THE FIRST LYAPUNOV COEFFICIENT FOR A CLASS OF SYSTEMS

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Abstract: In this paper the first Lyapunov coefficient of the emerging periodic solution in the Hopf bifurcation is established. A change of coordinates is introduced to eliminate some quadratic and cubic terms in the dynamic equations and the center manifold theory is used to reduce the dynamics to dimension two.

Keywords: Hopf bifurcation, center manifold, First Lyapunov coefficient.

1. INTRODUCTION

The creation of periodic orbits from equilibria as a real parameter crosses a critical value, is one of the simplest variations of the phase space in parametrized differential equations. In (Hopf, 1942) was proved the commonly known Hopf bifurcation theorem, and from then, numerous papers have dealt with this kind of bifurcation.

In (Chow and Mallet-Paret, 1977) the method of averaging is employed to ensure the Hopf bifurcation. In (Schmidt, 1978; Alexander and York, 1978) is showed that the Lyapunov's Center theorem can be derived from the Hopf's theorem. In (Hassard and Wan, 1978) the center manifold theorem is used to derive a formulae for the first Lyapunov coefficient. In (Hsu, 1976) is showed that the classical Belousov-Zaikin-Zhabotinskii reaction undergoes the Hopf bifurcation.

In this paper a change of coordinates is introduced to try to simplify the formulae derived in (Hassard and Wan, 1978). The idea is follow the new coordinates until the two dimensional center manifold, eliminate the quadratic terms and simplify the cubic terms. The significance of this result is at the bifurcation control theory of nonlinear control systems that undergo the Hopf bifurcation, where is important to establish the stability of the emerging periodic solution. See (Verduzco and Alvarez, 2004; Hamzi *et al*., 2004).

2. HOPF BIFURCATION

Theorem 1. Suppose that the system

$$\dot{x} = f(x, \mu)$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ has an equilibrium (x_0, μ_0) at which the following properties are satisfied:

(A1) $D_x f(x_0, \mu_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

(A2) Let $\lambda(\mu)$, $\bar{\lambda}(\mu)$ be the eigenvalues of $D_x f(x_0, \mu_0)$ which are imaginary at $\mu = \mu_0$, such that

$$\frac{d}{d\mu}\left(Re(\lambda(\mu))\right)|_{\mu=\mu_0} = d \neq 0. \qquad (1)$$

Then there is a unique three-dimensional center manifold passing through $(x_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}$ and a smooth system of coordinates for which the Taylor expansion of degree three on the center manifold, in polar coordinates, is given by

$$\dot{r} = (d\mu + ar^2)r,$$
$$\dot{\theta} = \omega + c\mu + br^2.$$

If $a \neq 0$, there is a surface of periodic solutions in the center manifold which has quadratic tangency with

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the eigenspace of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu = -\frac{a}{d}r^2$. If $a < 0$, then these periodic solutions are stable limit cycles, while if $a > 0$, are repelling.

For bidimensional systems, there exists an expression to find the called first Lyapunov coefficient a. Consider the system

$$\dot{x} = Jx + F(x),$$

where $J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, $F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$, $F(0) = 0$ and $DF(0) = 0$. Then

$$a = \frac{1}{16\omega}(R_1 + \omega R_2), \tag{2}$$

where

$$\begin{aligned} R_1 = {} & F_{1x_1x_2}(F_{1x_1x_1} + F_{1x_2x_2}) \\ & -F_{2x_1x_2}(F_{2x_1x_1} + F_{2x_2x_2}) \\ & -F_{1x_1x_1}F_{2x_1x_1} + F_{1x_2x_2}F_{2x_2x_2} \end{aligned} \tag{3}$$

$$R_2 = F_{1x_1x_1x_1} + F_{1x_1x_2x_2} + F_{2x_1x_1x_2} + F_{2x_2x_2x_2} \tag{4}$$

3. PROBLEM FORMULATION

Consider the nonlinear system

$$\dot{\xi} = F(\xi, \mu) \tag{5}$$

where $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. The vector field $F(\xi, \mu)$ is assumed to be sufficiently smooth, with $F(0, 0) = 0$. We suppose that for $\mu \approx 0$, the matrix $DF(0, \mu)$ has the eigenvalues $\alpha(\mu) \pm i\omega(\mu)$, with $\alpha(0) = 0$, $\alpha'(0) \neq 0$ and $\omega(0) = \omega_0 > 0$, and the others $n - 2$ eigenvalues are reals, and negatives. Then, it follows that, from the Hopf theorem, the system (14) undergoes the called Hopf bifurcation at the origin $\xi = 0$ when $\mu = 0$.

We assume that $DF(0, 0)$ is in Jordan form, *i.e.*,

$$DF(0, 0) = \begin{pmatrix} J_H & 0 \\ 0 & J_S \end{pmatrix}$$

where $J_H = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}_{2\times 2}$, with $\omega_0 > 0$, and $J_S = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n-2} \end{pmatrix}$ where $\lambda_i < 0$ for $i = 1, \ldots, n - 2$.

Now, if $\xi = \begin{pmatrix} z \\ w \end{pmatrix}$, with $z \in \mathbb{R}^2$ and $w \in \mathbb{R}^{n-2}$, then, expanding system (14) around $\xi = 0$ and $\mu = 0$, yields

$$\dot{z} = J_H z + f(z, w) \tag{6}$$

$$\dot{w} = J_S w + g(z, w) \tag{7}$$

Then, our goal is to determine the stability of the emerging periodic solution in the system (6-7). We use the center manifold theorem to reduce the stability analysis to the planar equation (6), and then utilize (2) to calculate the first Lyapunov coefficient. Before that, we propose a change of coordinates to eliminate some quadratic terms in the system (6-7) in such way that $R_1 = 0$ and R_2 be simplest in (2).

4. CHANGE OF COORDINATES

Consider the change of coordinates

$$z = x + P_2(x) + P_3(x), \tag{8}$$

$$w = y + Q(x), \tag{9}$$

with $P_2, P_3 : \mathbb{R}^2 \to \mathbb{R}^2$, $Q : \mathbb{R}^2 \to \mathbb{R}^{n-2}$, where P_2 and Q are quadratic functions and P_3 is a cubic function. Then,

$$\begin{aligned} & z = x + P_2(x) + P_3(x) \\ \Leftrightarrow \quad & \frac{dz}{dt} = \left(I + \frac{\partial P_2}{\partial x}(x) + \frac{\partial P_3}{\partial x}(x) \right) \frac{dx}{dt} \\ \Leftrightarrow \quad & \frac{dx}{dt} = \left(I + \frac{\partial P_2}{\partial x}(x) + \frac{\partial P_3}{\partial x}(x) \right)^{-1} \frac{dz}{dt} \end{aligned} \tag{10}$$

But,

$$\left(I + \frac{\partial P_2}{\partial x}(x) + \frac{\partial P_3}{\partial x}(x) \right)^{-1} = I - \frac{\partial P_2}{\partial x}(x) - \frac{\partial P_3}{\partial x}(x) + \left(\frac{\partial P_2}{\partial x}(x) \right)^2 + \cdots,$$

$$J_H z = J_H(x + P_2(x) + P_3(x))$$

and,

$$\begin{aligned} f(z, w) = {} & \frac{1}{2} z^T \frac{\partial^2 f}{\partial z^2}(0, 0) z + z^T \frac{\partial^2 f}{\partial z \partial w}(0, 0) w \\ & + w^T \frac{1}{2} \frac{\partial^2 f}{\partial w^2}(0, 0) w + \frac{1}{6} \frac{\partial^3 f}{\partial z^3}(0, 0)(z, z, z) + \cdots \\ = {} & \frac{1}{2} (x + \cdots)^T \frac{\partial^2 f}{\partial z^2}(0, 0)(x + \cdots) \\ & + (x + \cdots)^T \frac{\partial^2 f}{\partial z \partial w}(0, 0)(y + Q(x)) \\ & + \frac{1}{2}(y + Q(x))^T \frac{\partial^2 f}{\partial w^2}(0, 0)(y + Q(x)) \\ & + \frac{1}{6} \frac{\partial^3 f}{\partial z^3}(0, 0)(x + \cdots, x + \cdots, x + \cdots) + \cdots \\ = {} & \frac{1}{2} x^T \frac{\partial^2 f}{\partial z^2}(0, 0) x + x^T \frac{\partial^2 f}{\partial z^2}(0, 0) P_2(x) \\ & + x^T \frac{\partial^2 f}{\partial z \partial w}(0, 0) Q(x) + x^T \frac{\partial^2 f}{\partial z \partial w}(0, 0) y \\ & + \frac{1}{6} \frac{\partial^3 f}{\partial z^3}(0, 0)(x, x, x) + \cdots, \end{aligned}$$

then, simplifying we obtain

$$\dot{x} = J_H x + \tilde{f}_2(x) + \tilde{f}_3(x) + \cdots \tag{11}$$

where

$$\begin{aligned}
\tilde{f}_2(x) &= J_H P_2(x) - \frac{\partial P_2}{\partial x}(x) J_H x + \frac{1}{2} x^T \frac{\partial^2 f}{\partial z^2}(0,0)x,\\
\tilde{f}_3(x) &= J_H P_3(x) - \frac{\partial P_3}{\partial x}(x)\\
&\quad + x^T \frac{\partial^2 f}{\partial z^2}(0,0)P_2(x) + x^T \frac{\partial^2 f}{\partial z \partial w}(0,0)Q(x)\\
&\quad + x^T \frac{\partial^2 f}{\partial z \partial w}(0,0)y + \frac{1}{6}\frac{\partial^3 f}{\partial z^3}(0,0)(x,x,x)\\
&\quad - \frac{\partial P_2}{\partial x}(x)\tilde{f}_2(x).
\end{aligned}$$

Now then, from (9),

$$\begin{aligned}
& w = y + Q(x)\\
\Leftrightarrow\ & \frac{dw}{dt} = \frac{dy}{dt} + \frac{\partial Q}{\partial x}(x)\frac{dx}{dt}\\
\Leftrightarrow\ & \frac{dy}{dt} = \frac{dw}{dt} - \frac{\partial Q}{\partial x}(x)\frac{dx}{dt}\\
\Leftrightarrow\ & \frac{dy}{dt} = J_S w + g(z,w)\\
& \quad - \frac{\partial Q}{\partial x}(x)\left(J_H x + \tilde{f}_2(x) + \tilde{f}_3(x) + \cdots\right),
\end{aligned}$$

but,

$$\begin{aligned}
g(z,w) &= \frac{1}{2} z^T \frac{\partial^2 g}{\partial z^2}(0,0)z + z^T \frac{\partial^2 g}{\partial z \partial w}(0,0)w\\
&\quad + w^T \frac{1}{2}\frac{\partial^2 g}{\partial w^2}(0,0)w + \cdots\\
&= \frac{1}{2} x^T \frac{\partial^2 g}{\partial z^2}(0,0)x + \cdots,
\end{aligned}$$

then, simplifying, we obtain

$$\dot{y} = J_S y + \tilde{g}_2(x) + \cdots, \tag{12}$$

where

$$\begin{aligned}
\tilde{g}_2(x) &= J_S Q(x) - \frac{\partial Q}{\partial x}(x) J_H x \\
&\quad + \frac{1}{2} x^T \frac{\partial^2 g}{\partial z^2}(0,0)x.
\end{aligned} \tag{13}$$

From the normal form theory, there exist $P_2(x)$ such that $\tilde{f}_2(x) \equiv 0$. If

$$P_2(x) = \begin{pmatrix} p_{21}(x) \\ p_{22}(x) \end{pmatrix},$$

where $p_{2i}(x) = a_{2i}x_1^2 + b_{2i}x_1x_2 + c_{2i}x_2^2$, and

$$\frac{1}{2} x^T \frac{\partial^2 f}{\partial z^2}(0,0)x = \begin{pmatrix} f_{21}(x) \\ f_{22}(x) \end{pmatrix},$$

where $f_{2i}(x) = k_{2i}x_1^2 + l_{2i}x_1x_2 + m_{2i}x_2^2$, then

$$\begin{aligned}
a_{21} &= -\frac{k_{22} + l_{21} + 2m_{22}}{3\omega_0},\\
a_{22} &= -\frac{-k_{21} + l_{22} - 2m_{21}}{3\omega_0},\\
b_{21} &= -\frac{-2k_{21} - l_{22} + 2m_{21}}{3\omega_0},\\
b_{22} &= -\frac{-2k_{22} + l_{21} + 2m_{22}}{3\omega_0},\\
c_{21} &= -\frac{2k_{22} - l_{21} + m_{22}}{3\omega_0},\\
c_{22} &= -\frac{-2k_{21} - l_{22} - m_{21}}{3\omega_0}.
\end{aligned}$$

Now, we are going to find $P_3(x)$ and $Q(x)$ such that

$$\begin{aligned}
0 &= J_H P_3(x) - \frac{\partial P_3}{\partial x}(x)\\
&\quad + x^T \frac{\partial^2 f}{\partial z^2}(0,0)P_2(x) + x^T \frac{\partial^2 f}{\partial z \partial w}(0,0)Q(x)\\
&\quad + \frac{1}{6}\frac{\partial^3 f}{\partial z^3}(0,0)(x,x,x),
\end{aligned} \tag{14}$$

i.e., $\tilde{f}_3(x) = x^T \frac{\partial^2 f}{\partial z \partial w}(0,0)y$. If

$$P_3(x) = \begin{pmatrix} p_{31}(x) \\ p_{32}(x) \end{pmatrix},$$

where $p_{3i}(x) = a_{3i}x_1^3 + b_{3i}x_1^2x_2 + c_{3i}x_1x_2^2 + d_{3i}x_2^3$,

$$Q(x) = \begin{pmatrix} q_1(x) \\ \vdots \\ q_{n-2}(x) \end{pmatrix},$$

where $q_i(x) = \delta_i x_1^2$, and

$$x^T \frac{\partial^2 f}{\partial z^2}(0,0)P_2(x) + \frac{1}{6}\frac{\partial^3 f}{\partial z^3}(0,0)(x,x,x) = \begin{pmatrix} F_{11}(x) \\ F_{22}(x) \end{pmatrix}$$

where $F_{ii}(x) = s_{i1}x_1^3 + s_{i2}x_1^2x_2 + s_{i3}x_1x_2^2 + s_{i4}x_2^3$; besides, if

$$f(z,w) = \begin{pmatrix} f_1(z,w) \\ f_2(z,w) \end{pmatrix},$$

then,

$$\begin{aligned}
f_{kzw}(0,0) &= \left(\frac{\partial^2 f_k}{\partial z \partial w}(0,0)\right)_{2\times(n-2)}\\
&= \begin{pmatrix} \dfrac{\partial^2 f_k(0,0)}{\partial z_1 \partial w_1} & \cdots & \dfrac{\partial^2 f_k(0,0)}{\partial z_1 \partial w_{n-2}} \\ \dfrac{\partial^2 f_k(0,0)}{\partial z_2 \partial w_1} & \cdots & \dfrac{\partial^2 f_k(0,0)}{\partial z_2 \partial w_{n-2}} \end{pmatrix},
\end{aligned} \tag{15}$$

now then,

$$f_{kzw}(0,0)Q(x) = \begin{pmatrix} \displaystyle\sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_1 \partial w_j}\delta_j x_1^2 \\ \displaystyle\sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_2 \partial w_j}\delta_j x_1^2 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{q}_{k1}x_1^2 \\ \tilde{q}_{k2}x_1^2 \end{pmatrix}$$

where

$$\tilde{q}_{ki} = \sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j}\delta_j, \quad (16)$$

therefore,

$$x^T f_{zw}(0,0)Q(x) = \begin{pmatrix} x^T f_{1zw}(0,0)Q(x) \\ x^T f_{2zw}(0,0)Q(x) \end{pmatrix} = \begin{pmatrix} \tilde{q}_{11}x_1^3 + \tilde{q}_{12}x_1^2x_2 \\ \tilde{q}_{21}x_1^3 + \tilde{q}_{22}x_1^2x_2 \end{pmatrix}.$$

Finally, the coefficients of the functions $P_3(x)$ and $Q(x)$ that satisfy the equation (14), must to satisfy the next system of equations:

$$\begin{aligned} \tilde{q}_{11} &= c_1 + \omega_0(a_{32} - c_{32}) \quad (17) \\ \tilde{q}_{12} &= c_2 - 3\omega_0(a_{31} - d_{32}) \\ \tilde{q}_{21} &= c_3 - w_0(a_{31} - d_{32}) \\ \tilde{q}_{22} &= c_4 - 3\omega_0(a_{32} - c_{32}) \\ 0 &= c_5 + (c_{32} + b_{31}) \\ 0 &= c_6 + (d_{31} + c_{32}) \\ 0 &= c_7 + (d_{32} - b_{32}) \\ 0 &= c_8 + (d_{32} - c_{31}) \end{aligned}$$

where c_i are constants for each $i = 1, \dots, 8$.

The last four equations have many solutions. If we fixed one of them, then, the first four equations have n unknown variables: a_{31}, a_{32}, and the $n-2$ δ_j that are implicit in the $\tilde{q}_{ki}$. The next hypothesis ensure us one solution of the subsystem of four equations with n unknown variables.

(H1) There exist i_r, j_r, k_r, such that

$$\frac{\partial^2 f_{k_r}(0,0)}{\partial z_{i_r} \partial w_{j_r}} \neq 0 \quad (18)$$

for $r = 1, 2$, where $j_1 \neq j_2$, and $i_1 \neq i_2$ or $k_1 \neq k_2$.

Summarizing: The system

$$\begin{aligned} \dot{z} &= J_H z + f(z, w) \\ \dot{w} &= J_S w + g(z, w) \end{aligned}$$

is transformed, by the change of coordinates

$$\begin{aligned} z &= x + P_2(x) + P_3(x), \\ w &= y + Q(x), \end{aligned}$$

in the system

$$\begin{aligned} \dot{x} &= J_H x + x^T \frac{\partial^2 f}{\partial z \partial w}(0,0)y + \cdots \quad (19) \\ \dot{y} &= J_S y + \tilde{g}_2(x) + \cdots, \quad (20) \end{aligned}$$

where $\tilde{g}_2(x)$ is giving by (13).

5. CENTER MANIFOLD

Consider the system (19-20), we seek a center manifold $h : \mathbb{R}^2 \to \mathbb{R}^{n-2}$, such that $y = h(x)$, with $h(0) = 0$, $Dh(0) = 0$ and

$$\frac{\partial h}{\partial x}(x)\,(J_H x + \cdots) - (J_S h(x) + \tilde{g}_2(x) \cdots) \equiv 0. \quad (21)$$

From (13),

$$\begin{aligned} \tilde{g}_2(x) &= J_S Q(x) - \frac{\partial Q}{\partial x}(x) J_H x \\ &+ \frac{1}{2} x^T \frac{\partial^2 g}{\partial z^2}(0,0)x \\ &= \begin{pmatrix} \tilde{g}_{21}(x) \\ \vdots \\ \tilde{g}_{2,n-2}(x) \end{pmatrix}. \end{aligned}$$

If

$$\frac{1}{2} x^T \frac{\partial^2 g}{\partial z^2}(0,0)x = \begin{pmatrix} p_1(x) \\ \vdots \\ p_{n-2}(x) \end{pmatrix}$$

where $p_i(x) = p_{i1}x_1^2 + p_{i2}x_1x_2 + p_{i3}x_2^2$, then

$$\tilde{g}_{2i}(x) = g_{i1}x_1^2 + g_{i2}x_1x_2 + g_{i3}x_2^2,$$

with $g_{i1} = p_{i1} + \lambda_i\delta_i$, $g_{i2} = p_{i2} + 2\omega_0\delta_i$, and $g_{i3} = p_{i3}$.

Now then, if $h_i(x) = \alpha_i x_1^2 + \beta_i x_1 x_2 + \gamma_i x_2^2$, substituting in (21), we obtain that

$$\begin{aligned} \alpha_i &= -\frac{\omega_0(g_{i2}\lambda_i + 2g_{i3}\omega_0) + g_{i1}(\lambda_i^2 + 2\omega_0^2)}{\lambda_i \Delta_i} \\ \beta_i &= \frac{-g_{i2}\lambda_i + 2(g_{i1} - g_{i3})\omega_0}{\Delta_i} \quad (22) \\ \delta_i &= \frac{\omega_0(g_{i2}\lambda_i - 2g_{i3}\omega_0) - g_{i3}(\lambda_i^2 + 2\omega_0^2)}{\lambda_i \Delta_i}, \end{aligned}$$

for $i = 1, \dots, n-2$.

Then, the dynamics on the center manifold is giving by

$$\dot{x} = J_H x + x^T \frac{\partial^2 f}{\partial z \partial w}(0,0)h(x) + \cdots$$

where each $h_i(x)$ is giving by (22).

6. THE FIRST LYAPUNOV COEFFICIENT

From (15),

$$f_{kzw}(0,0)h(x) = \begin{pmatrix} \sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_1 \partial w_j} h_j(x) \\ \sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_2 \partial w_j} h_j()x) \end{pmatrix},$$

but

$$\begin{aligned} \sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j} h_j(x) &= \left(\sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j} \alpha_j \right) x_1^2 \\ &+ \left(\sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j} \beta_j \right) x_1 x_2 \\ &+ \left(\sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j} \gamma_j \right) x_2^2 \\ &= \alpha_{ki} x_1^2 + \beta_{ki} x_1 x_2 + \gamma_{ki} x_2^2, \end{aligned}$$

where

$$\alpha_{ki} = \sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j} \alpha_j$$

$$\beta_{ki} = \sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j} \beta_j$$

$$\gamma_{ki} = \sum_{j=1}^{n-2} \frac{\partial^2 f_k(0,0)}{\partial z_i \partial w_j} \gamma_j$$

Therefore,

$$x^T f_{zw}(0,0)h(x) = \begin{pmatrix} \tilde{h}_1(x) \\ \tilde{h}_2(x) \end{pmatrix},$$

where

$$\begin{aligned} \tilde{h}_i(x) = \alpha_{i1} x_1^3 + (\beta_{i1} + \alpha_{i2}) x_1^2 x_2 \\ + (\gamma_{i1} + \beta_{i2}) x_1 x_2^2 + \gamma_{i2} x_2^3 \end{aligned}$$

From (2),

$$\begin{aligned} &R_1 = 0, \text{ and} \\ &R_2 = 6(\alpha_{11} + \gamma_{11}) + 2(\alpha_{22} + \gamma_{11} + \beta_{12} + \beta_{21}), \end{aligned}$$

then, the first Lyapunov coefficient is giving by

$$a = \frac{1}{8} \left(3(\alpha_{11} + \gamma_{11}) + \alpha_{22} + \gamma_{11} + \beta_{12} + \beta_{21} \right)$$

7. CONCLUSIONS

The first Lyapunov coefficient has been calculated for a particular class of nonlinear systems. This approach introduce a change of coordinates to simplify the dynamics on the center manifold, in a such way that dissappear the quadratic terms. This method permits some degree of manipulation in the dynamic equations, that, surely it is possible to find simpler expressions for the dynamics on the center manifold.

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