DELAYED TIME-VARYING $\mathcal{H}^{\infty}$ CONTROL DESIGN

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Abstract: The linear H^{∞} control problem is stated for time-varying plant with arbitrary delays and discontinuous coefficients. The image representation of the plant is explicitly written and full solution to the suboptimal control problem is presented. Necessary and sufficient conditions for existence of a solution are derived from the abstract principle of maximum. No Riccati equations are used. The existence of smooth kernels of optimal integral operators is proved and their properties are studied. A numerical method is presented for computation of the optimal kernels for the special case. The case is a model of a car autopilot on a concave road.

Keywords: Maximum principle, H-infinity control, Delay analysis, Numerical methods

1. INTRODUCTION

A solution to the linear time-varying $\mathcal{H}^{\infty}$ control problem is proposed which does not contain Riccati equations or factorization. The new approach is a generalization of the time-invariant Φ-approach proposed in [Barabanov and Ghulchak, 2000] for delayed systems with an essential reference to the abstract principle of maximum [Matveev and Yakubovich, 1994].

The solution proposed has the following features:

- it does not refer to operator equations, in particular, to the Riccati equations. Instead, a number of differential equations are to be solved;
- the conditions of the existence of a solution are necessary and sufficient;
- the image representation (see the behavioral approach in [Willems, 1991]) of all processes that satisfy the target inequality is presented;
- parameterization of all controllers that solve the $\mathcal{H}^{\infty}$ control problem is given;
- the plant equation is of the general type. It may contain discontinuous coefficients and not only arbitrary pure delays with time-varying coefficients but also arbitrary bounded operators from L^2 to L^2;
- the solution does not contain auxiliary variables. The kernel functions in the controller equation and in the image representation are determined directly from the differential equations.

In this paper the solution is presented for the full information case, but it can be directly general-

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ized to the input-output problem by the standard duality and separation technique.

The solution can be used for study of the nonlinear $\mathcal{H}^\infty$ control problem. It gives necessary and sufficient conditions for local optimality of the process that is not assumed to be in a small vicinity of the origin.

2. PROBLEM STATEMENT

Definition 1. A mapping $G : L^2(0,\infty) \to L^2(0,\infty)$ is called causal if for any functions $x_1, x_2 \in L^2(0,\infty)$ and any $t > 0$ it follows from the condition $x_1(s) = x_2(s)$ almost everywhere on $s \in (0,t)$ that $(G(x_1))(s) = (G(x_2))(s)$ almost everywhere on $s \in (0,t)$.

The control system is described by the equation in generalized functions

$$\dot{x} = \mathcal{A}x + \mathcal{B}u + \mathcal{C}v + f,$$

where $u \in L^2_{n_u}(0,\infty)$, $v \in L^2_{n_v}(0,\infty)$. The function f is in the space $(W^2_{1,n_x}(0,\infty))^*$ that is the adjoint space to the Hilbert space of Sobolev $W^2_{1,n_x}(0,\infty)$. The operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ from $L^2(0,\infty)$ to $L^2(0,\infty)$ are causal, bounded and exponentially decreasing. No other restrictions are assumed for these operators.

The initial conditions of the system are defined by the function $f \in (W^2_1(s,\infty))^*$. The set of all initial conditions at the instant s is defined as $\mathcal{Z} = (W^2_1(s,\infty))^*$. The set of all solutions of the system equation after the instant s with fixed $f \in W^2_1(s,\infty)$ and $v = 0$ is denoted by

$$\mathcal{M}_s(f) = \{(x,u) \in L^2(s,\infty) \mid \dot{x} = \mathcal{A}x + \mathcal{B}u + f\}.$$

Definition 2. An admissible feedback strategy for u is defined as a mapping $S : L^2_{n_x}(0,\infty) \times \mathcal{Z} \to L^2_{n_u}(0,\infty)$ such that $S(\cdot, Z)$ is causal, and for any $s \geq 0$ and any function $v \in L^2_{n_v}(s,\infty)$ the system equation with the initial data Z at the instant s has a unique solution $(x(\cdot), u(\cdot)) \in L^2(s,\infty)$ for which $u = S(x, Z)$.

The set of all admissible feedback strategies is denoted by $\mathcal{S}$. Note that these strategies can be nonlinear and time-varying. Fix a quadratic cost function

$$\mathcal{F}_s = \int_s^\infty (x(t)^* Q_0 x(t) + |u(t)|^2 - \lambda |v(t)|^2)\, dt.$$

Assumption 1. (Positive definiteness of the cost function.) There exists a number $\varepsilon > 0$ such that for any $(x,u) \in \mathcal{M}_0(0)$ it holds

$$\mathcal{F}_0(x(\cdot), u(\cdot), 0) \geq \varepsilon(\|x\|^2_{L^2(0,\infty)} + \|u\|^2_{L^2(0,\infty)}).$$

Assumption 2. (Minimal stability condition [Fomin, *et al.*, 1981].) There exists a linear bounded operator S_{stable} which is an admissible feedback and such that the closed loop system is uniformly exponentially stable.

3. PRINCIPLE OF MAXIMUM

The maximin control problem contains two consecutive LQ problems. The inner LQ problem seems to be standard but the solution is not easy for the general operators in L^2. The resulting operator is proved to be continuous in the appropriate norm of the dual space to the Sobolev space W^2_1.

Theorem 1. Let Assumptions 1 and 2 hold. Then for any $s \geq 0$ and any function $f \in (W^2_1(s,\infty))^*$ there exists a unique process (x^0, u^0) on which the cost function $\mathcal{F}_s(x, u, 0)$ is minimal on the set $\mathcal{M}_s(f)$.

Moreover, the mapping $(x^0(\cdot), u^0(\cdot)) = T_s(f)$ is a linear operator. It is bounded uniformly for $s \geq 0$ in the norms $(W^2_1(s,\infty))^* \to L^2(s,\infty)$.

The outer maximization problem is solved in the next Theorem. The authors do not know a solution that does not refer to the abstract principle of maximum presented in [Matveev and Yakubovich, 1994]. According to the next Theorem any stationary point of the Lagrange function is proved to provide a solution to the maximin control problem.

For any $s \geq 0$ define

$$\Lambda_s = \inf\{\lambda \mid \sup_{v \in L^2(s,\infty)} \inf_{(x,u) \in \mathcal{M}_s(\mathcal{C}v)} \mathcal{F}_s < \infty\}.$$

Theorem 2. Let $s \geq 0$, $f \in (W^2_1(s,\infty))^*$ and $\lambda > \Lambda_s$. Suppose Assumptions 1 and 2 hold. Then there exist a unique quadruple (x^0, u^0, v^0, ψ^0) in the sets $(x,u,v) \in L^2(s,\infty)$, $\psi \in W^2_1(s,\infty)$ which is a stationary point of the Lagrange function

$$\mathcal{L}_s(x,u,v,\psi) = \int_s^\infty [F(x(t), u(t), v(t)) \\ -2\psi^*(t)(\dot{x}(t) - \mathcal{A}x(t) - \mathcal{B}u(t) - \mathcal{C}v(t) - f(t))]dt$$

and satisfies the condition $(x,u) \in \mathcal{M}_s(\mathcal{C}v + f)$. Moreover, the function v^0 and the process (x^0, u^0) solve the maximin control problem

$$\max_{v \in L^2(s,\infty)} \min_{(x,u) \in \mathcal{M}_s(\mathcal{C}v + f)} \mathcal{F}_s,$$

the function ψ^0 is the dual variable in the problem of inner minimization by (x,u) with fixed $v = v^0$, and the mapping $P_s : f \to (x^0, u^0, v^0, \psi^0)$ is linear and continuous in the norms $f \in (W^2_1(0,\infty))^*$, $(x^0, u^0, v^0) \in L^2(s,\infty)$, $\psi^0 \in W^2_1(s,\infty)$. The norms of the operators P_s are bounded uniformly for $s \geq 0$. □

4. BASIC OPERATORS

Following the behavioral approach of Willems [1991] introduce the state variable $X = \mathrm{col}(x, u, v)$, the matrix $Q = \mathrm{diag}\{Q_0, I_{n_u}, -\lambda I_{n_v}\}$ of the quadratic form and the plant operator $\mathcal{R} = (\partial/\partial t - \mathcal{A}, -\mathcal{B}, -\mathcal{C})$.

It follows from Theorem 2 that the optimal process in the maximin control problem satisfies the standard set of the optimality conditions and the plant equation

$$QX(t) = \mathcal{R}^*\psi(t),$$
$$\mathcal{R}X(t) = f(t).$$

This set of equations can be derived also from the standard Pontryagin principle of maximum. Its generalization, the abstract principle of maximum [Matveev and Yakubovich, 1994] was used for the proof that the adjoint variable ψ belongs to the space $L^2(s, \infty)$. This condition is important for solution of the corresponding minimax game that is well known as the H^∞ control problem for the time-varying delayed systems.

The optimality equations are the same as in the linear functional equation that is basic in the Φ-approach to the H^∞ control problem. Therefore, a generalization of the direct approach to the infinite dimensional game problem can be obtained from the equations that leads to the numerical algorithms similar to [Afanassieva *et al*, 2001].

The operators $f \to (x, u, v, \psi)$ are proved to be integral. Denote there kernels by $K_s(t, r)$. They can be explicitly transformed to the kernel functions from the following Theorem.

Theorem 3. There exists an integral operators $\mathcal{K}$, $\mathcal{K}f(t) = \int_0^\infty K(t, r)f(r)\,dr$, from $L^2_{n_u+n_v}(0, \infty)$ to $L^2_{n_x+n_u+n_v}(0, \infty)$, and $\mathcal{K}^\psi : f \to \psi$ from $L^2_{n_x}(0, \infty)$ to $L^2_{n_x}(0, \infty)$ with the kernel $K^\psi(t, r) = \mathrm{col}(K^\psi_u(t, r), K^\psi_v(t, r))$ such that

1. The operators $\mathcal{K}$ and $\mathcal{K}^\psi$ are bounded.

2. The operator $\mathcal{K}$ is causal, that is, $K(t, r) = 0$ for $t < r$.

3. For almost all $(t, r) \in (0, \infty) \times (0, \infty)$ the optimality conditions hold

$$QK(t, r) = \mathcal{R}^* K^\psi(t, r)$$

where the operator $\mathcal{R}^*$ is applied to the function of t under fixed r.

4. For almost all $r \geq 0$ the plant equation holds for the generalized functions of t

$$\mathcal{R}K(t, r) = (\mathcal{B}, \mathcal{C})\delta(t - r). \quad □$$

The optimality conditions and plant equation can be written in the operator form: $Q\mathcal{K} = (\mathcal{R}^*\mathcal{K}^\psi)_+$ and $\mathcal{R}\mathcal{K} = (\mathcal{B}, \mathcal{C})$ where the operation $(\cdot)_+$ means taking the causal part of the operator.

Let $n_{uv} = n_u + n_v$ and E_{uv} be the identity operator in the space $L^2_{n_{uv}}(0, \infty)$. Denote the dimension of the manifest variable X by $n_X = n_x + n_u + n_v$. The basic operator from $L^2_{n_{uv}}(0, \infty)$ to $L^2_{n_X}(0, \infty)$ is defined as

$$\Phi = \begin{pmatrix} 0 \\ E_{uv} \end{pmatrix} + \mathcal{K}.$$

The matrix of the quadratic form X^*QX under $x = 0$ is $J = \mathrm{diag}\{I_{n_u}, -\lambda I_{n_v}\}$. The next assertion is crucial for the image representation of the plant.

Theorem 4. It holds $\mathcal{R}\Phi = 0$ and $\Phi^*Q\Phi = J$.

5. MAIN RESULTS

Theorem 5. (Image representation.) The set of functions $\hat{z} = \mathrm{col}(U, V) \in L^2_{n_{uv}}(0, \infty)$ is in the one to one correspondence with the set $\mathcal{L}$ of all solutions of the plant equation $X = \mathrm{col}(x, u, v) \in L^2(0, \infty)$ under zero disturbances. This correspondence $X \leftrightarrow \hat{z}$ is causal both sides and gives the image representation

$$X = \Phi\hat{z}, \qquad \hat{z} = J^{-1}\Phi^*QX,$$

or in the explicit form

$$x(t) = \int_0^t [K^x_u(t, r)U(r) + K^x_v(t, r)V(r)]\,dr,$$
$$u(t) = U(t) + \int_0^t [K^u_u(t, r)U(r) + K^u_v(t, r)V(r)]\,dr,$$
$$v(t) = V(t) + \int_0^t [K^v_u(t, r)U(r) + K^v_v(t, r)V(r)]\,dr$$

and back in the causal form

$$U(t) = u(t) + (K^\psi_u(t, t))^* x(t) + \int_0^t [((\mathcal{A}^* K^\psi_u)(s, t))^* x(s) + ((\mathcal{B}^* K^\psi_u)(s, t))^* u(s) + ((\mathcal{C}^* K^\psi_u)(s, t))^* v(s)]\,ds,$$
$$V(t) = v(t) - \lambda^{-1}(K^\psi_v(t, t))^* x(t) - \lambda^{-1} \int_0^t [((\mathcal{A}^* K^\psi_v)(s, t))^* x(s) + ((\mathcal{B}^* K^\psi_v)(s, t))^* u(s) + ((\mathcal{C}^* K^\psi_v)(s, t))^* v(s)]\,ds.$$

Furthermore, the cost function can be represented as

$$\mathcal{F}(X) = \int_0^\infty F(x(t), u(t), v(t))\,dt = \|\hat{z}\|^2_J = \|U\|^2_{L^2(0,\infty)} - \lambda\|V\|^2_{L^2(0,\infty)}.$$

□

Split Φ in accordance to the dimensions of x, u, v:

$$\Phi = \begin{pmatrix}\Phi_u & \Phi_v\end{pmatrix} = \begin{pmatrix}\Phi_{xu} & \Phi_{xv}\\ \Phi_{uu} & \Phi_{uv}\\ \Phi_{vu} & \Phi_{vv}\end{pmatrix}.$$

Lemma 1. Assume $\lambda > \Lambda$. Then the inverse operator Φ_{vv}^{-1} exists, is bounded and causal. Moreover, $\|\Phi_{vv}^{-1}\Phi_{vu}\| \le \lambda^{-1/2}$.

Lemma 2. Let $\lambda > \Lambda$ and $D : L^2_{n_v}(0,\infty) \to L^2_{n_u}(0,\infty)$ be an arbitrary causal Lipschitz mapping with the Lipschitz constant less than $\sqrt{\lambda}$. Then the mapping $\Phi_{vv} + \Phi_{vu}D$ is invertible and the inverse mapping is Lipschitz and causal.

Definition 3. Let $\gamma>0$. A mapping $T_{z/v}\colon (v, Z) \to z=(x,u)$ from $L^2_{n_v}(0,\infty)\times\mathcal{Z}$ to $L^2_{n_z}(0,\infty)$ is called γ-contracting if there exist $\varepsilon > 0$ such that for any $v_1, v_2 \in L^2_{n_v}(0,\infty)$ and any $Z \in \mathcal{Z}$ if $(x_1,u_1) = T_{z/v}(v_1,Z)$ and $(x_2,u_2) = T_{z/v}(v_2,Z)$ then

$$\int_0^\infty F(x_1(t) - x_2(t), u_1(t) - u_2(t), 0)\, dt \le (\gamma^2 + \varepsilon)\|v_1 - v_2\|^2.$$

An admissible feedback strategy (respectively, a generalized disturbance feedback) is called γ-contracting if the closed loop mapping $(v, Z) \to z = (x, u)$ is γ-contracting.

Theorem 6. (Small gain.). Let $\gamma > \sqrt{\Lambda}$. Then the set of all γ-contracting generalized disturbance feedbacks coincides with the set of mappings

$$u = (\Phi_{uv} + \Phi_{uu}D_Z) \circ (\Phi_{vv} + \Phi_{vu}D_Z)^{-1}(v) + u_Z,$$

where D_Z for any $Z \in \mathcal{Z}$ is an arbitrary causal mapping from $L^2_{n_v}(0,\infty)$ to $L^2_{n_u}(0,\infty)$ with the Lipschitz constant $L_D < \gamma$; the function $u_Z = S^0(Z, 0)$ depends on the initial conditions Z and S^0 is an arbitrary generalized disturbance feedback. □

Remark. Under the conditions of Theorem 6 the solution of the plant equation under a fixed γ-contracting feedback is written as

$$x = (\Phi_{xv} + \Phi_{xu}D) \circ (\Phi_{vv} + \Phi_{vu}D)^{-1}(v) + x_Z,$$
$$u = (\Phi_{uv} + \Phi_{uu}D) \circ (\Phi_{vv} + \Phi_{vu}D)^{-1}(v) + u_Z.$$

If the operator $\Phi_{xv} + \Phi_{xu}D$ is continuously invertible then the generalized disturbance feedback can be replaced by the admissible feedback strategy

$$u = (\Phi_{uv} + \Phi_{uu}D) \circ (\Phi_{xv} + \Phi_{xu}D)^{-1}(x - x_Z) + u_Z.$$

All the γ-contracting feedback strategies can be obtained in this way.

6. SOLUTION FOR SYSTEMS WITH PURE DELAYS

The plant is described by the linear scalar time-varying delayed equation

$$a(p,\tau)y(t) = b(p,\lambda)u(t) + c(p,\mu)v(t),$$

where y is the output, u is the control, v is the disturbance. The operators a, b and c are time-varying and depend on the differentiation operator $p = d/dt$ and on the set of nonnegative delays $\tau = (\tau_k)_{k=1}^n$, $\lambda = (\lambda_k)_{k=0}^m$ and $\mu = (\mu_k)_{k=0}^m$ with $m < n$ in the following way:

$$a(p,\tau)y(t) = \sum_{k=1}^{n} a_k(t) y^{(n-k)}(t - \tau_k) + y^{(n)}(t),$$
$$b(p,\lambda)u(t) = \sum_{k=0}^{m} b_k(t) u^{(n-k)}(t - \lambda_k),$$
$$c(p,\mu)v(t) = \sum_{k=0}^{m} c_k(t) v^{(n-k)}(t - \mu_k).$$

The coefficients a_k, b_k and c_k are assumed to be bounded. They may have jumps.

Under zero initial conditions at $t = 0$ and with fixed $\gamma_0 > 0$ it is required to describe all solutions of the system that belong to $L^2(0,\infty)$ and satisfy the target inequality

$$\int_0^\infty (|y(t)|^2 + |u(t)|^2)\, dt < \gamma_0^2 \int_0^\infty |v(t)|^2\, dt.$$

It is also required to find all controllers that provide the target inequality for any function $v \in L^2(0,\infty)$.

The full solution is presented in section 5 for the more general case. For the case considered in this section it can be reduced to the following algorithm. First, define the adjoint operators $a^*(p,\tau)$, $b^*(p,\lambda)$, $c^*(p,\mu)$ as

$$a^*(p,\tau)\psi(t) = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dt^k}(a_{n-k}(t)\psi(t + \tau_{n-k})),$$
$$b^*(p,\lambda)\psi(t) = \sum_{k=0}^{m} (-1)^k \frac{d^k}{dt^k}(b_{m-k}(t)\psi(t + \lambda_{m-k})),$$
$$c^*(p,\mu)\psi(t) = \sum_{k=0}^{m} (-1)^k \frac{d^k}{dt^k}(c_{m-k}(t)\psi(t + \mu_{m-k})).$$

The main computational problem contains a solution to the next two series of the differential equations. For any fixed $r \ge 0$ it is required to find a function $\psi \in L^2(-\infty,\infty)$ and a solution to the disturbed plant equation on $t \in \mathsf{R}$:

$$a(p,\tau)\widetilde{y}(t) = b(p,\lambda)\widetilde{u}(t) + c(p,\mu)\widetilde{v}(t) + r(p)\delta(t - r)$$

where $r(p) = b(p,\lambda)$ or $r(p) = c(p,\mu)$, that satisfy on $t \in [r,\infty)$ the relations

$$\begin{pmatrix} \widetilde{y}(t) \\ \widetilde{u}(t) \\ \widetilde{v}(t) \end{pmatrix} = \begin{pmatrix} a^*(p,\tau) \\ -b^*(p,\lambda) \\ \gamma^{-2}c^*(p,\mu) \end{pmatrix} \psi(t)$$

and the functions $(\widetilde{y}(t), \widetilde{u}(t), \widetilde{v}(t), \psi(t))$ are zero for $t < r$. Denote the solutions by $K_u(t,r) = (K_u^y(t,r), K_u^u(t,r), K_u^v(t,r))$ if $r(p) = b(p,\lambda)$, and by $K_v(t,r) = (K_v^y(t,r), K_v^u(t,r), K_v^v(t,r))$ if $r(p) = c(p,\mu)$, respectively.

The equations are linear in ψ. It can be proved that a solution in $L^2(r,\infty)$ is unique if exists.

Theorem 7. 1. A solution to the problem stated exists if and only if the solutions $K_u(\cdot,r)$ and $K_v(\cdot,r)$ exist in $L^2(r,\infty)$ for any $\gamma \geq \gamma_0$ and for any $r \geq 0$.
2. Assume a solution exists and fix $\gamma = \gamma_0$. Then the set of all solutions (y,u,v) of the target inequality is in the one-to-one correspondence to the set of the pairs $(U,V) \in L^2(0,\infty)$ such that $\|U\| < \gamma\|V\|$. The relation is given by

$$\begin{aligned} y(t) &= \int_0^t [K_u^y(t,r)U(r) + K_v^y(t,r)V(r)]\,dr, \\ u(t) &= U(t) + \int_0^t [K_u^u(t,r)U(r) + K_v^u(t,r)V(r)]\,dr, \\ v(t) &= V(t) + \int_0^t [K_u^v(t,r)U(r) + K_v^v(t,r)V(r)]\,dr. \end{aligned}$$

3. Denote the integral operators with the kernels $(K_u^u, K_u^v, K_v^u, K_v^v)$ by $(\mathcal{K}_u^u, \mathcal{K}_u^v, \mathcal{K}_v^u, \mathcal{K}_v^v)$, respectively. Then all linear controllers that provide the target inequality for any nonzero function $v \in L^2(0,\infty)$ can be parameterized by the equation

$$\mathcal{K}_v^y u(t) = \mathcal{K}_v^u y(t) + (\mathcal{K}_u^y \mathcal{D} u(t) - \mathcal{D} y(t) - \mathcal{K}_u^u \mathcal{D} y(t))$$

where $\mathcal{D}$ is an arbitrary linear causal operator from $L^2(0,\infty)$ to $L^2(0,\infty)$ such that $\|\mathcal{D}\| < \gamma$. □

The assertion of Theorem 7 follows from Theorem 5. The main linear equation in the function ψ can be solved numerically, for instance, by the sweep method.

7. SPECIAL CASE

Consider a simple model of the car autopilot. A car is riding on the road at a constant speed. The road surface is not flat, the edges of the road are lower than the middle line for the rain water to leave the road. This slope of the road surface generates a force that tries to turn a car if it is not located at the middle line. The autopilot measures the deviation and tries to compensate it.

The road profile is approximated by the function $-ay^2$ where y is the distance between the car mass centre and the road middle line. The coefficient of concavity a is not constant along the road. The system is described by the equation

$$\ddot{y}(t) = a(t)y(t) + b_0u(t) + w(t)$$

where u is the control and w is the disturbance. For simplicity assume that $a(t) = a_0$ for $t \in [0,T]$ and $a(t) = a_1$ for $t > T$. The disturbance w reflects small nonflatness or small obstacles on the road. Since any small stone or nonflatness acts on the front wheels of a car and then acts on the rear wheels the disturbance contains a pure delay

$$w(t) = c_0v(t) + c_1v(t-\tau)$$

where $\tau = d/V$ is the time for riding the distance d between the front and the rear wheels. Hence, we have a time-varying control system with a pure delay in the disturbance. Under zero initial conditions it is required to design a controller that provides for all nonzero $v \in L^2(0,\infty)$ the inequality

$$\|y\|^2_{L^2(0,\infty)} + \|u\|^2_{L^2(0,\infty)} < \gamma^2\|v\|^2_{L^2(0,\infty)}.$$

In accordance to Theorems 3 and 5 the problem is equivalent to solution in $L^2(0,\infty)$ of the system

$$\begin{aligned} \ddot{\psi}(t) &= a(t)\psi(t) + y(t), \\ u(t) &= -b_0\psi(t), \\ v(t) &= \gamma^{-2}(c_0\psi(t) + c_1\psi(t+\tau)) \end{aligned}$$

together with the plant equation on the ray $[r,\infty)$. For calculation of the function $K^u(t,r)$ the initial conditions are $y(r) = 0$, $\dot{y}(r) = b_0$ and all variables are zero for $t < r$. For calculation of the function $K^v(t,r)$ the last initial condition is replaced by $\dot{y}(r) = c_0$ and the function $\dot{y}$ has a jump of the amplitude c_1 at the time $r + \tau$.

First consider the delay free special case $\tau = 0$ and assume a solution exists. The system is time-invariant on the interval $[T,\infty)$ and the characteristic polynomial admits the factorization

$$z^4 - 2a_1z^2 + a_1^2 + b_0^2 - \gamma^{-2}(c_0+c_1)^2 = f(z)f(-z)$$

where $f(z) = z^2 + g_1z + h_1$ is a Hurwitz polynomial. Since $\psi \in L^2(T,\infty)$ it must satisfy the equation

$$f(p)\psi = \ddot{\psi} + g_1\dot{\psi} + h_1\psi = 0, \quad t > T,$$

that has 2 degrees of freedom. The second and the third derivatives of ψ have jumps at the point $t = T$,

$$\begin{aligned} \ddot{\psi}(T+0) - \ddot{\psi}(T-0) &= (\Delta a)\psi(T), \\ \psi^{(3)}(T+0) - \psi^{(3)}(T-0) &= (\Delta a)\dot{\psi}(T). \end{aligned}$$

It remains to solve the equation

$$\psi^{(4)} - 2a_0\ddot{\psi} + (a_0^2 + b_0^2 - \gamma^{-2}(c_0+c_1)^2)\psi = 0$$

on (t,T) under 4 mixed boundary conditions. Denote the solution by $(y^0(t,r), \psi^0(t,r))$.

Consider the general case $\tau > 0$ and assume $r + \tau < T$. Then the function ψ on the ray $[T, \infty)$ satisfies the equation

$$\psi^{(4)}(t) - 2a_1\ddot{\psi}(t) + [a_1^2 + b_0^2 - \gamma^{-2}(c_0^2 + c_1^2)]\psi(t) \\ -\gamma^{-2}c_0c_1[\psi(t-T) + \psi(t+T)] = 0$$

with the initial data $\psi(s), s \in [T-\tau, T]$, $\psi(T) = \psi_T$, $\dot{\psi}(T) = \psi'_T$ and the condition $\psi \in L^2(T, \infty)$. If a solution exists then the operator in the left hand side of the equation admits spectral factorization and the differential equation takes the form $K_1(p)K_1(-p)\psi(t) = 0$ where the operator K_1 is stable and $K_1(p)\psi(t)$ is equal to

$$\ddot{\psi}(t) + g_1\dot{\psi}(t) + h_1\psi(t) + \int_0^{t-T} \xi_1(s)\psi(t-s)\,ds.$$

Since the operator $K_1(-p)$ is antistable the equation on the ray $[T, \infty)$ can be written as $K_1(p)\psi(t) = 0$. It can be proved similarly to results of [Afanassieva, *et al.*, 2001] that the support set of the function ξ_1 is located in $[0, \tau]$. The function ξ_1 and positive numbers g_1, h_1 are uniquely determined from the equation

$$\ddot{\xi}_1(t) - g_1\dot{\xi}_1(t) + h_1\xi_1(t) + \int_0^{\tau-t} \xi_1(s)\xi_1(s+t)\,ds = 0$$

with the boundary conditions

$$\xi_1(\tau) = 0, \quad \dot{\xi}_1(\tau) = \gamma^{-2}c_0c_1, \quad g_1^2 = 2(h_1 + a_0), \\ 2\dot{\xi}_1(0) = a_1^2 + b_0^2 - \gamma^{-2}(c_0^2 + c_1^2).$$

The function ψ satisfies also the equation

$$K_0(p)K_0(-p)\psi(t) = 0, \qquad t \in [r+\tau, T],$$

where the causal operator K_0 is defined similar to K_1 by equations where the parameter a_1 is replaced by a_0. The operator K_0 is described by the parameters g_0, h_0 and the function ξ_0 defined on the segment $[0, \tau]$. The function ψ has the same jump at $t = T$ as derived for the case $\tau = 0$.

Since $v(t-\tau) = 0$ for $t < r + \tau$ the function ψ satisfies

$$K_0(p)K_0(-p)\psi(t) = -\gamma^{-2}c_1^2\psi(t), \qquad t \in [r, r+\tau].$$

The function ψ and two its derivatives are continuous at $t = \tau$. It holds also $\psi(t) = 0$ for $t < r$ and $y(r) = \ddot{\psi}(r) - a_0\psi(r) = 0$.

When calculating $K^u(t, r)$ the function $\psi^{(3)}$ is continuous at $t = r + \tau$ and $\dot{y}(r) = \psi^{(3)}(r) - a_0\dot{\psi}(r) = b_0$. When calculating $K^v(t, r)$ the third derivative $\psi^{(3)}$ has a jump at $t = r + \tau$ of the amplitude c_1 and $\dot{y}(r) = \psi^{(3)}(r) - a_0\dot{\psi}(r) = c_0$.

The recursive procedure for solution of the basic system of equations with $\tau > 0$ consists of the operations mentioned above. First, the functions $y_0 = y^0$ and $\psi_0 = \psi^0$ are determined from the delay-free model.

Assume ψ_j is computed. The recursion contains solution of the equations

$$K_1(p)\psi_{j+1}(t) = 0, \qquad t \in (T, \infty),$$
$$K_0(p)K_0(-p)\psi_{j+1}(t) = 0, \qquad t \in (r+\tau, T),$$
$$K_0(p)K_0(-p)\psi_{j+1}(t) = -\gamma^{-2}c_1^2\psi_j(t), \ t \in (r, r+\tau).$$

Each equation contains boundary conditions that are algebraic equations containing the values of ψ_{j+1} at the boundary point. Each equation also requires the values of ψ on the interval of the length τ that is located outside the interval of the equation. For those initial conditions the previous values of ψ_j are substituted. The second equation is solved in two steps. First, an equation with the causal operator $K_0(p)$ is solved from the left edge $t = r + \tau$ to the right edge $t = T$ and with the initial condition $\psi_j(s)$, $s \in (r, r+\tau)$. Then the next equation with the anticausal operator $K_0(-p)$ and initial condition $\psi_j(s)$, $s \in (T, T+\tau)$, is solved in the backward direction.

Theorem 8. The described algorithm exponentially converges for all sufficiently small $\tau > 0$. □

It was noticed from experiments that the algorithm successfully converges for the values of τ that are not so small. The method can be easily generalized to the plants with a finite number of jumps in coefficients. The regulator equation is obtained in the integral form. It is causal and can be approximated by a regulator with a finite number of pure delays.

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