DYNAMIC FEEDBACK LINEARIZATION OF TWO INPUT NONLINEAR SYSTEMS

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Abstract: The paper deals with dynamic feedback linearization of two input continuous time affine systems. A constructive procedure is proposed which leads to necessary and sufficient conditions.

Keywords: Dynamic feedback laws, linearization, differential geometry.

1. INTRODUCTION

The static state feedback linearization problem has been widely studied both in continuous and discrete time (see Brockett [1978], Jakubczyk et al. [1980], Isidori et al. [1981], Hunt et al. [1983], Marino [1986], Monaco et al. [1986], Jakubczyk [1987], Lee et al. [1987], Isidori [1989], Nijmeijer et al. [1990], Califano et al. [1999]). Dynamic solutions were first considered in Isidori et al. [1986], Monaco et al. [1987] and Charlet et al. [1989]. In Charlet et al. [1991], sufficient conditions were given for the solvability of the problem via prolongations and diffeomorphism. A different approach based on algebraic techniques was proposed in Fliess et al. [1992], Fliess et al. [1995], where differentially flat systems were introduced. Necessary and sufficient conditions for the solvability of the problem were given in Aranda-Bricaire et al. [1995]. However these conditions are not constructive thus not allowing a direct computation of the dynamic compensator. Finally in Battilotti et al. [2003] an algorithm for the computation of a dynamic compensator consisting of prolongations was proposed for two input continuous affine systems. The general multi input case was considered in Battilotti et al. [2004].

In the present paper we extend the results proposed in Battilotti et al. [2003] for two input systems, by considering regular dynamic compensators. The proposed algorithm leads to necessary and sufficient conditions.

2. PRELIMINARIES

Consider the continuous time analytic system

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 \tag{1}$$

where $x \in \mathbb{R}^n$, and $f(x), g_1(x), g_2(x)$ are smooth maps defined on a open set of $\mathbb{R}^n$. The following notation will be used: given two smooth vector fields f and g_i, $ad_f g_i := [f, g_i] = \dfrac{\partial g_i}{\partial x} f - \dfrac{\partial f}{\partial x} g_i$, and $ad_f^k g_i = ad_f(ad_f^{k-1} g_i)$. We will denote by $g = (g_1, g_2)$, by $\mathcal{G}_i := \text{span}\{g, \cdots, ad_f^i g\}$, by $\bar{\mathcal{G}}_i$, the involutive closure of $\mathcal{G}_i$. Let us recall that the involutivity and constant dimensionality of the distributions $\mathcal{G}_i$ together with the controllability of the given dynamics are necessary and sufficient conditions for linear static feedback equivalence.

Assume now that (1) is linearizable over an open and dense set $\mathcal{U}_0 \ni (x_0, 0)$ with a regular dynamic controller of the form

$$\begin{aligned} \dot{\zeta} &= \eta(x, \zeta) + \delta(x, \zeta)v \\ u &= \alpha(x, \zeta) + \beta(x, \zeta)v \end{aligned} \tag{2}$$

We will first show in the present section some properties of the dynamics (1) and of the regular dynamic feedback (2), which allow to define an algorithm for the computation of a solution.

2.1 The dynamic feedback properties

The following result holds true

Lemma 1. If $\beta(x,\zeta)$ is invertible over an open and dense set $\mathcal{U}_0 \ni (x_0, 0)$ then (1) is static feedback equivalent to a linear system over $\mathcal{U}_0$.

Proof Since $\beta(x,\zeta)$ is invertible over an open and dense set $\mathcal{U}_0 \ni (x_0, 0)$, then we can consider the static state feedback $v = \beta(x,\zeta)^{-1}(w - \alpha(x,\zeta))$. The obtained closed–loop dynamics

$$\begin{aligned} \dot{x} &= f(x) + g(x)w \\ \dot{\zeta} &= \bar{\eta}(x,\zeta) + \bar{\delta}(x,\zeta)w \end{aligned} \tag{3}$$

will be static feedback equivalent to a linear system. Consequently the distributions $\mathcal{G}_i^e$ defined on the extended system must be involutive and of constant dimension locally around $(x_0, 0)$. Let us now note that since

$$F = \begin{pmatrix} f(x) \\ \bar{\eta}(x,\zeta) \end{pmatrix}, \quad G_i^e = \begin{pmatrix} g_i(x) \\ \bar{\delta}_i(x,\zeta) \end{pmatrix}, \ i = 1, 2,$$

consequently $ad_F^j G_i(.) = \begin{pmatrix} ad_f^j g_i(x) \\ * \end{pmatrix}$, $i = 1, 2$, $j \geq 0$. Consider now two elements $ad_F^{j_1} G_{i_1}(.)$, $ad_F^{j_2} G_{i_2}(.)$, $j_1, j_2 \leq j$, which belong to the distribution $\mathcal{G}_j^e$. The Lie bracket

$$\begin{aligned} [ad_F^{j_1} G_{i_1}, ad_F^{j_2} G_{i_2}] &= \begin{pmatrix} [ad_f^{j_1} g_{i_1}, ad_f^{j_2} g_{i_2}](x) \\ * \end{pmatrix} \\ &\in \text{span}\left\{ \begin{pmatrix} g_i(x) \\ * \end{pmatrix}, \cdots, \begin{pmatrix} ad_f^j g_i(x) \\ * \end{pmatrix}, \ i = 1, 2 \right\} \end{aligned}$$

which implies that $[ad_f^{j_1} g_{i_1}, ad_f^{j_2} g_{i_2}] \in \mathcal{G}_j, \forall j_1, j_2 \leq j$, i.e. the involutivity of the $\mathcal{G}_j$'s. Moreover the constant dimensionality of $\mathcal{G}_j^e$ implies the constant dimensionality of $\mathcal{G}_j$ over an open and dense set $\mathcal{U}_0' \subset \mathcal{U}_0$ so that (1) is static feedback linearizable on $\mathcal{U}_0'$ which ends the proof. ◁

The previous result can be used to enlighten some properties of the class of dynamic feedback which can be considered in order to achieve linearization. As we will show hereafter if we consider a dynamic controller of minimal dimension in appropriate coordinates it can be written as a combination of a feedback which depends only on the state variables of the given system plus an integrator.

Lemma 2. Assume that (1) is dynamic feedback linearizable with the regular dynamic feedback (2) of dimension ν. Let $\rho = \text{rank } \beta(x,\zeta) \leq 2$. Then there exists a diffeomorphism such that in the new coordinates, and after a possible reordering of the inputs, (2) can be written as

$$\begin{aligned} \dot{\chi}_i &= \bar{\eta}_i(x,\chi) + \bar{\delta}_i(x,\chi)v, \qquad i = a, b \\ u_a &= \bar{\alpha}_a(x,\chi) + \bar{\beta}_a(x,\chi)v \\ u_b &= \chi_a + \bar{M}(x,\chi)\left(\bar{\alpha}_a(x,\chi) + \bar{\beta}_a(x,\chi)v\right) \end{aligned} \tag{4}$$

with χ_a of dimension ρ and correspondingly χ_b of dimension $\nu - \rho$.

Proof By assumption in (2), $\rho = \text{rank } \beta(x,\zeta) \leq 2$. Moreover the controller (2) is regular so that

$$\text{rank}\left(\frac{\partial \alpha}{\partial \zeta} \;\Big|\; \beta\right) = 2.$$

Consequently, after a possible reordering of the inputs, there exists a partition of the input vector (u_a^T, u_b^T) with u_a of dimension ρ and u_b of dimension $2 - \rho$ such that the feedback $u = \alpha(x,\zeta) + \beta(x,\zeta)v$ can be rewritten as

$$\begin{aligned} u_a &= \alpha_a(x,\zeta) + \beta_a(x,\zeta)v \\ u_b &= \alpha_b(x,\zeta) + M(x,\zeta)u_a \end{aligned}$$

with β_a of full row rank ρ and rank $\frac{\partial \alpha_b}{\partial \zeta} = 2 - \rho$. Consequently we can consider the coordinates change $\chi_a = \alpha_b(x,\zeta)$, and χ_b such that $(x^T, \chi_a^T, \chi_b^T)^T$ is an independent coordinate set. In these coordinates (2) reads (4). ◁

Proposition 1. If the dynamic controller (4) achieves linearization then also the dynamic controller

$$\begin{aligned} \dot{\chi}_i &= \bar{\eta}_i(x,\chi) + \bar{\delta}_i(x,\chi)v \qquad i = a, b \\ u_a &= \bar{\alpha}_a(x,\chi) + \bar{\beta}_a(x,\chi)v \\ u_b &= \chi_a + \bar{M}(x,0)\left(\bar{\alpha}_a(x,\chi) + \bar{\beta}_a(x,\chi)v\right) \end{aligned} \tag{5}$$

achieves linearization.

Proof The proof, which is omitted for space reasons, is based on the consideration of the linear approximations of the closed loop system obtained by first considering thye controller (4) and then the controller (5), and by showing that the same output functions achieve defined relative degree $(r_1,\ r_2)$ with $\sum_{i=1}^{2} r_i = n + \nu$. ◁

The dynamic controller (5) can be rewritten as a regular static state feedback plus an integrator

$$\begin{aligned} u_a &= w_a, \quad u_b = \bar{M}(x,0)w_a + \chi_a \\ \dot{\chi}_a &= w_b \end{aligned} \tag{6}$$

and a reduced dynamic feedback

$$\begin{aligned} \dot{\chi}_b &= \bar{\eta}_b(x,\chi) + \bar{\delta}_b(x,\chi)v \\ w_a &= \bar{\alpha}_a(x,\chi) + \bar{\beta}_a(x,0)v \\ w_b &= \bar{\eta}_a(x,\chi) + \bar{\delta}_a(x,\chi)v \end{aligned} \tag{7}$$

Iterating the procedure on the residual dynamics (7) we can rewrite the dynamic controller (5) as the composition of a dynamic feedback given by a chain of regular static state feedback laws and integrators, which characterize a compensator of minimal order plus a residual dynamics.

2.2 The original dynamics properties

In order to enlighten the properties of the dynamics (1), assume that it isn't static feedback linearizable. Then there exists an index k such that the distributions $\mathcal{G}_{k+i}$ are involutive and of constant dimension on an open and dense subset $\mathcal{U}_0$, for any $i \geq 0$ whereas $\mathcal{G}_{k-1}$ is not involutive. Let ρ_{k-1} be the dimension of $\mathcal{G}_{k-1}$ and $\rho_{k-1}+s$ the dimension of its involutive closure $\bar{\mathcal{G}}_{k-1}$, with $s \leq 2$. Then there exist s indipendent vector fields $\tau_i,\ i=1,\ s$ which belong to $\mathcal{G}_k$ such that

$$\bar{\mathcal{G}}_{k-1} = \mathcal{G}_{k-1} + \text{span}\{\tau_i,\ i=1,\ s\} \qquad (8)$$

For simplicity we will assume throughout the paper that

H1: $\quad \bar{\mathcal{G}}_{k-1} = \mathcal{G}_{k-1} + \text{span}\{\tau_1\}$

In fact if $\bar{\mathcal{G}}_{k-1} = \mathcal{G}_{k-1} + \text{span}\{\tau_1, \tau_2\}$, then rank $\mathcal{G}_k = n$. This case can be held by adding a precompensator which ensures that **H1** is satisfied on the extended system.

Denoting by $k > 0$ the greatest index s.t. $\mathcal{G}_{k+l}$, $\forall l \geq 0$ is involutive, whereas $\mathcal{G}_{k-1}$ is not involutive we can then introduce the following definitions.

Definition 1. The Non-Characteristic set $NC_{\tau_1}^{k-1}$, associated with τ_1 is given by

$$NC_{\tau_1}^{k-1} = \{(ad_f^l g_{s_j}, ad_f^r g_{s_t}), s_j, s_t \in [1,2], l, r \leq k-1 : [ad_f^l g_{s_j}, ad_f^r g_{s_t}] \notin \mathcal{G}_{k-1}\}$$

The j–th channel is eligible if there exists at least one pair $(ad_f^l g_j, ad_f^r g_{s_t}) \in NC_{\tau_1}^{k-1}$.

Definition 2. The j–th channel is said to be unlocked if it is eligible and either $ad_f^k g_j \in \mathcal{G}_{k-1} + \text{span}\{ad_f^k g_i,\ i \neq j\}$ or $ad_f^k g_j \notin \bar{\mathcal{G}}_{k-1}$. The j–th channel is potentially locked if it is not unlocked and eligible. It is locked, if it is not unlocked and not eligible.

Definition 3. Two indices i_1 and i_2 are τ_1–redundant at step k if there exists $[r, s]$ with $1 \leq r \leq 2$, $0 \leq s \leq k$ and an index l, such that both $[ad_f^l g_{i_1}, ad_f^s g_r] \in \bar{\mathcal{G}}_k$, $[ad_f^l g_{i_2}, ad_f^s g_r] \in \bar{\mathcal{G}}_k$, and $[ad_f^l g_{i_1}, ad_f^s g_r] \notin \mathcal{G}_k$, $[ad_f^l g_{i_2}, ad_f^s g_r] \notin \mathcal{G}_k$. We define the Redundant set $R_{\tau_1}^l := \{i_1, i_2\}$.

The following result was proven in Battilotti et al. [2004].

Proposition 2. Let $k > 0$ be the greatest index such that $\mathcal{G}_{k+i}$ is involutive for any $i \geq 0$ whereas $\mathcal{G}_{k-1}$ is not involutive. Assume that its involutive closure $\bar{\mathcal{G}}_{k-1}$ is given by $\bar{\mathcal{G}}_{k-1} = \mathcal{G}_{k-1} + \text{span}\{\tau_1\} = \mathcal{G}_{k-1} + \text{span}\{ad_f^k g_1\}$. Then for any $i \geq 0$ the distribution $\mathcal{G}_{k-1+i} + \text{span}\{ad_f^{k+i} g_1\}$ is involutive and of constant dimension.

2.3 The static state feedback action

In the present section we will enlighten the role played by the static state feedback. We can recognize two relevant kind of feedback laws: the direction feedback and the reduction feedback.

As far as the direction feedback is concerned, let us consider the greatest index k such that the distribution $\mathcal{G}_{k+i}$ is involutive for any $i \geq 0$ whereas $\mathcal{G}_{k-1}$ is not involutive. Let γ_{i_1}, γ_{i_2} be appropriate coefficients, where without any loss of generality $\gamma_{i_1} \neq 0$, and $i_1 \neq i_2$, such that setting $\tau_1 = \left(\gamma_{i_1} ad_f^k g_{i_1} + \gamma_{i_2} ad_f^k g_{i_2}\right)$ and $\tau_2 = \left(\gamma_{i_1} ad_f^k g_{i_1} - \gamma_{i_2} ad_f^k g_{i_2}\right)$ the involutive closure $\bar{\mathcal{G}}_{k-1}$ satisfies $\bar{\mathcal{G}}_{k-1} := \mathcal{G}_{k-1} + \text{span}\{\tau_1\}$, and $\bar{\mathcal{G}}_{k-1} \neq \mathcal{G}_{k-1} + \text{span}\{\tau_2\}$. This feedback is used in order to modify the given dynamics so that $\bar{\mathcal{G}}_{k-1} := \mathcal{G}_{k-1} + \text{span}\{ad_f^k \tilde{g}_{i_1}\}$, whereas $\bar{\mathcal{G}}_{k-1} \neq \mathcal{G}_{k-1} + \text{span}\{ad_f^k \tilde{g}_{i_2}\}$. We can then consider the

Direction feedback:

$$u_{j_1} = v_{j_1}, \quad u_{j_2} = v_{j_2} + \frac{\gamma}{\gamma_{i_1}} v_{j_1} \qquad (9)$$

where

- If $ad_f^k g_{i_2} \notin \text{span}\{ad_f^k g_{i_1}\} + \mathcal{G}_{k-1}$, $j_1 = i_1$, $j_2 = i_2$ and $\gamma = \gamma_{i_2}$, which corresponds to set on the closed loop system $\dot{x} = f + \tilde{g}_{i_1} v_{i_1} + \tilde{g}_{i_2} v_{i_2}$,

  $$\tilde{g}_{i_1} = g_{i_1} + \frac{\gamma_{i_2}}{\gamma_{i_1}} g_{i_2},\ \tilde{g}_{i_2} = g_{i_2}$$

  so that $ad_f^k \tilde{g}_{i_1} = \left(ad_f^k g_{i_1} + \frac{\gamma_{i_2}}{\gamma_{i_1}} ad_f^k g_{i_2}\right)|_{mod\ \mathcal{G}_{k-1}}$, $ad_f^k \tilde{g}_{i_2} = ad_f^k g_{i_2}$ and $\bar{\mathcal{G}}_{k-1} := \mathcal{G}_{k-1} + \text{span}\{ad_f^k \tilde{g}_{i_1}\}$
- If $ad_f^k g_{i_2} \in \text{span}\{ad_f^k g_{i_1}\} + \mathcal{G}_{k-1}$, $j_1 = i_2$, $j_2 = i_1$ and $\gamma = -\gamma_{i_2}$ which corresponds to set on the closed loop system $\dot{x} = f + \tilde{g}_{i_1} v_{i_1} + \tilde{g}_{i_2} v_{i_2}$,

  $$\tilde{g}_{i_1} = g_{i_1},\ \tilde{g}_{i_2} = g_{i_2} - \frac{\gamma_{i_2}}{\gamma_{i_1}} g_{i_1}$$

  so that $ad_f^k \tilde{g}_{i_1} = ad_f^k g_{i_1}$, $ad_f^k \tilde{g}_{i_2} \in \mathcal{G}_{k-1}$ and $\bar{\mathcal{G}}_{k-1} := \mathcal{G}_{k-1} + \text{span}\{ad_f^k \tilde{g}_{i_1}\}$

As for the reduction feedback, let us assume that at step k there exist an index l and an element $ad_f^s g_i \in \mathcal{G}_k$ such that the redundant set $R_{\tau_1}^l = \{i_1, i_2\}$, $i_1 \neq i_2$, i.e. $[ad_f^l g_{i_1}, ad_f^s g_i] = \tau_1$,

$[ad_f^l g_{i_2}, ad_f^s g_i] = \alpha\tau_1|_{mod\ \mathcal{G}_{k-1}}$, $\alpha \neq 0$. We can consider the

Reduction feedback:

$$u_{i_1} = v_{i_1} - \alpha v_{i_2}, \quad u_{i_2} = v_{i_2} \tag{10}$$

which corresponds to set on the closed loop system $\dot{x} = f + \tilde{g}_{i_1} v_{i_1} + \tilde{g}_{i_2} v_{i_2}$,

$$\tilde{g}_{i_1} = g_{i_1},\ \tilde{g}_{i_2} = g_{i_2} - \alpha g_{i_1}$$

so that $[ad_f^l g_{i_2}, ad_f^s g_i] \in \mathcal{G}_{k-1}$.

2.4 The dynamic extension action

The dynamic extension may be used in two different situations, which correspond respectively to the case in which one channel is unlocked and to the case in which there are no unlocked channels. These situations are discussed hereafter.

Let us consider the greatest index k such that the distribution $\mathcal{G}_{k+i}$ is involutive for any $i \geq 0$ whereas $\mathcal{G}_{k-1}$ is not involutive.

One unlocked channel

Assume that the direction feedback (9) has been used so that, without any loss of generality we have that $\bar{\mathcal{G}}_{k-1} := \mathcal{G}_{k-1} + \mathrm{span}\{ad_f^k \tilde{g}_{i_1}\}$, and $\bar{\mathcal{G}}_{k-1} \not\equiv \mathcal{G}_{k-1} + \mathrm{span}\{ad_f^k \tilde{g}_{i_2}\}$. Set

$$u_{i_2} = \chi_1, \qquad \dot{\chi}_1 = \bar{v}_{i_2} \tag{11}$$

The extended system

$$\begin{aligned} \dot{x} &= f(x) + \tilde{g}_{i_2}\chi_1 + \tilde{g}_{i_1} v_{i_1} \\ \dot{\chi}_1 &= \bar{v}_{i_2} \end{aligned}$$

is characterized by the set of distributions

$$\mathcal{G}_j^e = \mathrm{span}\{\frac{\partial}{\partial \chi_1}, \tilde{g}\frac{\partial}{\partial x}, \cdots, ad_f^{j-1}\tilde{g}\frac{\partial}{\partial x}, ad_f^j \tilde{g}_{i_1}\frac{\partial}{\partial x}\}$$

As a consequence due to Proposition 2, we have that $\mathcal{G}_{k+j}^e$ is involutive for any $j \geq 0$.

No unlocked channels

Assume that **H1** holds true but there are no unlocked channels. We can then seek (if there exist) for the smallest index $j \leq n$ such that $\mathcal{G}_{k-1} + \mathrm{span}\{ad_f^k g_{i_1}, \cdots, ad_f^{k+j} g_{i_1}\} \equiv \mathcal{G}_{k+j}$, and the i_2 channel is potentially locked. Set

$$u_{i_2} = \chi_1, \qquad \begin{cases} \dot{\chi}_1 &= \chi_2 \\ &\vdots \\ \dot{\chi}_j &= \bar{v}_{i_2} \end{cases} \tag{12}$$

which after the direction feedback is used, leads to the one unlocked channel situation.

3. MAIN RESULT

We will now propose an algorithm for the computation of a dynamic feedback which renders the extended system equivalent to a linear system. We will then show that the convergency of the algorithm is a necessary and sufficient condition for the solvability of the problem.

The dynamic feedback linearization algorithm

Step 0 Let k be the first index such that $\mathcal{G}_{k+i}$ is involutive for any $i \geq 0$ and $\mathcal{G}_{k-1}$ is not involutive, compute its involutive closure $\bar{\mathcal{G}}_{k-1}$ and assume **H1** satisfied. If there are no unlocked channels, then apply the dynamic extension (12), and go back to Step 0, else go to next step.

Step 1 Compute the Noncharacteristic set $NC_{\tau_1}^{k-1}$.
Consider the set of unlocked channels
$\mathcal{I} = \{i \in [1,2] : \text{the } i\text{-th channel is unlocked}\}$
and define recursively

$$\mathcal{A}_{\tau_1}^{k-1} := \{(ad_f^{k-1} g_i, ad_f^r g_{s_t}) : [ad_f^{k-1} g_i, ad_f^r g_{s_t}] \notin \mathcal{G}_{k-1}, i \in \mathcal{I}\}$$

$$\vdots$$

$$\mathcal{A}_{\tau_1}^{l} := \mathcal{A}_{\tau_1}^{l+1} \cup \{(ad_f^l g_i, ad_f^r g_{s_t}) : [ad_f^l g_i, ad_f^r g_{s_t}] \notin \mathcal{G}_{k-1}\}, i \in \mathcal{I}\},\ l < k-1$$

Let $\hat{l}$ be the first index such that $\mathcal{A}_{\tau_1}^{\hat{l}+1} \not\equiv \mathcal{A}_{\tau_1}^0$, while $\mathcal{A}_{\tau_1}^{\hat{l}} \equiv \mathcal{A}_{\tau_1}^0$ and consider the following associated sets:

- the index set $\mathcal{I}_{\tau_1}^{\hat{l}} := \{i \in \mathcal{I} :\ (ad_f^{\hat{l}} g_i, ad_f^r g_{s_t}) \in \mathcal{A}_{\tau_1}^{\hat{l}}\}$
- the redundant set $R_{\tau_1}^{\hat{l}}$.

Choose $i_2 \in \mathcal{I}_{\tau_1}^{\hat{l}}$ and apply the reduction feedback (10) and the direction feedback (9). Set

$$u_{i_2} = \zeta_{i1}, \quad \dot{\zeta}_{i1} = v_{i_2}.$$

Go back to Step 0

We will now state our main result.

Theorem 1. Assume that **H1** is satisfied. The dynamics (1) is dynamic feedback equivalent to a linear system on an open and dense set $\mathcal{U}_0 \ni (x_0, 0)$, if and only if the dynamic feedback linearization algorithm converges.

Proof Sufficiency. If the algorithm converges, then the distributions $\mathcal{G}_{k-1}$, $0 \leq k \leq n^e - 1$, defined on the extended system of dimension n^e are involutive and of constant dimension with dim $\mathcal{G}_{n^e-1} = n^e$. Thus the extended system is static feedback equivalent to a linear system.

Necessity. For space reasons we will only give the general lines of the proof. Assume that there exists a dynamic compensator of the form (5) which solves the problem. Let the extended static feedback equivalent system be

$$\dot{x}^0 = \tilde{f}^0(x^0) + \tilde{g}_1^0(x^0)\tilde{u}_1^0 + \tilde{g}_2^0(x^0)\tilde{u}_2^0 \quad (13)$$

According to (6-7) denoting by $x^0 = ((x^1)^T, \chi^1)^T$, the closed loop system can then be rewritten as the combination of an integrator plus a regular static state feedback, i.e. (13) is given by

$$\dot{x}^0 = f^0(x^0) + g_1^0(x^0)u_1^0 + g_2^0(x^0)u_2^0 \quad (14)$$

$$u^0 = \alpha^0(x^0) + \beta^0(x^0)\tilde{u}^0 \quad (15)$$

where $\beta^0(x^0)$ in (15) is locally invertible, and (14) is given by

$$\dot{x}^1 = f^1(x^1) + g_1^1(x^1)u_1^1 + g_2^1(x^1)u_2^1 \quad (16)$$

$$\dot{\chi}_1^1 = u_2^0 \quad (17)$$

$$u_1^1 = u_1^0,\ u_2^1 = \chi_1^1 \quad (18)$$

Consequently $f^0(x^0) = \left(f^1(x^1) + g_2^1(x^1)\chi_1^1\right)\dfrac{\partial}{\partial x^1}$, $g_1^0(x^0) = g_1^1(x^1)\dfrac{\partial}{\partial x^1}$, $g_2^0(x^0) = \dfrac{\partial}{\partial \chi_1^1}$.

By assumption the $\tilde{\mathcal{G}}_k^0$'s computed on the extended system (13) are involutive and of constant dimension for any $k \geq 0$. Consequently also the $\mathcal{G}_k^0$'s computed on the extended system (14) are involutive and of constant dimension for any $k \geq 0$ since the two systems differ from a regular static state feedback. This implies on the reduced system (16) that the distribution

$$\mathcal{G}_{k-1}^1 + \mathrm{span}\{ad_{f^1}^k g_1^1\} \quad (19)$$

is involutive and of constant dimension. Moreover since the dynamics (16) is not static feedback linearizable, then there must exists an index k_1, such that the distribution $\mathcal{G}_{k_1-1}^1$ is not involutive, whereas $\mathcal{G}_{k_1+i}^1$ is involutive for any $i \geq 0$.

Finally since the problem is solvable in one step, i.e. adding an integrator on the second input, then necessarily we have that for some $s_2 \in [0, k_1 - 1]$ and $j_2 \in [1, 2]$

$$[ad_{f^1}^{k_1-1}g_2^1, ad_{f^1}^{s_2}g_{j_2}^1] = \alpha_1 ad_{f^1}^{k_1}g_1^1|_{mod\ \mathcal{G}_{k_1-1}^1} \notin \mathcal{G}_{k_1-1}^1 \quad (20)$$

whereas for any $s_1 < k_1 - 1$, $j_2 \in [1, 2]$, with $(s_2, j_2) \neq (k_1 - 1, 2)$, $[ad_{f^1}^{s_1}g_2^1, ad_{f^1}^{s_2}g_{j_2}^1] \in \mathcal{G}_{k_1-1}^1$. ◁

Accordingly we will show that any choice operated by the algorithm, solves the problem in one step. By assumption at step $k_1 - 1$, $\mathcal{A}_{\tau_1}^{k_1-1} \equiv NC_{\tau_1}^{k-1}$ and the second input is unlocked.

If in (20) $s_2 < k_1-1$, then the algorithm chooses to extend the second input channel, thus obtaining the same dynamic compensator considered in (17-18). The only ambiguous case occurs when both channels are unlocked and there is only one element which causes the loss of involutivity given by $[ad_{f^1}^{k_1-1}g_2^1, ad_{f^1}^{k_1-1}g_1^1]$.

We will show that after the application of the direction feedback (9), the algorithm may choose to extend the first input channel, and this will solve the problem as well.

In fact, assume that this is not the case, then by setting an integrator on the first input channel, we would have that for some index $\bar{k}_1 < k_1$, the distribution $\tilde{\mathcal{G}}_{\bar{k}_1-1}^1 + \mathrm{span}\{ad_{f^1}^{k_1}\tilde{g}_2^1\}$ is not involutive whereas $\tilde{\mathcal{G}}_{\bar{k}_1+i}^1 + \mathrm{span}\{ad_{f^1}^{\bar{k}_1+i}\tilde{g}_2^1\}$ is involutive for any $i \geq 0$.

Thus there would exists an element

$$[ad_{f^1}^{s_1}\tilde{g}_{j_1}^1, ad_{f^1}^{s_2}\tilde{g}_{j_2}^1] \notin \tilde{\mathcal{G}}_{\bar{k}_1-1}^1 + \mathrm{span}\{ad_{f^1}^{\bar{k}_1}\tilde{g}_2^1\}$$

where without any loss of generality we can assume $s_1 \geq s_2 \geq 0$ and $s_1 \in [0, \bar{k}_1 - 1]$, or $s_1 > s_2 \geq 0$ and $(s_1, j_1) = (\bar{k}_1, 2)$.

For $s_1 \in [0, \bar{k}_1 - 1]$, by assumption

$$[ad_{f^1}^{s_1}\tilde{g}_{j_1}^1, ad_{f^1}^{s_2}\tilde{g}_{j_2}^1] = \gamma_1 ad_{f^1}^{\bar{k}_1}\tilde{g}_1^1|_{mod\ \tilde{\mathcal{G}}_{\bar{k}_1-1}^1}$$

we would then have that

$$ad_{f^1}^{k_1-\bar{k}_1}[ad_{f^1}^{s_1}\tilde{g}_{j_1}^1, ad_{f^1}^{s_2}\tilde{g}_{j_2}^1] = \gamma_1 ad_{f^1}^{k_1}\tilde{g}_1^1|_{mod\ \tilde{\mathcal{G}}_{\bar{k}_1-1}^1} = \sum_{i=0}^{k_1-\bar{k}_1}\binom{k_1-\bar{k}_1}{i}[ad_{f^1}^{s_1+k_1-\bar{k}_1-i}\tilde{g}_{j_1}^1, ad_{f^1}^{s_2+i}\tilde{g}_{j_2}^1]$$

which is contrast with the assumption that $[ad_{f^1}^{k_1-1}\tilde{g}_2^1, ad_{f^1}^{k_1-1}\tilde{g}_1^1]$ is the only element which does not belong to $\tilde{\mathcal{G}}_{k_1-1}^1$. Assume now that

$$[ad_{f^1}^{\bar{k}_1}\tilde{g}_2^1, ad_{f^1}^{s_2}\tilde{g}_{j_2}^1] = \gamma_1 ad_{f^1}^{\bar{k}_1+1}\tilde{g}_1^1|_{mod\ \tilde{\mathcal{G}}_{\bar{k}_1}^1}$$

The distribution $\tilde{\mathcal{G}}_{\bar{k}_1}^1$ cannot be involutive, otherwise by construction also the distribution $\tilde{\mathcal{G}}_{k_1}^1$ would be involutive, according to Proposition 2. It follows that $\gamma_1 \neq 0$.

Consequently we would have that the involutive closure of $\mathcal{G}_{k_1-1}^1 + \mathrm{span}\{ad_{f^1}^{\bar{k}_1}g_2^1\}$ is given by $\mathcal{G}_{k_1-1}^1 + \mathrm{span}\{ad_{f^1}^{\bar{k}_1}g_2^1\} + \mathrm{span}\{ad_{f^1}^{\bar{k}_1}g_1^1,\ ad_{f^1}^{\bar{k}_1+1}g_1^1\}$ which would be in contrast with the assumption that $[ad_{f^1}^{k_1-1}g_2^1, ad_{f^1}^{k_1-1}g_1^1]$ is the only element which does not belong to $\mathcal{G}_{k_1-1}^1$ thus proving the result.

The result can be proven iteratively by considering that the dynamics (16) is given by a reduced dynamics plus a feedback and an integrator. We can then rewrite it as

$$\dot{x}^1 = \tilde{f}^1(x^1) + \tilde{g}_1^1(x^1)v_1^1 + \tilde{g}_2^1(x^1)v_2^1 \quad (21)$$

$$v^1 = \alpha^1(x^1) + \beta^1(x^1)u^1 \quad (22)$$

where (21) is given by

$$\dot{x}^2 = f^2(x^2) + g_{i_1}^2(x^2)u_{i_1}^2 + g_{i_2}^2(x^2)u_{i_2}^2 \quad (23)$$

$$\dot{\chi}_1^2 = v_2^1 \quad (24)$$

$$u_{i_1}^2 = v_1^1, \quad u_{i_2}^2 = \chi_1^2 \quad (25)$$

As a consequence we can define a link between the distributions $\mathcal{G}_i^2$ and $\mathcal{G}_i^1$ and the indices k_1 and k_2 which characterize the loss of involutivity of such distributions. This link can be deduced by noting that

$$\tilde{f}^1 = \left(f^2 + g_{i_2}^2 \chi_1^2\right) \frac{\partial}{\partial x^2}, \ \tilde{g}_1^1 = g_{i_1}^2 \frac{\partial}{\partial x^2}, \ \tilde{g}_2^1 = \frac{\partial}{\partial \chi_1^2}$$

and by computing the distributions $\mathcal{G}_i^1$ in terms of the $ad_{f^2}^s g_i$'s. It can be finally deduced that if the given dynamics is dynamic feedback linearizable, then necessarily the algorithm converges.

Example. Consider the system

$$\dot{x}_1 = x_2 + x_3 x_5, \ \dot{x}_2 = x_3 + x_1 x_5, \ \dot{x}_3 = u_1 + x_2 x_5$$
$$\dot{x}_4 = x_5, \ \dot{x}_5 = x_6, \ \dot{x}_6 = u_1 + u_2.$$

Step 0. The distributions $\mathcal{G}_i$, are given by

$$\mathcal{G}_0 = \text{span}\left\{\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_6}\right\},$$
$$\mathcal{G}_1 = \mathcal{G}_0 + \text{span}\left\{-x_5 \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_5}\right\},$$

and $\mathcal{G}_2 \equiv \mathbb{R}^6$ with

$$ad_f^2 g_1 = (1 + x_3 - x_1 x_5 - x_6)\frac{\partial}{\partial x_1} + x_5^2 \frac{\partial}{\partial x_2} + (2x_2 + x_3 x_5 + x_5 - x_5 x_6)\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}$$

and $ad_f g_2 = (x_3 - x_1 x_5)\frac{\partial}{\partial x_1} + x_3 x_5 \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}$.

The distribution $\mathcal{G}_1$ is not involutive, since

$$[ad_f g_1, ad_f g_2] = \frac{\partial}{\partial x_1} = \gamma(ad_f^2 g_1 - ad_f^2 g_2)|_{mod\ \mathcal{G}_1}.$$

Its involutive closure $\bar{\mathcal{G}}_1 = \mathcal{G}_1 + \text{span}\{\frac{\partial}{\partial x_1}\}$.

Step 1. We have $NC_{\tau_1}^1 = (ad_f g_1, ad_f g_2)$. Moreover both channels are unlocked and $A_{\tau_1}^1 \equiv NC_{\tau_1}^1$. We choose $i_2 = 2$. Since $\tau_1 = \gamma(ad_f^2 g_1 - ad_f^2 g_2)$, we apply the direction feedback

$$u_1 = v_1, \quad u_2 = v_2 - v_1$$

which corresponds to set $\tilde{g}_1 = g_1 - g_2$ and $\tilde{g}_2 = g_2$, and correspondingly we set $v_1 = w_1$, $v_2 = \zeta_1$, $\dot{\zeta}_1 = w_2$.

The extended dynamics is static feedback equivalent to a linear system as it can be easily verified. ◁

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