ON COMPUTING THE WORST-CASE NORM OF CONVOLUTION SYSTEMS: A COMPARISON OF CONTINUOUS-TIME AND DISCRETE-TIME APPROACHES

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Abstract: This paper compares two approaches to compute the worst-case norm of finite-dimensional convolution systems. All admissible inputs are defined to have bounded magnitude and limited rate of change. Due to physical and mathematical reasons, the inputs are also specified to start from zero. The first approach is based on continuous-time optimal control formulation. Necessary conditions obtained via the Pontryagin's maximum principle provide a systematic means to characterize and construct the worst-case input. The second approach is based on discretization of the norm-computation problem which results in a large-scale finite-dimensional linear programming. We also investigate computational errors including truncation errors and discretization errors. Although the second approach seems to be simpler, the first approach is deemed to yield better accuracy.

Keywords: Worst-case norm, Magnitude bound, Rate limit, Optimal control, Linear programming, Truncation error, Discretization error.

1. INTRODUCTION

The *worst-case norm* of a convolution system is defined as the maximum peak magnitude of an output that can be generated when an input is subject to certain conditions by which an admissible input collection is characterized. In this paper, such collection consists of all inputs with bounded magnitudes and limited rates of change. Specifically, this worst-case norm is defined as follows. Let $h(t)$ be an impulse response of a convolution system. For simplicity, only strictly proper convolution systems are considered. For all t, let the system input $w(t)$ be continuous, and its derivative $\dot{w}(t)$ be piecewisely continuous. The input set $\mathcal{W}$ is characterized by a magnitude bound and a rate limit constraints as

$$\begin{aligned}\mathcal{W} \triangleq \{w(t) : w(t) = 0,\ \forall t \le 0; \\ |w(t)| \le M, |\dot{w}(t)| \le D,\ \forall t > 0\},\end{aligned} \tag{1}$$

where $0 < M < \infty$ and $0 < D < \infty$. Denote the system output by $z(t, w)$. The second argument emphasizes the dependency on $w(t)$, namely, $z(t, w) = h(t) * w(t)$ where $*$ stands for a convolution operation. The maximum magnitude of $z(t, w)$ at each time instant is defined as

$$\xi(t) \triangleq \sup_{w \in \mathcal{W}} |z(t, w)|. \tag{2}$$

The worst-case norm is defined as the maximum peak magnitude expressing in terms of $\xi(t)$ as follows.

$$\|h\|_{\rm wc} \triangleq \sup_{t \ge 0} \xi(t). \tag{3}$$

The input $w \in \mathcal{W}$ causing the maximum peak magnitude is referred to as *the worst-case input.* It is noted that the definition of the input set in (1) implies an initial condition $w(0) = 0$ for

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all admissible inputs. Although this condition is not common, it does make sense because the concept of imposing a rate limit, in addition to a magnitude bound, originates from the physical requisition of input continuity at $t > 0$. Thus, assuming $w(t)$ to be continuous at $t = 0$ is agreeable. In the later section, we will show that this assumption is not only reasonable, but also beneficial.

The concept of using system norms as performance measures indicating *sizes* of control systems was pioneered by Narendra and Goldwyn (1964), and Zames (1966), for example. A relaxed version of the worst-case norm, by omitting the rate limit D, is a product of M and the well-known $\mathcal{L}_1$-norm. Dahleh and Diaz-Bobillo (1995) have provided comprehensive exposition on $\mathcal{L}_1$ theory. The development of the worst-case norm can be traced back to Birch and Jackson (1959) who studied the problem of computing the maximum peak magnitude of an output of a second-order convolution system by constructing the worst-case input. Thereafter, Chang (1962), Horowitz (1963), and Bongiorno Jr. (1967) have given necessary and sufficient conditions for the worst-case input of finite-dimensional convolution systems, but the construction of such input was not clearly mentioned. In 1962, Chang has first associated the computation of the worst-case norm to optimal control problems. Boyd and Barratt (1991) subsequently pointed out that the associated optimal control problem pertains a *free* terminal time, and may be numerically solved by optimal control approaches.

Furthermore, Lane (1992) obtained the necessary and sufficient conditions for the worst-case input of convolution systems and the rules to construct it. Still the proposed method is rather tedious and needs to determine an involved auxiliary function. An additional relevant literature presented by Saridis and Rekasius (1966) incorporated the same input set with a slightly modified performance measure called an error criterion. They exploited an optimal control formulation which resulted in a nonlinear two-point boundary value problem, and applied the combined numerical-analytical method to construct the worst-case input. Nevertheless, the convergence of such method is not guaranteed.

The most recent result by Khaisongkram and Banjerdpongchai (2004) also employed an optimal control formulation. Necessary conditions acquired by the Pontryagin's Maximum Principle were analyzed in a straightforward manner to determine the practical characterization of the worst-case input for finite-dimensional convolution systems. Such characterization was exploited to derive the systematic method that constructs the worst-case input. However, the optimal control formulation therein assumed a *fixed* terminal time which actually yields a sub-optimal solution. In other words, the performance measure previously proposed is only a sharp upper bound for the worst-case norm.

In this paper, we discuss two practical means to compute the worst-case norm of convolution systems; one is based on an analytical deduction; the other is based on linear programming. The first approach is the refinement of the main result in Khaisongkram and Banjerdpongchai (2004). We improve the norm computation by imposing an initial condition $w(0) = 0$ in the input collection which allows the assumption on *fixed* terminal time to be mathematically sound. The previous optimality conditions and the worst-case-input construction have been slightly modified. As for the second approach, the norm computation is based on input and output discretization. The resulting problem can be simply cast as a large-scale finite-dimensional linear programming with high sparsity. The main contribution is to derive the bounds on approximation errors from both approaches so as to compare their computational accuracy.

The paper is organized as follows. In the next section, the worst-case norm definition is simplified. In Section 3, the continuous-time optimal control formulation is discussed. The characterization and the construction of the worst-case input are briefly mentioned. In Section 4, the computation problem of the worst-case norm is alternatively formulated as a linear-programming problem. Then, in Section 5, the computational errors are presented. A short numerical example is given in Section 6. Finally, some concluding remarks are summarized in Section 7.

2. PRELIMINARIES

The definition of the maximum output magnitude in (2) can be simplified by the fact that the term $h(t) * w(t)$ is linear in $w(t)$ and the bounding conditions on $w(t)$ are symmetrically defined, *i.e.*, for all $t > 0$

$$\begin{aligned} -M \leq w(t) \leq M, \\ -D \leq \dot{w}(t) \leq D. \end{aligned} \tag{4}$$

Hence, we can omit the absolute-value operator in (2), and obtain the equivalent definition as

$$\xi(t) = \sup_{w \in \mathcal{W}} \{h(t) * w(t)\}. \tag{5}$$

Furthermore, $\xi(t)$ is actually a nondecreasing function of time (Lane, 1992). To show this, let $t_1 < t_2$, and let $w_1(t)$ yield $\xi(t_1)$, *i.e.*, $\xi(t_1) = h(t_1) * w_1(t_1)$. Define $w_2(t)$ by shifting $w_1(t)$ with $\triangle t = t_2 - t_1$. That is

$$w_2(t) \triangleq w_1(t - \triangle t). \tag{6}$$

By simple integration, it follows that

$$h(t_2) * w_2(t_2) = h(t_1) * w_1(t_1) = \xi(t_1).$$

Evidently, we have $\xi(t_2) \geq h(t_2) * w_2(t_2) = \xi(t_1)$ by definition (5). As mentioned earlier, the initial condition $w(0) = 0$ plays an important role in the previous argument. If there is no such a condition, $w_1(0)$ may not equal zero, resulting in a discontinuity of $w_2(t)$ at $t = \triangle t$. Consequently, $w_2(t)$ is excluded from $\mathcal{W}$, and then $\xi(t_2)$ may not have any explicit relation with $\xi(t_1)$.

The benefit of the assumption on initial condition is that the definition (3) is equivalent to

$$\|h\|_{\text{wc}} \quad = \quad \lim_{t\to\infty} \xi(t). \tag{7}$$

This implies that we can approximate $\|h\|_{\text{wc}}$ by $\xi(T)$ with arbitrary degree of accuracy by taking T sufficiently large. The discussion on the choice of T will be presented in Section 5.

It is worth noting that all the given propositions are not valid if the worst-case norm in (3) is not finite. The necessary and sufficient condition for finiteness of the worst-case norm is that a convolution system is BIBO stable (Lane, 1992). The proof for necessity is omitted since it needs extra space. However, the sufficiency simply follows from the well-known fact that $M\|h(t)\|_1$ is an upper bound of the worst-case norm where $\|h(t)\|_1$ stands for an $\mathcal{L}_1$-norm of $h(t)$.

3. CONTINUOUS-TIME FORMULATION

Assume that the convolution system is strictly proper and has a finite dimension n with the minimal realization (A, B, C), and a state vector $x(t) \in \mathrm{R}^n$. To compute $\xi(T)$, we define an auxiliary state variable $x_{n+1}(t)$ and a control signal $u(t)$ as follows.

$$\begin{aligned} x_{n+1}(t) &\triangleq w(t), \\ u(t) &\triangleq \dot{w}(t). \end{aligned} \tag{8}$$

The fixed-terminal-time optimal control problem can be posed as

$$\begin{array}{lll} \sup\limits_{u} & Cx(T) & \\ \text{s.t.} & \dot{x}(t) = Ax(t) + Bx_{n+1}(t) & x(0) = 0 \\ & \dot{x}_{n+1}(t) = u(t) & x_{n+1}(0) = 0 \\ & -M \le x_{n+1}(t) \le M & 0 \le t \le T \\ & -D \le u(t) \le D & 0 \le t \le T. \end{array} \tag{9}$$

Notice that the initial time is 0, the terminal time is T, the objective cost is $z(T, w) = Cx(T)$, and the initial condition of $x_{n+1}(0)$ is 0. To define the Hamiltonian function, the inequality constraint on $x_{n+1}(t)$ is converted to

$$x_{n+1}^2(t) + \alpha^2(t) \quad = \quad M^2, \tag{10}$$

where $\alpha(t)$ is a real-valued auxiliary Lagrange variable. The Hamiltonian function is defined as

$$\begin{aligned} &\mathcal{H}(x, x_{n+1}, u, \alpha, p, p_{n+1}, p_{n+2}) \\ &\quad \triangleq p^T(Ax + Bx_{n+1}) + p_{n+1}u \\ &\quad\quad + p_{n+2}(M^2 - x_{n+1}^2 - \alpha^2) \end{aligned} \tag{11}$$

where $p(t)$, $p_{n+1}(t)$, and $p_{n+2}(t)$ are Lagrange multipliers corresponding to $\dot{x}(t)$, $\dot{x}_{n+1}(t)$ and the constraint (10), respectively. The necessary conditions are similar to those in Khaisongkram and Banjerdpongchai (2004), that is,

$$\dot{x}(t) = Ax(t) + Bx_{n+1}(t), \tag{12}$$

$$\dot{x}_{n+1}(t) = u(t), \tag{13}$$

$$\dot{p}(t) = -A^T p(t), \tag{14}$$

$$\dot{p}_{n+1}(t) = -B^T p(t) - 2p_{n+2}(t)x_{n+1}(t), \tag{15}$$

$$M^2 = x_{n+1}^2(t) + \alpha^2(t), \tag{16}$$

$$0 = \alpha(t)p_{n+2}(t). \tag{17}$$

In accordance with the Pontryagin's Maximum Principle (Pontryagin *et al.*, 1962), the optimal control signal $u(t)$ is chosen to maximize the Hamiltonian function in (11), *i.e.*, the control signal should have the same sign as $p_{n+1}(t)$, and its magnitude should be as large as possible. This yields the following control law

$$u(t) \quad = \quad D\text{sgn}\{p_{n+1}(t)\}. \tag{18}$$

The transversality conditions in Khaisongkram and Banjerdpongchai (2004) are modified to

$$p(T) = C^T, \tag{19}$$

$$x_{n+1}(0) = 0, \tag{20}$$

$$p_{n+1}(T) = 0. \tag{21}$$

The control law on singular arc [2] can be deduced step-by-step from the necessary conditions. It is found that the control law in this case takes the similar form as (18). In addition, the corner condition, which requires the continuity of $\mathcal{H}$ and all Lagrange multipliers, implies that $p_{n+1}(t)$ must be continuous everywhere.

The key idea is to classify sub-intervals of $[0, T]$ into two types. The first type, called a transition region, is a time interval in which $|w(t)| < M$. The other type, called a saturation region, is a time interval in which $|w(t)| = M$. Then, the worst-case input within each region is characterized separately by means of the optimality conditions (12)-(21).

To construct the worst-case input, the procedure given in Khaisongkram and Banjerdpongchai (2004) is slightly modified to give the worst-case input with $w(0) = 0$. The modification is made at how to start constructing the worst-case input. In brief, the initial value $p_{n+1}(0)$ is varied from zero to either positive or negative sides, depending on the input characterization. The initial region is attained when the cumulative summation reaches either M/D or $-M/D$. The reader is referred to Khaisongkram and Banjerdpongchai (2004) for detailed definitions.

The input construction needs discretization of the responses $h(T - t)$ and $s(T - t)$ since the computation is implemented on digital computers. As the discretized time series of $h(T-t)$ is used in numerical integration, the discretized time series of $s(T - t)$ is used as consecutive searching steps for the adjacent region. However, despite the need in sampling $s(T - t)$, each switching instant of the worst-case input can be precisely obtained via simple bisection algorithm, independent of the length of sampling period.

4. DISCRETE-TIME FORMULATION

It is observed that even for the continuous-time formulation, the algorithm to compute the worst-case norm still requires discretization of some time responses. This fact motivated us to consider the

[2] Singular control occurs when $p_{n+1}(t) = 0$, which causes $\mathcal{H}$ to be independent of $u(t)$

formulation in discrete-time domain. Let $h[k]$ be the discrete equivalent of $h(t)$ obtained by passing $h(t)$ into a sampler with a sampling period of τ, *i.e.*, $t = k\tau$. The maximum magnitude of output in (5) is modified as

$$\xi[k] \triangleq \tau \sup_{w \in \mathcal{W}} \{h[k] * w[k]\}, \tag{22}$$

Recall that the discrete convolution is given by $\sum_{i=0}^{k} h[k-i]w[i]$. The presence of τ in (22) is due to the fact that $\xi[k]$ must be an estimate of $\xi(t)$. Then, the worst-case norm in (7) is approximated as

$$\|h\|_{\mathrm{wc}} \approx \lim_{k \to \infty} \xi[k]. \tag{23}$$

The bounding conditions (4) are inherited into the discrete-time domain as follows. For all $k \geq 0$,

$$-M \leq w[k] \leq M,$$
$$-\tau D \leq w[k+1] - w[k] \leq \tau D.$$

where $w[k] = 0,\ \forall k \leq 0$. Notice that the initial condition $w[0] = 0$ is still assumed for the validity of definition (23).

According to Section 2, suppose that a sufficiently large terminal time T can be selected to yield an acceptable approximation error. Let $N+1$ be the number of sampling points. The sampling period τ is then computed as T/N. Let c and q be vectors in R^{N+1} representing the time series of $h[N-k]$ and $w[k]$, respectively. That is, for $k = 0, 1, \ldots, N$

$$c_{k+1} = h[N-k],$$
$$q_{k+1} = w[k].$$

The norm-computation problem can be formulated as linear programming:

$$\begin{aligned} \max_{q \in \mathrm{R}^{N+1}} \ & \tau c^T q \\ \text{s.t.} \ & q_1 = 0 \\ & -M \leq q_i \leq M, \\ & q_{i+1} - q_i \leq \tau D, \\ & q_{i+1} - q_i \geq -\tau D \\ & i = 1, \ldots, N. \end{aligned} \tag{24}$$

It is easy to check that the constraint matrix of this problem is highly sparse with a structure of 2×2 block diagonal consisting of one diagonal block and the other bidiagonal block. Thus, the number of nonzero elements in such matrix increases only linearly with N. Intuitively, the solution of $\xi[N]$ should move closer to the worst-case norm as N becomes larger. This increases the size of linear programming (24) and potentially causes numerical difficulty, so it is preferable to apply the large-scale linear-programming solvers that make use of the primal-dual method with interior-point algorithm (Mehrotra, 1992). The known sparsity and structure of (24) will help alleviate the difficulty of this large-scale problem.

5. ERROR ANALYSIS

In computing the worst-case norm on digital computers, some approximation errors arise inevitably. In this section we discuss some details pertaining the sources and bounds of these errors.

5.1 Truncation Error

In the process to compute $\xi(T)$ or $\xi[N]$, the choice of terminal time T may be selected by considering a *truncation error* which is the difference between $\|h\|_{\mathrm{wc}}$ and $\xi(T)$. From (5) and (7), it is easy to see that the truncation error can be bounded by $M \int_T^\infty |h(t)|dt$. Nevertheless, this quantity cannot be efficiently computed in practice.

To determine a bound of the truncation error with computable quantity, recall that (A, B, C) is defined as a minimal realization of a convolution system. Since the left-shifted impulse response $h(t+T) = Ce^{At}e^{AT}B$ can be viewed as an auxiliary impulse response $\tilde{h}(t)$ with a new realization $(A, e^{AT}B, C)$, a bound of the truncation error is derived as follows (Zhou and Doyle, 1998).

$$\int_T^\infty |h(t)|dt = \int_0^\infty |\tilde{h}(t)|dt \leq 2\sum_{i=1}^{n} \sigma_i \tag{25}$$

where n is the system order, and σ_i is the Hankel singular value of $\tilde{h}(t)$. The discrete version of this inequality, which will be applied to a discrete impulse response in the succeeding subsection, appears in Balakrishnan and Boyd (1992). Since the Hankel singular values of $h(t)$ are the square roots of the eigenvalues of the product of the observability and controllability gramians of $(A, e^{AT}B, C)$, this error bound can be readily calculated by solving two Lyapunov equations together with eigenvalue computation.

5.2 Discretization Error

Besides the truncation error, discretizing $h(t)$ causes an additional approximation error in the norm computation. Henceforth, we will refer to this error as a *discretization error*. First, consider the continuous-time approach. In Section 3, we mentioned that the step response $s(T-t)$ needs to be discretized, but the resulting worst-case input can be arbitrarily precise. Nevertheless, in calculating the worst-case output, it needs to carry out a numerical integration between the worst-case input and $h(T-t)$, which gives rise to the discretization error.

For consistency, we assume that $h(T-t)$ is sampled at the same rate as the discrete-time approach, *i.e.*, the sampling period τ. This yields the sample-and-hold impulse response $h_d(T-t)$ where $h_d(T-t) = h[N-k]$, for $k\tau \leq t < (k+1)\tau$, and $T = N\tau$. As mentioned previously, we assume that the error between the exact worst-case input and the computed input is insignificant. For notation simplicity, let denote the worst-case input corresponding to $\xi(T)$ by $w(t)$, and represent $h_d(t) * w(t)$ by $\xi_d(T)$. The discretization error for the continuous-time case is defined as $|\xi(T) - \xi_d(T)|$.

Theorem 1.

$$|\xi(T) - \xi_d(T)| \leq \tau M \|h[k] - h[k-1]\|_1 \tag{26}$$

where $\|h[k] - h[k-1]\|_1$ is the l_1-norm of $h[k] - h[k-1]$.

Proof. To verify this theorem, we begin with

$$\begin{aligned}\xi(T) - \xi_d(T) &= \int_0^T h(T-t)w(t)dt - \int_0^T h_d(T-t)w(t)dt \\ &= \int_0^T (h(T-t) - h_d(T-t))w(t)dt \\ &= \sum_{k=0}^{N} \int_{k\tau}^{(k+1)\tau} (h(T-t) - h_d(T-t))w(t)dt.\end{aligned}$$

Assume that the sampling frequency is sufficiently high, *i.e.*, τ is small enough, so that $h(T-t)$ is nearly monotonic in the interval $[(N-k)\tau, (N-k+1)\tau]$, and hence

$$h(T-t) \leq \max\{h[N-k], h[N-k+1]\}. \quad (27)$$

Recall that $T = N\tau$ and $t = k\tau$. Then, the bound of discretization error is obtained by

$$\begin{aligned}&|\xi(T) - \xi_d(T)| \\ &\leq M \sum_{k=0}^{N} \int_{k\tau}^{(k+1)\tau} |h(T-t) - h_d(T-t)|dt \\ &= M \sum_{k=0}^{N} \int_{k\tau}^{(k+1)\tau} |h(T-t) - h[N-k]|dt \\ &\leq M \sum_{k=0}^{N} \tau\, |h[N-k] - h[N-k-1]| \\ &\leq \tau M \|h[k] - h[k-1]\|_1.\end{aligned}$$

The proof is completed. ■

Due to the demand of good precision, N becomes larger, and τ becomes smaller. This causes $h[k] - h[k-1]$ to approach $\tau\dot{h}(k\tau)$. Hence, according to the bound (26), the discretization error diminishes at least quadratically with τ.

For the discrete-time approach, the discretization error is defined as $|\xi(T) - \xi[N]|$. Let $w(t)$ and $w[k]$ be the worst-case inputs corresponding to $\xi(T)$ and $\xi[N]$, respectively.

Theorem 2.

$$\begin{aligned}|\xi(T) - \xi[N]| \leq\; &\tau M \|h[k] - h[k-1]\|_1 \\ &+ \tau D \|h(t)\|_1\end{aligned} \quad (28)$$

where $\|h[k] - h[k-1]\|_1$ is the l_1-norm of $h[k] - h[k-1]$, and $\|h(t)\|_1$ is the $\mathcal{L}_1$-norm of $h(t)$.

Proof. According to (22), we have

$$\begin{aligned}&\xi(T) - \xi[N] \\ &= \int_0^T h(T-t)w(t)dt - \tau \sum_{k=0}^{N} h[N-k]w[k] \\ &= \int_0^T h(T-t)w(t)dt \\ &\quad - \sum_{k=0}^{N} w[k] \int_{k\tau}^{(k+1)\tau} h(N-k)dt.\end{aligned} \quad (29)$$

Consider the following fact:

$$\sum_{k=0}^{N} w[k] \int_{k\tau}^{(k+1)\tau} h(T-t)dt = \int_0^T h(T-t)w_d(t)dt$$

where $w_d(t) = w[k],\ k\tau \leq t < (k+1)\tau$. The signal $w_d(t)$ can be regarded as an output of a zero-order hold when an input is $w[k]$. Subtracting $\sum_{k=0}^{N} w[k] \int_{k\tau}^{(k+1)\tau} h(T-t)dt$ from the first term of (29), and adding it to the second term of (29) yields

$$\begin{aligned}\xi(T) - \xi[N] &= \int_0^T h(T-t)(w(t) - w_d(t))dt \\ &+ \sum_{k=0}^{N} w[k] \int_{k\tau}^{(k+1)\tau} h(T-t) - h(N-k)dt.\end{aligned} \quad (30)$$

From (30), we have $|\xi(T) - \xi[N]| \leq |e_1| + |e_2|$, where

$$\begin{aligned}e_1 &= \int_0^T h(T-t)(w(t) - w_d(t))dt, \\ e_2 &= \sum_{k=0}^{N} w[k] \int_{k\tau}^{(k+1)\tau} h(T-t) - h(N-k)dt.\end{aligned}$$

Now, it is straightforward to show that

$$\begin{aligned}|e_1| &\leq \int_0^T |h(T-t)|\, |w(t) - w_d(t)|\, dt \\ &\leq \tau D \int_0^T |h(T-t)|\, dt \\ &\leq \tau D \|h(t)\|_1.\end{aligned} \quad (31)$$

As for e_2, it can be shown that

$$\begin{aligned}|e_2| &\leq M \sum_{k=0}^{N} \int_{k\tau}^{(k+1)\tau} |h(T-t) - h(N-k)|\, dt \\ &\leq M \sum_{k=0}^{N} \tau\, |h[N-k] - h[N-k-1]| \\ &\leq \tau M \|h[k] - h[k-1]\|_1.\end{aligned} \quad (32)$$

The condition (27) is also assumed here. Finally, the error bound (28) simply follows from (31) and (32). ■

It is observed that, the first term of (28) decreases by the rate of τ^2. However, the second term of (28) decreases by the rate of τ. This suggests that the discretization error in the discrete-time case diminishes at least linearly with τ.

Remark here that the error bounds (26) and (28) are effective within limited ranges of D. For instance, if D becomes extremely large so that $\tau D > M$, the second term of (28) may exceed $M\|h(t)\|_1$, which is the upper bound of the worst-case norm itself. In addition, with a superficial look at the bounds (26) and (28), one may be misled to conclude that the difference $|\xi_d(T) - \xi[N]|$ is bounded by $\tau D\|h(t)\|_1$. Nevertheless, this is not the case. It can be shown after some nontrivial algebra that this difference can be bounded by $\tau^2 D\|h[k]\|_1$.

Furthermore, to make all the bounds on the discretization errors computable, we employ the same technique as that in (25). Let σ_{d_k} be the Hankel singular value of the discrete impulse response $h[k] - h[k-1]$, and let σ_{c_k} be the Hankel singular value of $h(t)$. The computable bounds on discretization errors for the continuous-time and the discrete-time approaches become

$$2\tau\left(M\sum_{k=1}^{n}\sigma_{d_k}\right) \quad \text{and}$$

$$2\tau\left(M\sum_{k=1}^{n}\sigma_{d_k}+D\sum_{k=1}^{n}\sigma_{c_k}\right),$$

respectively. Comparing these bounds, it is obvious that the discretization-error bound for the continuous-time approach is less than that of the discrete-time approach provided that $h(t)$ is essentially nonzero.

6. NUMERICAL EXAMPLE

We consider a nonminimum-phase linear system described by

$$H(s)=\frac{50-s^2}{(s+1)(s^2+2s+50)}.$$

Let the input bound $M=1$, and the rate limit $D=1$. The truncation error is a nonincreasing function of the terminal time T (Zhou and Doyle, 1998), so a simple bisection method can find an appropriate value of T for a given truncation error bound. In this example, we choose $T=6$ which makes the truncation error of the worst-case norm computation less than 0.0075. Select the number of sampling intervals $N=1000$ so the sampling period τ is $6/1000=0.006$.

The worst-case norm via continuous-time approach is computed to be 1.0328 and via discrete-time approach is 1.0306. The difference of these values is 0.0022 which is less than the derived bound $\tau^2 D\|h[k]\|_1=0.0180$. The discretization error bounds for continuous-time and discrete-time approaches are 0.1027 and 0.1206, respectively.

7. CONCLUSIONS

Two practical methods for computing the worst-case norm of finite-dimensional convolution system are presented. The initial condition on the system input is proved to have a great merit on computational simplification. The bounds on computational errors of both methods are calculated. While the discrete-time approach can be easily implemented provided that an adequate linear-programming solver is available, the continuous-time approach seems to yield better accuracy when the same sampling interval is used.

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