fact that the control objective should be realizable, i.e., some (unperturbed) motion of system (2.1) corresponding to the curve S should exist. This is possible only if S is sufficiently smooth.

Note that the control law of the WS does not necessarily have discontinuous character of a feedback of type (4.4). A continuous WS control law $u=u_z$ can be obtained, for example, following the schemes of constructing b_z and a_z in relations (4.4). However, in this case, it should be assumed that the function F is known, unlike the original assumption. Moreover, it is possible to take into account the dynamics of the WS drive in the general form

$$\dot b=F\left(b,u,t\right),\qquad \dot u=\psi\left(b,u,U,t\right).$$

Here, u is considered as the drive state variable $\{b,u\}$. The new control U should provide $u\to u_z$.

5. ALLOWANCE FOR ERRORS OF MEASURING THE WHEELED SYSTEM STATE

In the contrast of the last section, the WS control problem is solved now with regard for system state measurement errors. In distinction to (4.4), the control law

$$u=-h\,\mathrm{sign}\left(b+\Delta b-b_z\right), \tag{5.1}$$

$$v\,\mathrm{tg}(b_z)/L\overset{\text{def}}{=}\dot a_z+\varphi(a+\Delta a-a_z),\ \sin\left(a_z-A^*\right)\overset{\text{def}}{=}-f\left(r+\Delta r\right)$$

includes $\Delta x,\ldots$, considered as the errors of measuring the coordinates $x,\ldots$ The errors is assumed to be smooth and bounded

$$|\Delta x|\le\delta,\qquad |\Delta y|\le\delta,\qquad |\Delta a|\le\delta,\qquad |\Delta b|\le\delta, \tag{5.2}$$

$$|\Delta\dot x|\le C,\ldots,|\Delta\dot y|\le C,\quad |\Delta\dot a|\le C,\quad \delta,\ C=\mathrm{const}\ge0.$$

Theorem 2. Assume that for the system (2.1), the numbers $\varepsilon, x^0, y^0, a^0, b^0$ ($0<b^0<\pi/2$) are given, and

$$|x(0)+\Delta x(0)| \le x^0, ..., |b(0)+\Delta b(0)| \le b^0. \qquad (5.3)$$

Then, there exist numbers $\overline{A}', \overline{A}''$ from proposition (3.2), numbers δ in (5.2) and a control of the form (5.1) such that

$$|r| \le \gamma |r(0)|, \quad 0\le t\le\tau, \ |r| \le \varepsilon, \quad t\ge\tau, \quad \exists \gamma,\tau = \text{const}. \quad (5.4)$$

The prove of the theorem follows along the lines of theorem 1 and is not presented here (see Ref. [*Matyukhin, 2006*]). The theorem implies that developed control law (4.4) maintains the stability of WS motion when the system is subjected to perturbations due to errors of measured state variables. Also, it turns out that according to (5.4) the deviation r from given trajectory S will be small if the measurement error is small according to (5.2). The conditions for the initial values of WS state have the form (5.3) of nonrestrictive constraints that is they do not virtually exist, the case with the theorem 1.

6. ACCOUNT FOR WHEELES SLIPPING

Note that the ideal WS model (2.1) can be violated, for example, if the WS moves along a surface with a noticeable slope. Modern wheeled systems are designated for motion along surfaces with a slope of an order of 15^0 and higher [*Cordesses et al., 2000*]. The reason is related to a slip of the WS back wheels along their axes. So the WS slipping phenomenon must be taken into account. In this case, the equations (2.1) take the following form

$$\dot{x}/v = \cos a - d\sin a, \quad \dot{y}/v = \sin a + d\cos a, \qquad (6.1)$$

$$\dot{a} = (\text{tg}b - d)v/L, \quad \dot{b} = F(b,u,t).$$

The quantity d characterizes the slip of the WS back wheels along their axes (Fig. 1).

The slipping phenomenon, as a rule, was not taken into account, and the perturbed kinematic WS model of the form (6.1), in essence, was not studied [*Kolmanovsky et al., 1995*]. To take into account the slipping phenomenon, in some studies it was assumed that d=const [*Canudas et al., 1995*]. Unlike these studies, in system (6.1) d is assumed to be a function of the WS state, i.e., $d=d(x,y,a,b)$. Note that in the general case, the slip characteristic d may depend on the inertia arising in sharp turns of the WS [*Ackermann et al., 1995*]. This can be described in terms of the forces acting on the WS that are not contained in model (6.1).

The slip value is described in the form

$$d = k\sin(\gamma)\sin(a-\beta). \qquad (6.2)$$

To take slipping into account of, we assume that the WS moves along a surface close to the plane X,Y, but not coincides with it (Figs. 1 and 2). We denote by Π the plane tangent to the WS wheels. The angle between the plane Π and the horizontal plane X,Y is denoted by γ. If γ=0, it is assumed that WS slipping is absent, d=0, which follows from (6.2). The angle β in (6.2) determines the slope direction of the plane Π (or the orientation of this plane) in the system $\{X,Y\}$. If a=β, the back wheels axle is horizontal, and WS slipping is absent according to (6.2). The coefficient k in (6.2) determines the slip amplitude and has an experimental character.

The basic assumption concerning the functions k, γ, β, has the form

$$|k(x,y)| \le \delta_0, \qquad (6.3)$$

$$|\partial k/\partial x| \le \delta_1, |\partial k/\partial y| \le \delta_1, \ldots, |\partial^2 k/\partial y^2| \le \delta_2,$$

$$|\partial\gamma/\partial x| \le C, \ldots, |\partial\beta^2/\partial y^2| \le C, \qquad \delta_i,\ C = \text{const} \ge 0.$$

Condition (6.3) means that the surface along which the WS moves is sufficiently smooth. The number C is assumed to be given, and the constants δ_i will be chosen for ensuring that the WS motion is along the curve S. Introduced assumptions (6.2) and (6.3) correspond to physical reasons of slipping under study, which are related to limited slope of the surface along which the WS moves.

These model representations of the phenomenon of the slipping of wheeled machines have an approximate character. For example, the slip value may increase on wet surfaces, which is not taken into account in (6.2), (6.3). To take into account this circumstance, the coefficient of friction with the road bed is introduced [*Ackermann et al., 1995*].

Theorem 3. Let WS equations have the form (6.1), numbers $\varepsilon, x^0, y^0, a^0, b^0$ ($0<b^0<\pi/2$) are given, and

$$|x(0)| \le x^0, \quad |y(0)| \le y^0, \quad |a(0)| \le a^0, \quad |b(0)| \le b^0. \ (6.4)$$

Then, there exist numbers $\overline{A}', \overline{A}''$ from proposition (3.2), numbers δ_i in (6.3) and a control of the form (4.4) such that the following relations are valid:

$$|r(t)| \le \Upsilon |r(0)|, \quad 0\le t\le\tau, \quad |r(t)| \le \varepsilon, \quad t\ge\tau, \quad \exists \Upsilon,\tau = \text{const}. \ (6.5)$$

The prove of the theorem follows along the lines of theorem 1 and is not presented here (see Ref. [*Matyukhin, 2006*]). The basic result of theorem 3 is the substantiation of the stability of WS motion under the perturbed conditions. The perturbations are related to side slip of wheels (along their axes) are considered. It is assumed that the slipping can occur due to a slope of the surface along which the WS moves.

Namely, the theorem 3 proves the fact that control law (4.4) constructed in the previous section 4 provides a small deviation of r from a given trajectory S according to (6.5). For this, it is sufficient that the slip amplitude is small according to assumption (6.3). Moreover, the condition of (6.3) that is the surface along which WS moves is sufficiently smooth is taken into account. Note that d is considered as an unknown perturbation acting on the system [*Malkin, 1952, Krasovskii, 1959*]. In other words, theorem 3 states the stability of WS motion if the perturbations which act constantly are taken into account [*Matyukhin, 2001*]. Note also that similar to theorem 1, the constraints on the initial values ($x(0)$,$y(0)$,$a(0)$,$b(0)$) of the system have the form (6.4) of unrestrictive constraints, i.e., are practically absent.

6.1 Account of slipping in the control law

The WS control problem is solved now under the assumption that the slip d is known. The control law explicitly containing the function $d(x,y,a,b)$ and providing a possibility of ensuring asymptotically an exact WS motion along the trajectory S is constructed.

Note that in the general case, d cannot be measured directly. In practice, the estimate of the values of d can be obtained from, for example, measurements of (spatial) angular WS position. Here, it is taken into account that the deviation of the angular position of the back WS axle from the horizontal position provides a possibility of slope estimating of the surface along which the WS moves, i.e., estimating of the value of the slipping d. The value of d can also be estimated using the results of state observation of the WS. Taking into account this fact, the control law is constructed in the form

$$u = -h\,\mathrm{sign}(b-b_z),\quad v\,\mathrm{tg}(b_z)/L \overset{\text{def}}{=} \dot a_z + vd/L + \varphi(a-a_z),\qquad (6.6)$$
$$\sin\left(a_z - A^*\right) + d\cos(a-A^*) \overset{\text{def}}{=} -f(r)\cdot$$

This law, unlike the control law (4.4), contains $d(x,y,a,b)$ explicitly. In this case, the proposition similar to theorem 1 is valid.

Theorem 4. Assume that the motion equations of the WS have form (6.1) and arbitrary numbers x^0,y^0,a^0,b^0 $(0<b^0<\pi/2)$ are given, and

$$|x(0)| \le x^0,\quad |y(0)| \le y^0,\quad |a(0)| \le a^0,\quad |b(0)| \le b^0\cdot \qquad (6.7)$$

Then, there exist such numbers $\bar A', \bar A''$ in (3.2), δ_i in (6.3), and the control of form (6.6) that the motion r=0 of the system is exponentially stable.

The proof of the theorem is similar to the proof of theorem 1 and is not mentioned here (see Ref. [*Matyukhin, 2006*]). According to theorem 4, control law (6.6) provides WS stability motion when the slip d is taken into account. For this, it is sufficient that the slip amplitude is small and the surface along which WS moves is sufficiently smooth according to assumption (6.3). The difference of theorem 4 from theorem 3 is that for sufficiently smooth surfaces, an exact WS motion is possible due to control (6.6) containing the slip $d(x,y,a,b)$ explicitly.

6.2 Simulation

The goal of simulation is to illustrate the practical orientation of theoretical results presented above. The basic simulation result is that the WS motion is stable with account of slipping. This is substantiated for dynamical WS parameters typical, for example, for agricultural tractors [*Cordesses et al., 2000*].

The system described in (6.1),(6.6) was studied numerically. The slope surface was assumed to be flat and directed along the axis X, i.e. β=0, γ=const, d=0.2sin(a) (Fig. 3). The following values of the basic WS parameters were chosen: the tractor length L=3 m, the velocity v=2 m/s, the front axle drive performance $|\dot b| \le 1/s$. Sinusoids with an amplitude of up to 1 m and a period of 10 m (dashed line in Fig. 3) were considered as given trajectories S. The curvature radius of these curves reached 3 m, the initial deviation $|r(0)|$ reached 1 m, the deviation $|a(0)|$ reached 0.3 rad.

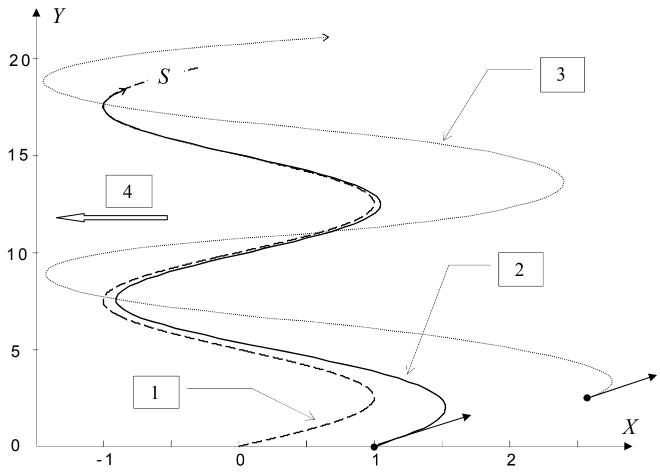

Fig. 3. Trajectory of WS motion from the point (x,y)=(1,0). Velocity v=2 m/s; S - sinusoid with an amplitude of 1 m, a period of 10 m; slope d=0.2sin(a) along the axis X. After 15 m, the deviation is smaller than 0.02 m. Key: 1. Given trajectory S; 2. WS trajectory; 3. Front axle; 4. Slope

The basic result of the simulation is that the WS motion turned out to be stable in the sense of theorem 4. Rather short transition process is provided in the system for considerable slope, when d =0.2sin(a). For example, the initial deviation $r(0)$ of an order of 1 m was compensated practically already after 15 m, and was smaller than 0.02 m. Then, the deviation tended to zero (Fig. 3). If the slip characteristic d was not taken into account in the control law (as in law (4.4) of theorem 3), oscillations with an amplitude of 0.06 m arised.

APPENDIX

Scheme of proof of the theorem 1. The idea of the proof is connected with the steering of system (2.1), (4.4)

$$\dot x = v\cos a,\quad \dot y = v\sin a,\quad \dot a = v\,\mathrm{tg}(b)/L,\quad \dot b = F(b,u,t),\qquad (1)$$
$$u = -h\,\mathrm{sign}(b-b_z),$$
$$v\,\mathrm{tg}(b_z)/L \overset{\text{def}}{=} \dot a_z + \varphi(a-a_z),\quad \sin\left(a_z - A^*\right) \overset{\text{def}}{=} -f(r)$$

under study to the sliding mode.

Lemma 2. In system (1), the sliding mode of the following form arises:

$$|b-b_z| \le |b(0)-b_z(0)|,\quad 0 \le t \le t^1,\quad b=b_z,\quad t \ge t^1.\qquad (2)$$

In this mode, the following estimates are satisfied:

$$|b_z| < \bar b_z,\quad |b| < \bar b,\quad t \ge 0,\quad \bar b_z, \bar b = \mathrm{const} < \pi/2.\qquad (3)$$

The proof of the lemma is given below. In sliding mode (2) taking into account (4.5), the following equalities are valid: $\dot e = \varphi(e),\ e = a - a_z$. This implies

$$|e(t)| \le \Lambda |e(t^1)| \exp\left(-\lambda(t-t^1)\right),\quad t \ge t^1.\qquad (4)$$

The constants λ,Λ>0 exist due to the assumed stability of the system $\dot e = \varphi(e)$. Let us take into account relations (4.5) written in the form

$$d|r|/dt \le -v|f| + R,\quad d|e|/dt \le -|\varphi| + R_1,\qquad (5)$$

$$R = v\left|\sin(a_z - A^*) - \sin(a - A^*)\right| = v\left|\cos(\tilde{a})\, e\right| \le v|e|,$$

$$R_1 = v\left|\mathrm{tg}b - \mathrm{tg}b_z\right| / L.$$

Due to (4), the following relations are valid:

$$d\,|r|\,/\,dt \le -v\,|f| + v\Lambda\,\left|e(t^1)\right|\exp(-\lambda(t - t^1)) \cdot \qquad (6)$$

On this basis, the exponential stability of the motion r=0 of system (1) under study, i.e., theorem 1, is established. Thus, theorem 1 follows from lemma 2.

Proof of lemma 2. Let us introduce the Lyapunov function $G = |b - b_z|$ and show that the function G decreases in the course of motion of the system (1) to zero. This will imply relation (2) for the sliding mode.

Indeed, for the derivative of the function G with respect to the system, we have

$$\begin{aligned} \dot{G} &= \left\{\dot{b} - \dot{b}_z\right\}\mathrm{sign}(b - b_z) = \\ &= \left\{F\left(b, -h\,\mathrm{sign}(b - b_z), t\right) - \dot{b}_z\right\}\mathrm{sign}(b - b_z) \end{aligned} \quad . \qquad (7)$$

If $b \ne b_z$, then $\dot{G} \le -H + |\dot{b}_z|$ which implies that

$$\dot{G} < -H/2, \qquad (8)$$

if the following inequality is satisfied:

$$|\dot{b}_z| < H/2 \cdot \qquad (9)$$

The solutions to differential inequality (8) decrease to zero. Therefore, the function $G(t)$ in equality (7) satisfies the relations given in the references [*Matyukhin, 2001, Filippov, 1985*]:

$$G \le G(0) - Ht/2, \quad \text{if } 0 \le t < t_1, \qquad G = 0, \quad \text{if } t \ge t_1, \ (10)$$

$t_1 = 2G(0)/H$. This implies correlation (2) of lemma 2.

For substantiation of condition (9), we take into account the following proposition.

Lemma 3. The condition (9) is valid for t=0, and it follows from the inequality (8) for t>0. Estimates (3) are also valid.

The proof of the lemma is given below. Lemma 3 proves inequality (8). Indeed, inequality (9) is satisfied for t=0 according to lemma 3. Therefore, for t=0, the inequality (8) is satisfied. Using the proof by contradiction, it is established that (8) is also satisfied for t>0. Thus, lemma 2, and, consequently, theorem 1 follow from lemma 3.

Proof of lemma 3. Relations (3.5) imply $v\dot{b}_z / L\cos^2 b_z = \ddot{a}_z + \langle \dot{a} - \dot{a}_z \rangle \varphi'$. Therefore, condition (9) follows from the inequality

$$\left\langle |\ddot{a}_z| + \left(|\dot{a}| + |\dot{a}_z|\right)|\varphi'| \right\rangle L/v < H/2, \qquad t \ge 0. \quad (11)$$

Here, the notation $\varphi' = d\varphi/dr$ is introduced and it is assumed that

$$|\varphi| \le \bar{\varphi}, \ \ |\varphi'| \le \bar{\varphi}', \ \ |f| \le \bar{f} < 1, \ \ |f'| \le \bar{f}', \ \ |f''| \le \bar{f}'', \ \ \forall r, \ (12)$$

where $\bar{\varphi}, ..., \bar{f}''$ are constants. The idea of the substantiation of inequality (11) is related to analysis of constraints of the form

$$|\ddot{a}_z| \le \bar{\ddot{a}}_z, \ \ |\dot{a}| \le \bar{\dot{a}}, \ \ |\dot{a}_z| < \bar{\dot{a}}_z, \ \ t \ge 0, \ \ \bar{\ddot{a}}_z, \bar{\dot{a}}, \bar{\dot{a}}_z = \mathrm{const} \cdot (13)$$

Condition (11) obviously follows from inequalities (13) if the assumptions (3.2) and (12), where the constants $\bar{\ddot{a}}_z, \bar{\varphi}'$ sufficiently small, are taken into account.

To prove (13), let us determine the constant $\bar{\dot{a}}_z$ by the relations

$$L\bar{\dot{a}}_z / v < \mathrm{tg}(\bar{b}_z)/2, \quad \bar{b}_z = (\pi/2 - b^0)/4 < \pi/2. \quad (14)$$

Here, the assumption of the theorem of the form $0<b^0<\pi/2$ is taken into account. The details of substantiation of inequality (13) are in Ref. [*Matyukhin, 2006*].

Thus, lemma 3, lemma 2, and theorem 1 are proved.

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