A GENERALIZATION OF MORSE'S THEOREM FOR NONLINEAR TIME-VARYING SYSTEMS

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Abstract: The paper investigates uniform asymptotic stability (UAS) for nonlinear time-varying (NTV) systems from the state-output viewpoint. Uniform Lyapunov stability (ULS) of the origin is first guaranteed by employing a new detectability condition and an integral inequality relating to output function. In addition to a newly developed criterion for attractivity, a novel result for UAS can then be proposed without assuming the ULS property in priori. It extends a theorem proposed by Morse to NTV systems. Moreover, the observability conditions often assumed in present literature can be relaxed by using detectability based on our approaches.

Keywords: Stability criterion, Lyapunov stability, detectability, uniform asymptotic stability, Morse's theorem.

1. INTRODUCTION

The paper investigates uniform asymptotic stability (UAS) for nonlinear time-varying (NTV) systems from the state-output viewpoint. Uniform Lyapunov stability (ULS) as well as uniform attractivity are treated simultaneously by employing detectability and an integral inequality relating to output function. The goal is to extend a theorem of Morse proposed in linear time-invariant systems (LTI) to NTV systems.

Morse's theorem is closely related to the following Lyapunov matrix equations:

$$PA + A^T P + C^T C = 0, \qquad (1)$$

where A and C are $p \times p$ and $q \times p$ matrices, respectively, and (C, A) is a detectable pair. It is well-known that A is stable if and only if (1) has a positive semidefinite solution P (Wonham, 1985). Moreover, the solution is unique and can be described as $P = \int_0^\infty (e^{At})^T C^T C e^{At} dt.$ If we define an output map as $y = Cx$, it can be verified that $x_0{}^T P x_0 = \int_0^\infty |y(t)|^2 dt, \quad \forall x_0 \in \Re^p,$ where $y(\cdot) = Cx(\cdot)$ and $x(\cdot)$ is a solution of $\dot{x} = Ax$ with $x(0) = x_0$. Then, the condition that $\int_0^\infty |y(t)|^2 dt < \infty$ is equivalent to the existence of positive semidefinite solutions of (1). Particularly, the following result first discovered by Morse is readable from the observation above (Morse, 1990; Byrnes and Martin, 1995). For the simplicity, it will be referred as Morse's theorem throughout this paper.

Theorem 1. Let A and C be $p \times p$ and $q \times p$ matrices, respectively. Then, A is stable if and only if (C, A) is detectable and the following inequality holds:

$$\int_0^\infty |y(t)|^2 \, dt < \infty, \qquad (2)$$

where $y(\cdot) = Cx(\cdot)$ and $x(\cdot)$ is a solution of $\dot{x} = Ax$.

Morse's theorem is very useful for those systems having an output function given in priori, such as dissipative systems. On the extension of Morse's theorem to NTV systems, the main problem is how to define a suitable detectability condition such that a similar result as in Theorem 1 holds. Implicitly, this is related to the positive definiteness of Lyapunov functions. To see such relation, let P be a solution of (1) and $V = x^T Px$ be the associated Lyapunov function. If (C, A) is observable, it is well-known that V is positive definite (Khalil, 1992). By contrast, it can only be guaranteed to be positive semidefinite under the detectability condition in general. Thus, the standard Lyapunov theory cannot be used due to lacking the positive definiteness.

In present literature, observability is frequently used to replace detectability in order to avoid the difficulty described in the above. In (Anderson and Moore, 1969), a concept of uniform complete observability (UCO) was defined. Moreover, they consider the following differential matrix equation

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) + C^T(t)C(t) = 0 \qquad (3)$$

and show that under the UCO condition, the origin of $\dot{x} = A(t)x$ is UAS if and only if (3) has a uniformly positive definite and bounded solution P. When the output map is defined as $y = C(t)x$, the latter can be described as

$$a|x_0|^2 \leq \int_{t_0}^{\infty} |y(\tau)|^2 \, d\tau \leq b|x_0|^2, \forall t_0 \geq 0, \forall x_0 \in \Re^p, (4)$$

for some positive constants a and b where $y(\cdot) = C(\cdot)x(\cdot)$ and $x(\cdot)$ is a solution of $\dot{x} = A(t)x$ with $x(t_0) = x_0$. In LTI systems, it can also be shown that (4) is equivalent to (2) under the observability condition. Thus, the result proposed in that paper can be viewed as a generalization of Morse's theorem to linear time-varying systems (LTV). Recently, Morse's theorem was further extended to nonlinear time-invariant systems (Byrnes and Martin, 1995). Under the ULS condition, it was shown that the origin of $\dot{x} = f(x)$ is UAS when the system is zero-state observable and (2) holds with the output map defined as $y = h(x)$. While these results both generalize Morse's theorem to certain classes of NTV systems by using some kind of observability, it still has a room to be improved. Particularly, it is interesting to ask if it is possible to use detectability rather than observability to guarantee the same result as in Theorem 1 for general NTV systems? As it was explained before, it is related to the use of *positive semidefinite* Lyapunov functions in guaranteeing ULS and UAS.

On this direction, several interesting results have been proposed (Bulgakov and Kalitine , 1978; Aeyels and Sepulchre, 1992; Iggidr and Sallet, 2003). Loosely speaking, they showed that the origin is ULS when there is a positive semidefinite Lyapunov function V, that is nonincreasing along with all trajectories of the system and such that, the UAS property holds for those trajectories constrained in the zero locus of V. In addition that $\dot{V} \leq 0$ and the origin is UAS restricted in the zero locus of $\dot{V}$, UAS of the origin can be guaranteed. Let us briefly connect such criteria to Morse's theorem. In fact, by defining a virtual output map as $y = \sqrt{-\dot{V}(t,x)}$, it can be seen that a similar inequality like (2) holds and the UAS property constrained in the zero locus of $\dot{V}$ just describes a detectability condition. This way, these results can be viewed as a vision of Morse's theorem in terms of Lyapunov functions.

In contrast to the approach of Lyapunov functions, the paper will study the ULS property of the origin by employing output functions. A new detectability condition that is necessary for UAS will be introduced. Then, we show that the origin is ULS provided that the system is detectable and satisfies an integral inequality related to the output function. In addition to a newly developed criterion given in (Lee and Chen, 2002), it will be shown that the origin is UAS when the system satisfies an integral inequality like (2). When reduce to continuous periodic systems, the proposed detectability condition is equivalent to the zero-state detectability condition plus a Lyapunov stable condition constrained in the zero locus of the output function. Particularly, the result given in (Byrnes and Martin, 1995) can be deduced from the proposed criterion without assuming Lyapuov stability in priori. For LTV systems, the proposed result also improves the theorem given in (Anderson and Moore, 1969) by relaxing certain conditions. Particularly, the uniformly positive definite property (the first inequality in (4)) of solutions of (3) is not necessary based on our approaches. In LTI systems, it will be further shown that the proposed detectability condition is equivalent to the classic one. Then, our main result is reduced to Theorem 1. This indicates that the proposed criterion extends Morse's theorem to NTV systems without using extra conditions. Due to a limited space, some results are only stated without proofs. Readers can contact the author for the detailed proofs.

2. PRELIMINARIES

2.1 A basic assumption and criterion

In this subsection, a basic assumption will be made. A related result that guarantees the basic assumption will be proposed. In this paper, we study a nonlinear time-varying system described as

$$\dot{x} = f(t, x) \tag{5}$$

$$y = h(t, x) \tag{6}$$

where x is contained in an open and connected subset (i.e., a domain) X of $\Re^p$ with $0 \in X$, $y \in \Re^q$, and f and h are all measurable functions defined on $[0,\infty) \times X$, such that $f(t,0)=h(t,0)=0$ for all $t \geq 0$. Assume that f satisfies the Caratheodory condition so that the existence theorem and extension theorem of solutions of (5) are satisfied (Hale, 1980). Throughout this paper, we denote $\phi(t, t_0, x_0)$ as a trajectory of (5) starting from x_0 at time $t = t_0$, $|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_p^2}$, $\forall v = (v_1, v_2, \cdots, v_p) \in \Re^p$, and the Lebesgue meaure of a set $S \subseteq \Re^p$ as $|S|$. Let I be any interval. It is said that a statement $P(t)$ holds for almost all t in I if $\left|\left\{ t \in I \middle| \, P(t) \text{ is false} \right\}\right| = 0$ (Lang, 1983).

A basic assumption that will be used throughout this paper is given as follows.

(A1) For any compact $K \subset X$, there exists a nondecreasing function $\mu_K : [0,\infty) \to [0,\infty)$, continuous at 0, with $\mu_K(0) = 0$ and such that, whenever $u : [a,b] \to K$ is continuous, the integral $\int_a^b f(\tau, u(\tau)) d\tau$ is well defined, and $f(\cdot,\cdot)$ satisfies the inequality

$$\left| \int_a^b f(\tau, u(\tau)) d\tau \right| \leq \mu_K(b-a). \tag{7}$$

Recall that a function g is uniformly bounded if for any compact $K \subset X$, there exists a positive constant $k_0(K)$ which dominates g over $[0,\infty)\times K$. The following lemma shows that (A1) is implied by the uniformly bounded property of *f*.

Lemma 1. Consider (5) with f being uniformly bounded. Then, (A1) holds.

Proof. For any compact $K \subset X$, let $k_0(K)$ be a positive constant which dominates *f* over $[0,\infty)\times K$. Define $\mu_K(s) = k_0 s, \forall s \ge 0$. Then, it can be directly checked that (7) holds and hence (A1) is true. This completes the proof of this lemma. ■

2.2 Asymptotic detectability: basic definitions and relations

Let us recall the weak zero-state detectability from (Lee and Chen, 2002) and define a stronger detectability condition as follows. The latter will be used to guarantee the ULS and UAS properties.

Definition 1. System (5)-(6) is weakly zero-state detectable (WZSD) if, for any unbounded sequence $\{t_n\}$ in $[0,\infty)$ and any sequence $\{\phi_n(t,t_n,x_n)\}$ of solutions of (5) which, lies within a compact subset *K* of *X* and satisfy the following equation

$$\lim_{n\to\infty} h(t+t_n, \phi_n(t+t_n,t_n,x_n)) = 0, \ a.e., \tag{8}$$

there is a subsequence $\{n_m\}$ of $\{n\}$ and a time sequence $\{s_m\}$ with $0 \le s_m \le n_m$ such that

$$\lim_{m\to\infty} \phi_{n_m}(s_m + t_{n_m}, t_{n_m}, x_{n_m}) = 0\,. \tag{9}$$

It is said to be locally weakly zero-state detectable if WZSD holds on some open neighborhood of the origin.

Definition 2. System (5)-(6) is asymptotically detectable (AD) if, the following conditions hold.

(a) For any $\varepsilon > 0$ and any $s > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $|x_n| < \delta$, $\forall n \in \aleph$, and the following equation

$$\lim_{n\to\infty} h(t+t_n, \phi_n(t+t_n,t_n,x_n)) = 0 \tag{10}$$

holds for almost all $t \in [0,s]$, we have

$$\liminf_{n\to\infty} |\phi_n(s+t_n,t_n,x_n)| < \varepsilon \tag{11}$$

where $\{t_n\}$ is any unbounded sequence in $[0,\infty)$ and $\{\phi_n(t,t_n,x_n)\}$ is any sequence of solutions of (5) such that $\phi_n(t+t_n,t_n,x_n) \in K$, $\forall n \in \aleph, \forall 0 \le t \le s,$ for some compact subset *K* of *X*.

(b) If $\{\phi_n(t,t_n,x_n)\}$ satisfies the following equation

$$\lim_{n\to\infty} h(t+t_n, \phi_n(t+t_n,t_n,x_n)) = 0 \tag{12}$$

for almost all t in $[0,\infty)$, there is a subsequence $\{n_m\}$ of $\{n\}$ and a time sequence $\{s_m\}$ with $0 \le s_m \le n_m$ such that

$$\lim_{m\to\infty} \phi_{n_m}(s_m + t_{n_m}, t_{n_m}, x_{n_m}) = 0 \tag{13}$$

where $\{t_n\}$ is any unbounded sequence in $[0,\infty)$ and $\{\phi_n(t,t_n,x_n)\}$ is any sequence of solutions of (5) such that $\phi_n(t+t_n,t_n,x_n) \in K$, $\forall n \in \aleph, \forall 0 \le t \le n,$ for some compact subset *K* of *X*.

It is said to be locally asymptotically detectable if AD holds on some open neighborhood of the origin.

From the definitions, it can be seen that AD is a stronger condition than WZSD. More interestingly, the following result shows that WZSD also implies AD under certain standard stability conditions.

Lemma 2. Consider a system of the form (5)–(6). Suppose the origin is uniformly Lyapunov stable. Then, the system is locally asymptotically detectable if and only if it is locally weakly zero-state detectable. In addition that $X = \Re^p$ and solutions are globally bounded, the system is asymptotically detectable if and only if it is weakly zero-state detectable.

Proof. Without loss of generality, it is sufficient to prove the "if" part. By the ULS property, for any $\varepsilon > 0$ there exists a constant $\delta(\varepsilon) > 0$ such that, for any sequence $\{\phi_n(t,t_n,x_n)\}$ of solutions of (5) with $|x_n| < \delta, \forall n \in \aleph,$ the inequality $|\phi_n(s+t_n,t_n,x_n)| < \varepsilon/2$ holds for all $s > 0$ and all $n \in \aleph$. This results in $\liminf_{n\to\infty} |\phi_n(s+t_n,t_n,x_n)| \le \varepsilon/2 < \varepsilon, \forall s > 0.$ Thus, condition (a) in Definition 2 holds. Now suppose that the systems is locally weak zero-state detectable. Then, there is an open set U containing the origin such that WZSD holds on *U*. Since the origin is uniformly Lyapunov stable, there is also an open subset $\widetilde{U}$ of *U* containing the origin such that, every solution $\phi(t,t_0,x_0)$ of (5) with $x_0 \in \widetilde{U}$ lies within a compact subset of *U*. Thus, (b) also holds on $\widetilde{U}$ in view of the WZSD condition. Thus, local weak zero-state detectability implies local asymptotic detectability under the uniformly Lyapunov stable condition. In addition that $X = \Re^p$ and solutions are globally bounded, all solutions having the initial conditions contained in a compact set of $\Re^p$ must be contained in another compact set of $\Re^p$. In this case, it is easy to see that (b) in Definition 2 is equivalent to the WZSD condition. This completes the proof of the lemma. ■

In (Lee and Chen, 2002), it was shown that WZSD is a necessary condition of UAS. In view of Lemma 2, it can be deduced that AD is also a necessary condition of UAS. Particularly, the following result is readable from Lemma 2 and the result given in that paper.

Lemma 3. Consider a system of the form (5)–(6). Then, the system is locally asymptotically detectable if the origin is uniformly asymptotically stable. In addition that $X = \Re^p$, the systems is asymptotically detectable if the origin is uniformly globally asymptotically stable (UGAS).

2.3 Specification to continuous periodic systems

In this subsection, the AD condition will be further characterized by the zero-state detectability condition and a Lyapunov stable condition constrained in the zero-locus of the output function for continuous periodic systems. First, let us extend the definition of zero-state detectability as follows (Byrnes and Martin, 1995).

Definition 3. System (5)-(6) is zero-state detectable (ZSD) if for any solution $\phi(t,t_0,x_0)$ of (5) satisfying $h(t,\phi(t,t_0,x_0))=0, \forall t \ge t_0,$ we have $\lim_{t\to\infty}\phi(t,t_0,x_0)=0.$

In addition that the following condition holds:

(C1) for any $\varepsilon>0$, there exists a $\delta(\varepsilon)>0$ such that for any $s>0$ and any solution $\phi(t,t_0,x_0)$ of (5) satisfying $|x_0|<\delta$ and $h(t,\phi(t,t_0,x_0))=0, \forall t_0 \le t \le t_0+s,$ we have $|\phi(t_0+s,t_0,x_0)|<\varepsilon$,
it is called as strongly zero-state detectable (SZSD).

It is said to be locally (strongly) zero-state detectable if (SZSD) ZSD holds for all x_0 contained in some open neighborhood of the origin.

Now, the following proposition can be proposed. Its proof is omitted due to a limited space

Proposition 1. Consider a system of the form (5)-(6) where f and h are both continuous periodic functions with the same period. Then, local asymptotic detectability is equivalent to local strong zero-state detectability. In addition that $X=\Re^p$ and solutions are globally bounded, the system is asymptotically detectable if and only if it is strongly zero-state detectable. ■

In LTI systems, it is straightforward to see that the usual detectability is equivalent to SZSD. Moreover, it can be directly checked that SZSD (AD) and local SZSD (AD) are the same in this case. Thus, the AD condition is equivalent to the classic detectability condition. Particularly, the following result is readable from Proposition 1.

Corollary 1. Consider a system of the form (5)-(6) where $f=Ax$ and $h=Cx$ for some matrices A and C. Then, it is AD if and only if (C, A) is detectable. ■

3. MAIN RESULTS

3.1 A new stability criterion for ULS

In this subsection, we study the ULS property with the help of output functions. First, let us consider the following technique assumption. Recall that a subset S of X is called as an invariant set if every solution $\phi(t,t_0,x_0)$ of (5) starting from $x_0\in S$ cannot leave S (Khalil,1992).

(A2) $S=\{0\}$ is an invariant set.

Remark 1. It is easy to see that (A2) holds when either the origin is ULS or the uniqueness theorem of solutions holds. ■

Under (A2), the following technique lemma can be proposed. Its proof is omitted here.

Lemma 4. Consider a system of the form (5)-(6). Suppose (A1)-(A2) hold. Let $\varepsilon>0$ be any constant, $\{t_n\}$ be an unbounded sequence in $[0,\infty)$ and $\{\phi_n(t,t_n,x_n)\}$ be any sequence of solutions of (5) satisfying $\lim_{n\to\infty}|x_n|=0$. Suppose there exists a sequence $\{s_n\}$ such that $s_n\ge t_n\ge 0$ and $|\phi_n(s_n,t_n,x_n)|=\varepsilon$, $\forall n\in\aleph$. Then, $\lim_{n\to\infty}s_n=\infty$.

To state the main result in this subsection, we need the following hypothesis. It plays a central role in guaranteeing the ULS property of the origin.

(H1) For any $\varepsilon>0$ and any compact $K\subset X$, there exists a $\delta(\varepsilon,K)>0$ such that for any $\hat{s}\ge s\ge t_0\ge 0$ and any solution $\phi(t,t_0,x_0)$ of (5) having $|x_0|<\delta$ and contained in K on the interval $[t_0,\hat{s}]$, the following inequality holds:

$$\int_s^{\hat{s}}|h(\tau,\phi(\tau,t_0,x_0))|^2 d\tau \le \varepsilon(\hat{s}-s+1). \quad (14)$$

Now, the following result can be proposed.

Theorem 2. Consider a system of the form (5)-(6) that satisfies (A1)-(A2). Suppose the system is locally asymptotically detectable and (H1) hold. Then, the origin is uniformly Lyapunov stable.

Proof. The theorem will be proven by contradiction. Suppose the origin is not uniformly Lyapunov stable. Then, there exists a $\varepsilon_0>0$ such that for each $n\in\aleph$, there exist a solution $\hat{\phi}_n(t,\hat{t}_n,\hat{x}_n)$ of (5) and a $\hat{s}_n\ge\hat{t}_n$ satisfying $|\hat{x}_n|<1/n$ and $|\hat{\phi}_n(\hat{s}_n,\hat{t}_n,\hat{x}_n)|\ge\varepsilon_0$. In the following, we would like to find a contradiction.

Since the system is locally asymptotically detectable, there is a positive constant $\hat{\varepsilon}_0<\varepsilon_0$ such that AD holds on $\{x\in\Re^p \mid |x|<\hat{\varepsilon}_0\}$. Then, there exist two positive constants $\varepsilon_1<\hat{\varepsilon}_0/2$ and T_1 only depending on $\hat{\varepsilon}_0$ such that, for any $t_0\ge 0$ and $|x_0|<\varepsilon_1$ every solution $\phi(t,t_0,x_0)$ satisfies $|\phi(t,t_0,x_0)|<\hat{\varepsilon}_0/2, \forall t_0\le t\le t_0+T_1$, by using (A1). In view of (a) in the definition of AD, there is also a $0<\delta_1<\varepsilon_1$ such that the conclusion of (a) is true for $\varepsilon=\varepsilon_1$. Let $\delta=\delta_1/2$ and N_0 be a large integer with $1/N_0\le\delta$. Since $|\hat{\phi}_n(\hat{s}_n,\hat{t}_n,\hat{x}_n)|\ge\varepsilon_0>\hat{\varepsilon}_0/2$ and $|\hat{x}_n|<1/n\le\delta, \forall n\ge N_0$, there exists two sequences $\{s_n\}$ and $\{t_n\}$, with $\hat{t}_n<t_n<s_n<\hat{s}_n$ and such that, $|\hat{\phi}_n(t_n,\hat{t}_n,\hat{x}_n)|=\delta$, $|\hat{\phi}_n(s_n,\hat{t}_n,\hat{x}_n)|=\hat{\varepsilon}_0/2$, $|\hat{\phi}_n(t,\hat{t}_n,\hat{x}_n)|<\hat{\varepsilon}_0/2, \forall\hat{t}_n\le t<s_n,$ and $|\hat{\phi}_n(\tau,\hat{t}_n,\hat{x}_n)|>\delta, \forall t_n<\tau\le s_n.$ Since $|\hat{\phi}_n(s_n,\hat{t}_n,\hat{x}_n)|=\hat{\varepsilon}_0/2$ and $|\hat{\phi}_n(t_n,\hat{t}_n,\hat{x}_n)|=\delta<\varepsilon_1$, we have $s_n\ge t_n+T_1$ for all $n\ge N_0$, by the choices of ε_1

and T_1. Using the fact that $\lim_{n\to\infty}|\hat{x}_n| \le \lim_{n\to\infty} 1/n = 0$ and Lemma 4, it can be seen that $t_n \to \infty$. Let $x_n = \hat{\phi}_n(t_n, \hat{t}_n, \hat{x}_n)$ and $\phi_n(t, t_n, x_n) = \hat{\phi}_n(t, \hat{t}_n, \hat{x}_n)$, $\forall n \ge N_0, \forall t \ge t_n$. Then, $\phi_n(t, t_n, x_n)$ is also a solution of (5) starting from x_n at time $t = t_n$ for each $n \ge N_0$. Moreover, we have $|\phi_n(t_n, t_n, x_n)| = |x_n| = \delta$, $|\phi_n(s_n, t_n, x_n)| = \hat{\varepsilon}_0 / 2$ and $\delta < |\phi_n(t, t_n, \tilde{x}_n)| < \hat{\varepsilon}_0 / 2, \forall t_n < t < s_n, \forall n \ge N_0$. Replacing the sequence $\{n\}$ by a suitable subsequence, we can assume that the limit $T = \lim_{n\to\infty}(s_n - t_n) \ge T_1$ exists. In the following, let us divide the proof into two parts according to the value of T.

(a) The case of $T < \infty$. Let $\hat{T} = T - T_1/2 \ge T_1/2 > 0$. We first claim that $\left|\phi_n(\hat{T} + t_n, t_n, x_n)\right| \ge \varepsilon_1$ for large enough *n*. Since $\lim_{n\to\infty}(s_n - t_n) = T$ and $\hat{T} < T < \hat{T} + T_1$, the inequality $\hat{T} + t_n < s_n < \hat{T} + t_n + T_1$ holds, for large enough *n*. If $\left|\phi_n(\hat{T} + t_n, t_n, x_n)\right| < \varepsilon_1$, we have $|\phi_n(s_n, t_n, x_n)| < \hat{\varepsilon}_0 / 2$ for large enough *n*, by the choices of ε_1 and T_1. This contradicts the fact that $|\phi_n(s_n, t_n, x_n)| = \hat{\varepsilon}_0 / 2$. Thus, the claim is true.

Moreover, the fact $\hat{T} + t_n < s_n$ implies that for large enough *n*, $\hat{\phi}_n(t, \hat{t}_n, \hat{x}_n)$ is contained in the compact subset $K_0 = \left\{x \in \Re^p \,\middle|\, |x| \le \hat{\varepsilon}_0 / 2\right\}$ of X for all t in $[\hat{t}_n, t_n + \hat{T}]$ by the choice of s_n. Let ε be any positive constant. In view of (H1), there exists a $\tilde{\delta}(\varepsilon, K_0) > 0$ such that (14) holds for all $\hat{s} \ge s \ge t_0 \ge 0$ and all solutions $\phi(t, t_0, x_0)$ of (7) having $|x_0| < \tilde{\delta}$ and contained in K_0 on the interval $[t_0, \hat{s}]$. Since $\lim_{n\to\infty} \hat{x}_n = 0$, (14) can be applied to the solution $\hat{\phi}_n(t, \hat{t}_n, \hat{x}_n)$ on the interval $[\hat{t}_n, t_n + \hat{T}]$ for large enough *n*. Thus, the following inequality holds (with $s = t_n$ and $\hat{s} = t_n + \hat{T}$):

$$\int_0^{\hat{T}} \left|h(\tau + t_n, \hat{\phi}_n(\tau + t_n, \hat{t}_n, \hat{x}_n))\right|^2 d\tau \le \varepsilon(\hat{T} + 1),$$

for large enough *n*. Since ε is arbitrary given, this results in

$$\lim_{n\to\infty} \int_0^{\hat{T}} \left|h(\tau + t_n, \phi_n(\tau + t_n, t_n, x_n))\right|^2 d\tau = 0.$$

From the theory of real analysis (Lang, 1983), we can replace the sequence $\{n\}$ by a suitable subsequence and assume that $\lim_{n\to\infty} h(t + t_n, \phi_n(t + t_n, t_n, x_n)) = 0$ for almost all t in $[0, \hat{T}]$. By the choice of δ_1 and $|x_n| = \delta < \delta_1$, we can concluded that

$$\liminf_{n\to\infty} \left|\phi_n(\hat{T} + t_n, t_n, x_n)\right| < \varepsilon_1,$$

by virtue of (a) in Definition 2. This violates the claim that $\left|\phi_n(\hat{T} + t_n, t_n, x_n)\right| \ge \varepsilon_1$ for large enough *n*. Thus, we reach a contradiction. The proof of this part is completed.

(b) The case of $T = \infty$. It can be proven along with a similar line by employing (b) in Definition 2, and the fact that $\left|\hat{\phi}_n(\tau, \hat{t}_n, \hat{x}_n)\right| > \delta, \forall t_n < \tau \le s_n$. This completes the proof of the theorem. ■

3.2 A generalized Morse's theorem

In this subsection, a generalized Morse's theorem will be proposed by combining Theorem 2 with a newly developed criterion given in (Lee and Chen, 2002). First, let us state a condition as follows.

(C2) For each compact $K \subset X$, there exists $M(K) > 0$ such that, for all solutions $\phi(t, t_0, x_0)$ of (5) lying within *K*,

$$\int_{t_0}^{t} \left|h(\tau, \phi(\tau, t_0, x_0))\right|^2 d\tau \le M, \ \forall\, t \ge t_0. \tag{15}$$

The following result comes from (Lee and Chen, 2002) by employing Theorem 1 and Proposition 3 in that paper.

Proposition 2. Consider a system of the form (5)-(6). Suppose the origin is uniformly Lyapunov stable and (C2) holds. Then, the origin is uniformly asymptotically stable when the system is locally WZSD. In addition, if $X = \Re^p$ and the solutions are globally uniformly bounded, then the origin becomes uniformly globally asymptotically stable under the WZSD condition. ■

Consider the following hypothesis like (2).

(H2) There exists a function $\beta : X \to [0, \infty)$, with $\beta(0) = 0$ and continuous at the origin such that, for all $s \ge t_0 \ge 0$ and all solutions $\phi(t, t_0, x_0)$ of (5) that can be defined on the interval $[t_0, s]$,

$$\int_{t_0}^{s} \left|h(\tau, \phi(\tau, t_0, x_0))\right|^2 d\tau \le \beta(x_0). \tag{16}$$

Now, the following result that generalizes Morse's theorem can be proposed based on Theorem 2 and Proposition 2.

Theorem 3. Consider a system of the form (5)-(6) where (A1)-(A2) holds. Suppose the system is locally asymptotically detectable and (H2) holds on some open neighborhood of the origin. Then, the origin is uniformly asymptotically stable. In addition that $X = \Re^p$ and solutions are uniformly bounded, the origin is uniformly globally asymptotically stable provided that the system is asymptotically detectable and (H2) hold with β being locally bounded.

Proof. Notice that AD implies WZSD. Thus, it is sufficient to show that (H2) implies (C2) and (H1) in view of Theorem 2 and Proposition 2. Since β is continuous at the origin and (H2) holds on some open neighborhood of the origin, we can replace X by a smaller open and connected neighborhood of the origin such that β is also locally bounded and (H2) holds on X in the local case. By the local boundedness of β, $M = \sup_{x \in K} \beta(x) < \infty$ for each compact $K \subset X$ (Lang, 1983). Then, it is straightforward to see that (16) implies (15). Thus, (C2) can be derived from (H2) when we replace X by a suitable open neighborhood of the origin. In the following, let us show that (H1) can also be deduced from (H2). Let $\varepsilon > 0$ be any constant and $K \subset X$ be any compact set. Since β is continuous at the origin, with $\beta(0) = 0$, there exists a $\delta(\varepsilon) > 0$ such that $\beta(x) < \varepsilon$ for all $|x| < \delta$. Let $\hat{s} \geq s \geq t_0 \geq 0$, $|x_0| < \delta$ and $\phi(t, t_0, x_0)$ be any solution of (5) contained in K on the interval $[t_0, \hat{s}]$. According to (H2), the following inequality holds:

$$\int_s^{\hat{s}} \left| h(\tau, \phi(\tau, t_0, x_0)) \right|^2 \, d\tau \leq \beta(x_0) < \varepsilon \leq \varepsilon(\hat{s} - s + 1).$$

Thus, (14) holds. The theorem follows from Theorem 2 and Proposition 2. ■

Remark 2. In continuous periodic systems, AD is equivalent to SZSD by Proposition 1. Thus, the result proposed in (Byrnes and Martin, 1995) can be derived by Theorem 3 where observability can be relaxed by detectability, and ULS of the origin can be replaced by a reduced condition-(C1). ■

In the remainder of this subsection, we assume that $X = \Re^p$, $f = A(t)x$ and $h = C(t)x$. If A is bounded, (A1) holds by Lemma 1. Moreover, the theorem of the uniqueness of solutions is true in this case (Hale, 1980). Then, (A2) also holds. To verify (H2), let us make the following condition.

(C3) For each $x_0 \in \Re^p$, the following inequality holds:

$$\sup_{t_0 \geq 0} \int_{t_0}^{\infty} \left| h(\tau, \phi(\tau, t_0, x_0)) \right|^2 \, d\tau < \infty, \tag{17}$$

for any solution $\phi(t, t_0, x_0)$ of (5).

Under (C3), let us check (H2). Let $\{e_1, e_2, \cdots, e_p\}$ be an orthonormal basis of $\Re^p$. Then, every vector $x \in \Re^p$ can be described as $x = \sum_{i=1}^{p} v_i e_i$. If the system is a LTV system, every solution $\phi(t, t_0, x_0)$ can be written as $\phi(t, t_0, x_0) = \sum_{i=1}^{p} v_i \phi(t, t_0, e_i)$ with $x_0 = \sum_{i=1}^{p} v_i e_i$. By employing Cauchy inequality, it can be shown that the following inequality holds:

$$\left| C(t)\phi(t, t_0, x_0) \right|^2 \leq |x_0|^2 \sum_{i=1}^{p} \left| C(t)\phi(t, t_0, e_i) \right|^2, \forall t \geq t_0 \geq 0. \tag{18}$$

Let $M_0 = \sum_{i=1}^{p} \sup_{t \geq 0} \int_t^{\infty} \left| h(\tau, \phi(\tau, t, e_i)) \right|^2 \, d\tau < \infty$ and $\beta(x) = M_0 |x|^2, \forall x \in \Re^p$. In view of (18), we have

$$\int_{t_0}^{\infty} \left| h(\tau, \phi(\tau, t_0, x_0)) \right|^2 \, d\tau \leq |x_0|^2 \sum_{i=1}^{p} \int_{t_0}^{\infty} \left| h(\tau, t_0, e_i) \right|^2 d\tau \leq \beta(x_0),$$

for all $t_0 \geq 0$ and all solutions $\phi(t, t_0, x_0)$ of (5). This implies (H2). Since UAS implies exponential stability in LTV systems (Khalil, 1992), the following result is readable from Theorem 3.

Proposition 3. Consider a linear time-varying system of the form (5)-(6) where $f = A(t)x$ and $h = C(t)x$. Suppose A is bounded, the system is asymptotically detectable and (C3) holds. Then, the origin is exponentially stable. ■

Remark 3. Notice that, (C3) can also be checked by employing differential matrix equation (3) given in section 1 where we only need the uniform bounded property (the second inequality of (4)) of the solution *P*, and the uniformly positive definite condition (the first inequality of (4)) is unnecessary. Moreover, by Corollary 1, the asymptotic detectability condition is equivalent to the usual one and (17) is equal to (2) in LTI systems. Thus, Proposition 3 is reduced to Theorem 1. Particularly, Theorem 3 generalizes Morse's theorem without using extra conditions. ■

4. CONCLUSIONS

A generalization of the celebrated Morse theorem has been obtained by employing a detectability condition. Future work may consider its new applications in NTV systems.

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