OPTIMAL AIRLINE SEAT INVENTORY CONTROL FOR MULTI-LEG FLIGHTS

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Abstract: For large commercial airlines, efficiently setting and updating seat allocation targets for each passenger category on each multi-leg flight is an extremely difficult problem. This paper presents static and dynamic models of airline seat inventory control for multi-leg flights with multiple fare classes, which allow one to maximize an expected contribution to profit. The dynamic model uses the most recent demand and capacity information and allows one to allocate seats dynamically and anticipatory over time.

Keywords: Aircraft, transportation, data, model-based control, optimization.

1. INTRODUCTION

It is common practice for airlines to sell a pool of identical seats at different prices according to different booking classes to improve revenues in a very competitive market. In other words, airlines sell the same seat at different prices according to different types of travelers (first class, business and economy) and other conditions. The question then arises whether to offer seats at a relatively low price at a given time with a given number of seats remaining or to wait for the possible arrival of a higher paying customer. Assigning seats in the same compartment to different fare classes of passengers in order to improve revenues is a major problem of airline seat inventory control. This problem has been considered in numerous papers. For details, the reader is referred to a review of yield management, as well as perishable asset revenue management, by Weatherford, *et al*. (1993), and a review of relevant mathematical models by Belobaba (1987). For a comprehensive and up-to-date overview of the area we refer to McGill and van Ryzin (1999) containing a bibliography of over 190 references.

A common approach to dealing with the above problem is to assume that lower-fare customers book before higher-fare customers (cf. Belobaba, 1989; Brumelle and McGill, 1993; Curry, 1990; Wollmer, 1992). Since customers from different fare classes do not necessarily arrive in the order of increasing fares, Robinson (1995) considered a somewhat more general case in which the customers of any given fare class remain clustered but the order of such clusters may not match that of the increasing fares. In practice, customers from different fare classes arrive concurrently rather than sequentially. Therefore, this case cannot be ignored. Hersh and Ladany (1978) studied an intermediate stop problem and developed a policy for allocating seats to passengers flying full or partial spans. Under the assumption that there are neither no-shows by accepted customers nor cancellations of booking, seat inventory problems with multiple fare classes have received a significant amount of attention. Using an approach similar to that proposed by Gerchak, *et al*. (1985), Lee and Hersh (1993) investigated a multiple fare classes seats allocation problem and showed that the booking policy can be represented using either a set of critical booking capacities, or critical decision periods.

The problem described here is usually considered in three stages according to increasing difficulty. First is the one-leg problem, which deals with one airplane for one takeoff and landing and ignores the potential revenue impact of other links of the passengers' itineraries. Second is the multi-leg problem, which deals with one airplane having multiple takeoffs and landings (still ignoring the impact of other links). The third is the origin-destination network (OD network) problem, which considers many airplanes having many takeoffs and landings on a routing network.

This paper deals with the problem of optimal airline seat inventory control under the following assumptions: (i) Multi-leg flight: In multi-leg flight seat inventory control, the complete flight offered by the airline is optimized simultaneously. One way to do this is to distribute the revenue of multi-leg flight over its passenger origin-destination (OD) combinations and apply seat inventory control to the individual OD combinations. Seats for each OD combination are reserved and offered at several fares according to different types of travelers (first class, business and economy). Assigning seats in the multi-leg flight to different passenger OD and fare class combinations is a major problem of multi-leg flight seat allocation. We seek an optimal policy that maximizes total expected revenue; (ii) Independent demands: The demands for the different passenger OD and fare class combinations are stochastically independent; (iii) Low before high demands: The lowest fare reservations requests arrive first, followed by the next lowest, etc., for each passenger OD combination; (iv) No cancellations: Cancellations, no-shows and overbooking are not considered; (v) Limited information: The decision to close a fare class is based only on the number of current bookings; (vi) Nested fare classes: Any fare class can be booked into seats not taken by bookings in lower fare classes (for the same OD combination).

2. MULTI-LEG FLIGHT MODEL FOR STATIC SEAT INVENTORY CONTROL

In order to obtain the multi-leg flight model for static seat inventory control, denote an origin-destination and fare class combination by ODF. The goal of seat inventory control for multi-leg flight is to maximize the expected revenue from its supply of ODF combinations, using estimated demand distributions. Each ODF in the multi-leg flight consists of one or more flight legs. The limited capacity on each flight leg has to be used in the most profitable way. This can be achieved by limiting the number of seats available to the less profitable classes. Therefore, let u_i denote the number of seats reserved for each separate OD:=*i*, and $u_{i,F}$ denote the number of seats protected for each separate ODF from all lower classes of the same OD. This definition implies that each seat on each flight leg is available for only one particular OD. Through this partitioned approach passengers are divided into homogeneous groups whose contribution to multi-leg flight revenue is clear, which is essential for the definition of the objective function. Let L be the total number of flight legs in the ODF multi-leg flight. $S_{i;l}$ denotes the set of OD combinations available on flight leg l. Let F_i be the number of fare classes for each separate OD:=*i*. The probabilistic demand for each ODF is denoted by $X_{i,F}$. Although demand is in fact a discrete variable, continuous approximations of the demand distributions are generally used. Furthermore, let $c_{i,F}$ be the fare required for an ODF, where $c_{i,1} > c_{i,2} > \ldots > c_{i,F_i}$, i.e., $c_{i,1}$ and c_{i,F_i} are the highest and lowest fare levels respectively, and let U denote the seat capacity of airplane. The u_i and $u_{i,F}$ are integer decision variables that should be chosen to maximize the expected profit of the multi-leg flight.

For each separate OD, if u_i is given, the problem is to find an optimal vector of individual protection levels $({}^*u_{i,1}, \ldots, {}^*u_{i,F_i-1})$ for fare classes 1, …, F_i–1 and booking limit ${}^*u_{i,F_i}$ for the lowest fare class,

$$({}^*u_{i,1}, \ldots, {}^*u_{i,F_i}) = \arg \max_{(u_{i,1}, \ldots, u_{i,F_i}) \in D_i} R_i(u_{i,1}, \ldots, u_{i,F_i}; u_i, F_i), \tag{1}$$

where

$$R_i(u_{i,1}, \ldots, u_{i,F_i}; u_i, F_i) = \int_0^{u_{i,F_i}} [c_{i,F_i} x_{i,F_i} + R_i(u_{i,1}, \ldots, u_{i,F_i-1} + u_{i,F_i} - x_{i,F_i}; u_i - x_{i,F_i}, F_i - 1)] \times f_{i,F_i}(x_{i,F_i}; \theta_{i,F_i}) dx_{i,F_i}$$
$$+ \int_{u_{i,F_i}}^{\infty} [c_{i,F_i} u_{i,F_i} + R_i(u_{i,1}, \ldots, u_{i,F_i-1}; u_i - u_{i,F_i}, F_i - 1)] \times f_{i,F_i}(x_{i,F_i}; \theta_{i,F_i}) dx_{i,F_i} \tag{2}$$

is the expected revenue, with $R_i(\cdot;\cdot,0) = 0$, $f_{i,F_i}(x_{i,F_i}; \theta_{i,F_i})$ is the probability density function of X_{i,F_i}, θ_{i,F_i} is a parameter (in general, vector),

$$D_i = \left\{ \begin{matrix} (u_{i,1}, \ldots, u_{i,F_i}) : \\ \sum_{F=1}^{F_i} u_{i,F} = u_i, u_{i,F} \geq 0, \forall F = 1(1)F_i \end{matrix} \right\}. \tag{3}$$

Theorem 1. The optimal protection levels can be obtained by finding ${}^*u_{i,1}, \ldots, {}^*u_{i,F_i-1}$ that satisfy

$$c_{i,2} = c_{i,1} \int_{{}^*u_{i,1}}^{\infty} f_{i,1}(x_{i,1};\theta_{i,1})dx_{i,1},$$

$$c_{i,3} = c_{i,2} \int_{{}^*u_{i,2}}^{\infty} f_{i,2}(x_{i,2};\theta_{i,2})dx_{i,2}$$

$$+ c_{i,1} \int_{0}^{{}^*u_{i,2}} f_{i,2}(x_{i,2};\theta_{i,2}) \int_{{}^*u_{i,1}+{}^*u_{i,2}-x_{i,2}}^{\infty} f_{i,1}(x_{i,1};\theta_{i,1})dx_{i,1}dx_{i,2},$$

$$c_{i,4} = c_{i,3} \int_{{}^*u_{i,3}}^{\infty} f_{i,3}(x_{i,3};\theta_{i,3})dx_{i,3}$$

$$+ c_{i,2} \int_{0}^{{}^*u_{i,3}} f_{i,3}(x_{i,3};\theta_{i,3}) \int_{{}^*u_{i,2}+{}^*u_{i,3}-x_{i,3}}^{\infty} f_{i,2}(x_{i,2};\theta_{i,2})dx_{i,2}dx_{i,3}$$

$$+ c_{i,1} \int_{0}^{{}^*u_{i,3}} f_{i,3}(x_{i,3};\theta_{i,3}) \int_{0}^{{}^*u_{i,2}+{}^*u_{i,3}-x_{i,3}} f_{i,2}(x_{i,2};\theta_{i,2})$$

$$\times \int_{{}^*u_{i,1}+{}^*u_{i,2}+{}^*u_{i,3}-x_{i,3}-x_{i,2}}^{\infty} f_{i,1}(x_{i,1};\theta_{i,1})dx_{i,1}dx_{i,2}dx_{i,3},$$

$$\vdots$$

$$c_{i,F} = c_{i,F-1} \int_{{}^*u_{i,F-1}}^{\infty} f_{i,F-1}(x_{i,F-1};\theta_{i,F-1})dx_{i,F-1}$$

$$+ c_{i,F-2} \int_{0}^{{}^*u_{i,F-1}} f_{i,F-1}(x_{i,F-1};\theta_{i,F-1})$$

$$\times \int_{{}^*u_{i,F-2}+{}^*u_{i,F-1}-x_{i,F-1}}^{\infty} f_{i,F-2}(x_{i,F-2};\theta_{i,F-2})dx_{i,F-2}dx_{i,F-1}$$

$$+ \ldots + c_{i,1} \int_{0}^{{}^*u_{i,F-1}} f_{i,F-1}(x_{i,F-1};\theta_{i,F-1})$$

$$\times \int_{0}^{{}^*u_{i,F-2}+{}^*u_{i,F-1}-x_{i,F-1}} f_{i,F-2}(x_{i,F-2};\theta_{i,F-2})$$

$$\cdots \int_{0}^{{}^*u_{i,2}+\ldots+{}^*u_{i,F-1}-x_{i,F-1}-\ldots-x_{i,3}} f_{i,2}(x_{i,2};\theta_{i,2})$$

$$\times \int_{{}^*u_{i,1}+{}^*u_{i,2}+\ldots+{}^*u_{i,F-1}-x_{i,F-1}-\ldots-x_{i,3}-x_{i,2}}^{\infty} f_{i,1}(x_{i,1};\theta_{i,1})dx_{i,1}\ldots dx_{i,F-1}, \quad (4)$$

where $F\in\{2, \ldots, F_i\}$.

Proof. The proof is a simple application of the Lagrange multipliers technique. □

One can see that the above equations are solved recursively for each fare class starting with the first fare class. This process is continued until we have the first $F=F^{\circ}$ such that

$$\sum_{F=1}^{F^{\circ}-1} {}^*u_{i,F} \le u_i \quad (5)$$

and

$$\sum_{F=1}^{F^{\circ}} {}^*u_{i,F} > u_i, \quad {}^*u_{i,F} > 0, \quad F^{\circ} \in \{1, \ldots, F_i - 1\}. \quad (6)$$

Then

$${}^*u_{i,F^{\circ}} = \max\left(0, u_i - \sum_{F=1}^{F^{\circ}-1} {}^*u_{i,F}\right), \quad (7)$$

and ${}^*u_{i,F} = 0$ for all $F>F^{\circ}$. Otherwise, if

$$\sum_{F=1}^{F_i-1} u_{i,F} \le u_i, \quad (8)$$

then the optimal booking limit for the lowest fare class, F_i, is

$${}^*u_{i,F_i} = \max\left(0, u_i - \sum_{F=1}^{F_i-1} u_{i,F}\right). \quad (9)$$

It follows from the above that, in general, an optimal set of individual protection levels (in general, non-integer) must satisfy the following conditions:

$$c_{i,2} = c_{i,1} \Pr\{X_{i,1} > {}^*u_{i,1}\},$$

$$c_{i,3} = c_{i,1} \Pr\{(X_{i,1} > {}^*u_{i,1})$$

$$\cap (X_{i,1} + X_{i,2} > {}^*u_{i,1} + {}^*u_{i,2})\}],$$

$$c_{i,4} = c_{i,1} \Pr\{(X_{i,1} > {}^*u_{i,1}) \cap (X_{i,1} + X_{i,2} > {}^*u_{i,1} + {}^*u_{i,2})$$

$$\cap (X_{i,1} + X_{i,2} + X_{i,3} > {}^*u_{i,1} + {}^*u_{i,2} + {}^*u_{i,3})\},$$

$$\vdots$$

$$c_{i,F} = c_{i,1} \Pr\{(X_{i,1} > {}^*u_{i,1}) \cap (X_{i,1} + X_{i,2} > {}^*u_{i,1} + {}^*u_{i,2})$$

$$\cap \ldots \cap (X_{i,1} + X_{i,2} + \ldots + X_{i,F-1}$$

$$> {}^*u_{i,1} + {}^*u_{i,2} + \ldots + {}^*u_{i,F-1})\}, \quad (10)$$

where $F\in\{2, \ldots, F_i\}$. Thus, the protection level for the two highest fare classes is obtained by summing two individual protection levels, $(u_{i,1}+u_{i,2})$, and so on. There is no protection level for the lowest fare class, F_i; u_{i,F_i} is the booking limit, or number of seats available, for class F_i at time prior to flight departure; class F_i is open as long as the number of bookings in class F_i remains less than this limit. Thus, $(u_{i,F}+\ldots+u_{i,F_i})$ is the booking limit, or number of seats available, for class F, $F\in\{1, \ldots, F_i\}$. Class F is open as long as the number of bookings in class F and lower classes remain less than this limit. It is possible, depending on the airplane capacity, fares, and demand distributions that some fare classes will not be opened at all.

Now the general problem can be formulated as:

Maximize

$$\sum_i R_i(u_{i,1},\ldots,u_{i,F_i};u_i,F_i) \qquad (11)$$

Subject to

$$\sum_{F=1}^{F_i} u_{i,F} = u_i \text{ for all } i, \qquad (12)$$

$$\sum_{i\in S_{i;l}} u_i = U \text{ for all flight legs } l=1, \ldots, L, \qquad (13)$$

$$u_i \geq 0,\ \ u_{i,F} \geq 0 \text{ integer for all } i \text{ and } F. \qquad (14)$$

Note that (11)-(14) is a non-linear optimization problem with a concave and separable objective functions. In general, the solution to this problem can be found in the following manner. For each integer value of $u_i \leq U$ maximize (2) with respect to $(u_{i,1},\ldots,u_{i,F_i})$ for all i in order to obtain the expressions

$$R_i({}^*u_{i,1},\ldots,{}^*u_{i,F_i};u_i,F_i) = \int_0^{{}^*u_{i,F_i}} [c_{i,F_i}x_{i,F_i}$$

$$+R_i({}^*u_{i,1},\ldots,{}^*u_{i,F_i-1}+{}^*u_{i,F_i}-x_{i,F_i};u_i-x_{i,F_i},F_i-1)]$$

$$\times f_{i,F_i}(x_{i,F_i};\theta_{i,F_i})dx_{i,F_i}$$

$$+\int_{{}^*u_{i,F_i}}^{\infty} [c_{i,F_i}{}^*u_{i,F_i}+R_i({}^*u_{i,1},\ldots,{}^*u_{i,F_i-1};u_i-{}^*u_{i,F_i},F_i-1)]$$

$$\times f_{i,F_i}(x_{i,F_i};\theta_{i,F_i})dx_{i,F_i} \qquad (15)$$

for all integer $u_i \leq U$ and i, where ${}^*u_{OD}^{(1)},\ldots,{}^*u_{OD}^{(F_{OD})}$ satisfy (1). In other words, at the first stage, an optimal vector $({}^*u_{i,1},\ldots,{}^*u_{i,F_i})$ will be obtained for each i and u_i.

At the second stage, it must be chosen all u_i such that

Maximize

$$\sum_i R_i({}^*u_{i,1},\ldots,{}^*u_{i,F_i};u_i,F_i) \qquad (16)$$

Subject to

$$\sum_{i\in S_{i;l}} u_i = U \text{ for all flight legs } l=1, \ldots, L, \qquad (17)$$

$$u_i \geq 0 \text{ integer for all } i. \qquad (18)$$

This problem can be treated by the functional equation method of dynamic programming.

3. NUMERICAL EXAMPLE

We consider a flight over two legs, each with a capacity U=20 seats. The airline offers all three possible OD (OD:=i) pairs. Passengers can choose from two different fare classes for each OD. Booking requests are accepted starting from 60 days before departure. The fare settings are given in Table 1.

Table 1. Fare settings

i	OD	Fare class 1	Fare class 2
		Fares	
1	A-B	$c_{1,1}=250$	$c_{1,2}=125$
2	A-C	$c_{2,1}=420$	$c_{2,2}=175$
3	B-C	$c_{3,1}=330$	$c_{3,2}=150$

Long haul flights are relatively cheap compared to single-leg flights. The highest fare class is relatively expensive. Let us assume that the probabilistic demand for each ODF follows an exponential distribution. Thus,

$$X_{i,F} \sim f_{i,F}(x_{i,F};\sigma_{i,F}) = \frac{1}{\sigma_{i,F}}\exp\left(-\frac{x_{i,F}}{\sigma_{i,F}}\right),$$

$$x_{i,F}\in(0,\infty),\ \sigma_{i,F}>0,\ \ i=1(1)3,\ \ F=1,2. \qquad (19)$$

The demand parameters are given in Table 2.

Table 2. Demand parameters

Itinerary i	Fare class 1	Fare class 2
	Parameters	
1	$\sigma_{1,1} = 7.213$	$\sigma_{1,2} = 5.70$
2	$\sigma_{2,1} = 6.853$	$\sigma_{2,2} = 7.5$
3	$\sigma_{3,1} = 5.073$	$\sigma_{3,2} = 6.25$

We summarize the results of optimal allocation of airline seats for a flight over two legs, each with a capacity U=20 seats, in Table 3.

Table 3. Optimal allocation of airline seats for a flight over two legs, each with a capacity U=20 seats

Itinerary i	${}^*u_i$	Fare class 1	Fare class 2
1	${}^*u_1$=12	${}^*u_{1,1}$=5	${}^*u_{1,2}$=7
2	${}^*u_2$=8	${}^*u_{2,1}$=6	${}^*u_{2,2}$=2
3	${}^*u_3$=12	${}^*u_{3,1}$=4	${}^*u_{3,2}$=8

4. MULTI-LEG FLIGHT MODEL FOR DYNAMIC SEAT INVENTORY CONTROL

It will be noted that the information on the actual demand process can reduce the uncertainty associated with the estimates of demand. Hence, repetitive use of a static policy over the booking period, based on the most recent demand and capacity information, is the general way to proceed and leads to a dynamic policy.

In this section, we consider a multi-leg flight for a single departure date with T predefined reading dates at which the dynamic policy is to be updated, i.e., the booking period before departure is divided into T readings periods determined by the T reading dates. These reading dates are indexed in decreasing order, t=T, …, 1, 0, where t=1 denotes the first interval immediately preceding departure, and t=0 is at departure. The T-th reading period begins at the initial reading date at the beginning of the booking period, and the t-th reading period begins at t-th reading date furthest from the departure date. Thus, the indexing of the reading periods counts downwards as time moves closer to the departure date. Typically, the reading periods that are closer to departure cover much shorter periods of time than those further from departure. For example, the reading period immediately preceding departure may cover 1 day whereas the reading period 1-month from departure may cover 1 week.

Let us suppose that the total seat demand for fare class F and each separate OD:=i at the t-th reading date (time t) prior to flight departure is $X_{i,F}^t$ ($F\in$\{1, 2, … , F_i\}), where $X_{i,1}^t$ corresponds to the highest fare class; $f_{i,F}^t(x_{i,F}^t;\theta_{i,F}^t)$ is the probability density function of $X_{i,F}^t$, where $\theta_{i,F}^t$ is a parameter (in general, vector). We assume that these demands are stochastically independent. The vector of demands is X_i^t=($X_{i,1}^t$, …, X_{i,F_i}^t). Each booking of a fare class F seat generates average revenue of $c_{i,F}$, where $c_{i,1}>c_{i,2}>\ \dots\ >c_{i,F_i}$. Let $u_{i,F}^t$, $F\in$\{1, …, F_i\} be an individual protection level for fare class F at time t prior to flight departure. These many seats are protected for class F from all lower classes. The protection for the two highest fare classes is obtained by summing two individual protection levels, ($u_{i,1}^t+u_{i,2}^t$), and so on. There is no protection level for the lowest fare class, F_i; u_{i,F_i}^t is the booking limit, or number of seats available, for class F_i at time t prior to flight departure; class F_i is open as long as the number of bookings in class F_i remains less than this limit. Thus, ($u_{i,F}^t+\ \dots\ +u_{i,F_i}^t$) is the booking limit, or number of seats available, for class F, $F\in$\{1, …, F_i\}. Class F is open as long as the number of bookings in class F and lower classes remain less than this limit. The maximum number of seats that may be booked by fare classes in the next period at time t prior to flight departure is the number of unsold seats U_t. Demands for the lowest fare class arrive first, and seats are booked for this class until a fixed time limit is reached, bookings have reached some limit, or the demand is exhausted. Sales to this fare class are then closed, and sales to the class with the next lowest fare are begun, and so on for all fare classes. It is assumed that any time limits on bookings for fare classes are prespecified. That is, the setting of such time limits is not part of the problem considered here. It is possible, depending on the airplane capacity, fares, and demand distributions that some fare classes will not be opened at all.

Now the general problem at the t-th reading date (time t, $t\in$\{T, …, 1\}) prior to flight departure can be formulated as:

Maximize

$$\sum_i R_i^t(u_{i,1}^t,\dots,u_{i,F_i}^t;u_i^t,F_i) \tag{20}$$

Subject to

$$\sum_{F=1}^{F_i} u_{i,F}^t = u_i^t \text{ for all } i, \tag{21}$$

$$\sum_{i \in S_{i,l}} u_i^t = U_t \text{ for all flight legs } l=1, \ldots, L, \quad (22)$$

$$u_i^t \geq 0, \;\; u_{i,F}^t \geq 0 \text{ integer for all } i \text{ and } F, \quad (23)$$

where

$$R_i^t(u_{i,1}^t, \ldots, u_{i,F_i}^t; u_i^t, F_i) = \int_0^{u_{i,F_i}^t} [c_{i,F_i} x_{i,F_i}^t$$

$$+ R_i^t(u_{i,1}^t, \ldots, u_{i,F_i-1}^t + u_{i,F_i}^t - x_{i,F_i}^t; u_i^t - x_{i,F_i}^t, F_i - 1)]$$

$$\times f_{i,F_i}^t(x_{i,F_i}^t; \theta_{i,F_i}^t) dx_{i,F_i}^t$$

$$+ \int_{u_{i,F_i}^t}^{\infty} [c_{i,F_i} u_{i,F_i}^t + R_i^t(u_{i,1}^t, \ldots, u_{i,F_i-1}^t; u_i^t - u_{i,F_i}^t; F_i - 1)]$$

$$\times f_{i,F_i}^t(x_{i,F_i}^t; \theta_{i,F_i}^t) dx_{i,F_i}^t. \quad (24)$$

This problem can be treated in the same way as it is described in Section 2.

5. CONCLUSIONS

The mathematical models described in this paper attempt to provide a consistent and valid approach to optimization of airline booking levels. Simulations and comparisons with existing simpler models from airline companies seem to indicate that the decision rules obtained from the above mentioned models form an efficient operational tool in the planning of an airline's booking policy.

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REFERENCES

Belobaba, P.P. (1987). Survey paper: Airline yield management – an overview of seat inventory control. *Transportation Science*, **21**, 63–73.

Belobaba, P.P. (1989). Application of a probabilistic decision model to airline seat inventory control. *Operations Research*, **37**, 183–197.

Brumelle, S.L. and J.I. McGill (1993). Airline seat allocation with multiple nested fare classes. *Operations Research*, **41**, 127–137.

Curry, R.E. (1990). Optimal airline seat allocation with fare classes nested by origins and destination. *Transportation Science*, **24**, 193–204.

Gerchak, Y., Parlar, M. and T. Yee (1985). Optimal rationing policies and production quantities for products with several demand classes. *Canadian Journal of Administrative Science*, **1**, 161–176.

Hersh, M. and S.P. Ladany (1978). Optimal seat allocation for flights with one intermediate stop. *Computers and Operations Research*, **5**, 31–37.

Lee, T.L. and M. Hersh (1993). A model for dynamic airline seat inventory control with multiple seat bookings. *Transportation Science*, **27**, 252–265.

McGill, J.I. and G.J. van Ryzin (1999). Revenue management: Research overview and prospects. *Transportation Science*, **33**, 233–256.

Robinson, L.W. (1995). Optimal and approximate control policies for airline booking with sequential nonmonotonic fare classes. *Operations Research*, **43,** 252–263.

Weatherford, L.R., Bodily, S.E. and P.E. Pferifer (1993). Modeling the customer arrival process and comparing decision rules in perishable asset revenue management situations. *Transportation Science*, **27**, 239–251.

Wollmer, R.D. (1992). An airline seat management model for a single leg route when lower fare classes book first. *Operations Research*, **40**, 26–37.