LINEAR SYMMETRIC DYNAMICAL SYSTEMS

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Abstract: The problem of characterizing those linear systems that exhibit a symmetric behavior was completely solved in the 1D case, ten years ago, in a few papers by F. Fagnani and J. C. Willems. Unfortunately, that theory could not be extended in its full generality to multidimensional systems: the techniques used in the proofs limited the results to systems of independent equations and to a restricted class of symmetries.
Using a recent result on symmetries of discrete systems, in this paper it is shown how to solve this open problem for other types of multidimensional systems, in particular, for systems of partial differential equations.

Keywords: Autoregressive models, partial differential equations, symmetries, behavioral approach

1. INTRODUCTION

The study of symmetries is a very important tool to comprehend and clarify many intrinsic properties of physical systems. Indeed, the knowledge of the symmetries of a dynamical system often leads to a simplification of its mathematical description as, for instance, the well-known Noether's Theorem shows.

Quite general results about symmetry of linear dynamical systems were presented by Fagnani and Willems (Fagnani and Willems, 1993; Fagnani and Willems, 1994*a*; Fagnani and Willems, 1994*b*) in the framework of the behavioral approach. However, the proofs carried out in the 1D case exploited a *regularity* property that is not necessarily verified by multidimensional (nD) systems. Moreover, for dynamical multidimensional systems defined by equations with real coefficients, only a smaller class of symmetries were allowed. In this paper we remove these restriction for generic continuous nD systems.

In Section 2 we briefly introduce the behavioral approach to nD discrete and continuous-time linear systems, i.e., system which can be described by a finite number of linear partial difference or differential equations. After having given the necessary concepts regarding symmetries and their representations in Section 3, in Section 4 we recall a fundamental result about symmetric discrete nD

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systems. In the following Section 5, this result is extended to continuous-time systems.

Eventually, we end by showing in Section 6 that the method which was used to prove the main result is a very powerful technique to demonstrate general properties of discrete and continuous dynamical systems.

2. DYNAMICAL SYSTEM AND BEHAVIORS

According to the behavioral approach, a dynamical system is a triple

$$\Sigma = (\mathbb{T}, W, \mathcal{B}),$$

where $\mathbb{T}$ denotes the *time set*, W the *signal space*, and $\mathcal{B}$, which is a subset of the set $W^{\mathbb{T}} = \{w : \mathbb{T} \to W\}$ of all the *trajectories*, represents the set of trajectories which are allowed by the definition of the system. This is called the *behavior* of the system.

We will consider discrete systems, with time set $\mathbb{T} = \mathbb{Z}^n$, and continuous systems, with time set $\mathbb{T} = \mathbb{R}^n$, where $n \in \mathbb{N}$, i.e., multidimensional (or nD) discrete and continuous-time systems. We also assume that they are linear, complete and shift-invariant. This amounts to say that $W = \mathbb{K}^q$ for some field $\mathbb{K}$, that the behavior $\mathcal{B}$ is a closed vector space (in the topology defined on the functions space), and that for any trajectory $w \in \mathcal{B}$ and $\tau \in \mathbb{T}$, $\sigma^\tau w \in \mathcal{B}$ where σ^τ is the shift operator such that $(\sigma^\tau w)(t) = w(t+\tau)$, $\forall \tau \in \mathbb{T}^n$.

Remark 1. The multi-index notation is used, i.e., if $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{T}^n$, then $\sigma^\tau = \sigma_1^{\tau_1} \cdots \sigma_n^{\tau_n}$ where σ_i is the shift operator acting on the i-th component of $t \in \mathbb{T}^n$. Analogously, for any $\tau \in \mathbb{Z}^n$, $s^\tau = s_1^{\tau_1} \cdots s_n^{\tau_n}$ is a monomial in the n indeterminates $s_1, \ldots, s_n$ and, if $\tau \in \mathbb{N}^n$, $\partial^\tau = \partial_1^{\tau_1} \cdots \partial_n^{\tau_n}$ is a partial differential operator of degree $\sum_i \tau_i$, defined as the composition of the operators $\partial_i^{\tau_i}$ acting on the i-th indeterminate of a function. The following notation will be used too: given a partition of the 'time set' $\mathbb{T} = \mathbb{T}_1 \times \mathbb{T}_2$, $w(t) = w(h,k)$ implicitly indicates a partition of the indeterminate $t \in \mathbb{T}$ into components $h \in \mathbb{T}_1$ and $k \in \mathbb{T}_2$.

Defining a dynamical system through its behavior leads to a theory which does not depend on a specific model for the defining equations, as for instance the input/state/output model of classical systems theory. Indeed, it is possible to characterize the trajectories of a dynamical system in many ways. The most studied are the *kernel* and the *image representations*, which define the behavior as the kernel and, respectively, as the image of an operator.

To be more precise, let us consider first the discrete case in detail. We need to introduce operators on trajectories $w \in W^{\mathbb{T}}$ of the form $\sum_{i \in \mathcal{I}} R_i w(t+i)$, where $\mathcal{I}$ is a finite subset of the time set $\mathbb{T}$ and R_i are constant matrices with entries in $\mathbb{K}$ and suitable dimensions. Notice that there is no loss of generality in considering just this class of operators since it can be proved that it coincides with the set of all linear operators which commute with every shift σ^τ.

By definition of σ, we may write

$$\sum_{i \in \mathcal{I}} R_i w(t+i) = \sum_{i \in \mathcal{I}} R_i \sigma^i w(t) = R(\sigma) w(t), \quad (1)$$

where $R(s)$ is a multivariate polynomial matrix that could contain negative powers of the indeterminates, called Laurent-polynomial matrix.

Therefore, we say that a behavior $\mathcal{B} \subseteq (\mathbb{K}^q)^{\mathbb{T}}$ is defined by a kernel representation if there exist a Laurent-polynomial matrix $R(s)$ with q columns such that

$$\mathcal{B} = \ker R = \{w \in (\mathbb{K}^q)^{\mathbb{T}} : R(\sigma) w = 0\},$$

i.e., if $\mathcal{B}$ is the set of solutions of a matrix difference equation. Note that, since we are considering shift-invariant systems, $R(s)$ and $s^\tau R(s)$ are kernel representation of the same behavior. Therefore, we may always suppose that $R(s) \in \mathbb{K}[s]^{p \times q}$, for some $p \in \mathbb{N}$, i.e., that $R(s)$ is effectively a polynomial matrix.

The same can be said of continuous-time systems. In this case a behavior can be defined as the kernel of a partial differential operator $R(\partial)$, where $R(s)$ is a polynomial matrix. Usually, a specific function space is chosen and in this paper we will deal with continuous-time systems whose trajectories are smooth functions, just for the sake of simplicity. Indeed, the most general case of Schwartz' distributions $\mathcal{D}'(\mathbb{T}, \mathbb{K})$ could be treated with only slight changes in the notation.

In the end, $R(s) \in \mathbb{K}[s]^{p \times q}$ is the kernel representation of a continuous-time behavior containing smooth trajectories, $\mathcal{B} \subseteq C^\infty(\mathbb{T}, \mathbb{K}^q)$, if

$$\mathcal{B} = \ker R = \{w \in C^\infty(\mathbb{T}, \mathbb{K}^q) : R(\partial) w = 0\}.$$

This is a very general representation since it can be proved that any (discrete or continuous) linear, closed and shift-invariant nD behavior admits a kernel representation (Oberst, 1990).

Remark 2. While every 1D behavior defined by a kernel representation admits a full row rank kernel representation, called *minimal*, this does not hold for nD behaviors. Behaviors that have a full row rank kernel representation are called *regular*.

3. SYMMETRIES AND THEIR REPRESENTATIONS

To define symmetries in a proper way we first have to briefly introduce group representations. For more details we refer the reader to (Serre, 1977).

Given a vector space $\mathcal{W}$ over the field $\mathbb{K}$, we will denote by $\mathrm{GL}(\mathcal{W})$ the group of $\mathbb{K}$-isomorphisms of $\mathcal{W}$.

Definition 3. A representation of the group G on the vector space $\mathcal{W}$ is a group homomorphism

$$\rho : G \to \mathrm{GL}(\mathcal{W}),\ g \mapsto \rho_g.$$

The degree of a representation is defined by $\deg \rho = \dim \mathcal{W}$.

Remark 4. We suppose in this paper that G is equipped with a topology that makes it into a Hausdorff compact topological group. Even if not mentioned, every representation will be assumed to be continuous. If G is finite, the discrete topology is employed, which trivially ensures continuity.

Note that, according to the definition, ρ_g is an isomorphism of $\mathcal{W}$ onto itself for every g. With a little abuse of notation we may implicitly assume that some basis of $\mathcal{W}$ has already been fixed and therefore we will confuse ρ_g with its matrix representation.

Definition 5. Given a representation ρ on $\mathcal{W}$, a subspace $\mathcal{U} \subseteq \mathcal{W}$ is ρ-symmetric if $\rho\mathcal{U} \subseteq \mathcal{U}$, i.e., $\rho_g\mathcal{U} \subseteq \mathcal{U}$ for any $g \in G$.

Note that when $\mathcal{U}$ is a ρ-symmetric subspace of $\mathcal{W}$, the restrictions of ρ_g to $\mathcal{U}$ are isomorphisms of $\mathcal{U}$ and thus $\rho|_{\mathcal{U}}$ is itself a representation which is called subrepresentation of ρ. It can be proved that in the case of finite-degree representations there exists another ρ-symmetric subspace $\mathcal{V}$ such that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$. We write also $\rho = \rho|_{\mathcal{U}} \oplus \rho|_{\mathcal{V}}$. A representation which does not admit proper symmetric subspaces, that is to say subrepresentations, is called irreducible. The decomposition of $\mathcal{W}$ into minimal symmetric subspaces gives then rise to a decomposition of ρ into irreducible subrepresentations. This decomposition becomes unique only if we identify different irreducible representations which are isomorphic, where ρ^1 is isomorphic to ρ^2, or also $\rho^1 \cong \rho^2$, if there exists an isomorphism π such that $\pi\rho^1_g = \rho^2_g\pi$ for every g.

Eventually, the standard way to write such a decomposition of ρ into subrepresentations is

$$\rho = m_1\eta^1 \oplus m_2\eta^2 \oplus \cdots \oplus m_r\eta^r, \tag{2}$$

where the notation $m_i\eta^i$ stands for the direct sum of m_i copies of subrepresentations isomorphic to η^i.

Group representations are used to define and analyze symmetries of dynamical systems. A quite general approach would be to choose $\mathcal{W} = W^T$, the set trajectories. Nevertheless, in this paper we deal with a simpler class of symmetries, called *static symmetries*, since they act on the coordinates of the trajectories, i.e., $\mathcal{W} = W$, the signal space of the behavior. Therefore, the representations are homomorphism of the type

$$\rho : G \to \mathrm{GL}(W).$$

Definition 6. Given a representation ρ on W, a behavior $\mathcal{B} \subseteq W^T$ is ρ-symmetric if

$$\rho\mathcal{B} \subseteq \mathcal{B}. \tag{3}$$

Note that, since ρ_g are isomorphisms, a behavior is ρ-symmetric if and only if $\rho\mathcal{B} = \mathcal{B}$

4. THE DISCRETE CASE

In (Fagnani and Willems, 1993) a characterization of symmetric continuous and discrete-time 1D systems was given in terms of their kernel representation. Here, we only write it for discrete-time systems.

Theorem 7. Given a representation of G on $W = \mathbb{K}^q$, the behavior $\mathcal{B} \subseteq W^{\mathbb{Z}}$ is ρ-symmetric if and only if it admits a minimal kernel representation provided by $R(s) \in \mathbb{K}[s]^{p\times q}$ such that

$$R(s)\rho = \rho' R(s)$$

where ρ' is a subrepresentation of ρ.

Under the restrictive assumption of regularity of the behavior, this theorem has been extended to nD systems (De Concini and Fagnani, 1993). Moreover, in the real case $\mathbb{K} = \mathbb{R}$, another hypotheses on the representation had to be added.

Remark 8. If $\mathcal{B} = \ker R$ is ρ-symmetric and the group G is finite with $|G|$ elements, then it is easy to construct a kernel representation which satisfies the condition of Theorem 7. Indeed, let

$$\tilde{R}(s) = \begin{bmatrix} R(s)\rho_0 \\ R(s)\rho_1 \\ \vdots \\ R(s)\rho_{|G|-1} \end{bmatrix} \text{ and } \rho'_g = \begin{bmatrix} 0 & I & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I \\ I & & & 0 \end{bmatrix}^g.$$

Then $\mathcal{B} = \ker \tilde{R}$ and $\tilde{R}(s)\rho = \rho'\tilde{R}(s)$. However, notice that the size (i.e., the number of rows) of $\tilde{R}$, depending on $|G|$, could be rather big.

In (Vettori, 2004), a complete extension of Theorem 7 was proved for nD discrete systems. In this section we just state that result after recalling some facts that will be used later in the paper.

Suppose that an nD discrete behavior $\mathcal{B}$ is given. For every trajectory $w \in \mathcal{B}$ define

$$w^i(h) = \begin{bmatrix} w(h,0) \\ w(h,1) \\ \vdots \\ w(h,i-1) \end{bmatrix}, \ \forall h \in \mathbb{Z}^{n-1},$$

for any positive integer i and let $\mathcal{B}^i = \{w^i : w \in \mathcal{B}\}$.

A priori, $\mathcal{B}^i$ is just a set of trajectories: $\mathbb{Z}^{n-1} \to \mathbb{K}^q$. However, by easily extending to the nD case a theorem about 2D systems stated in (Komorník *et al.*, 1991), it can be proved that $\mathcal{B}^i$ is an $(n-1)$D linear, shift-invariant and complete behavior. Moreover, if $\mathcal{B}$ is symmetric, an analogue property holds for $\mathcal{B}^i$.

Note that if $\mathcal{B} = \ker R$, where $R(s) = \sum_{i=0}^{N-1} R_i(s) s_n^i$ with $R_i(s) \in \mathbb{K}^{p\times q}[s_1,\ldots,s_{n-1}]$, then a kernel representation of $\mathcal{B}^N$ is given by $R^N(s) \in \mathbb{K}^{p\times Nq}[s_1,\ldots,s_{n-1}]$ defined by

$$R^N(s) = \begin{bmatrix} R_0(s) \ R_1(s) \ \cdots \ R_{N-1}(s) \end{bmatrix}.$$

Notice also that $R^N(s)$ could be defined implicitly by the equation $R(s) = R^N(s)\phi^N(s)$, with

$$\phi^i(s) = \begin{bmatrix} I \ Is_n \ \cdots \ Is_n^{i-1} \end{bmatrix}^\top, \qquad (4)$$

where i is any positive integer and I is the identity matrix with appropriate size.

On the other hand, knowing $\mathcal{B}^i$ it is possible to determine $\mathcal{B}$, as the following theorem states.

Theorem 9. For any nD linear, shift invariant and complete discrete behavior $\mathcal{B}$ there exists N such that, with the notation we introduced,

$$\mathcal{B}^N = \ker R^N \ \Rightarrow \ \mathcal{B} = \ker R^N \phi^N.$$

Remark 10. The number N, which represents the growth in the dimension of the behavior's signal space, is related to the order of the partial difference/differential equations in the variable s_n that is *eliminated* when passing from $\mathcal{B}$ to $\mathcal{B}^N$.

To sum up, starting from the nD symmetric behavior $\mathcal{B}$, we can construct $(n-1)$D symmetric behaviors $\mathcal{B}^i$. On the contrary, starting from the kernel representation of these, we can obtain a kernel representation of $\mathcal{B}$. So, an inductive reasoning shows that it is possible to reduce the problem to a 1D behavior $\tilde{\mathcal{B}}$ that still exhibits symmetry, apply Theorem 7, and go back to the nD case.

Theorem 11. Given a representation ρ of G on $W = \mathbb{K}^q$, the behavior $\mathcal{B} \subseteq W^{\mathbb{Z}^n}$ is ρ-symmetric if and only if it admits a kernel representation provided by $R(s) \in \mathbb{K}[s]^{p\times q}$ such that for a suitable representation ρ' of G,

$$R(s)\rho = \rho' R(s). \qquad (5)$$

Moreover, if $\rho = m_1\eta^1 \oplus \cdots \oplus m_r\eta^r$, then $\rho' = m_1'\eta^1 \oplus \cdots \oplus m_r'\eta^r$ and there exists $M \in \mathbb{N}$ such that

$$0 \le m_i' \le M m_i \text{ for every } i = 1,\ldots,r. \qquad (6)$$

Remark 12. As for the dimension of the matrix $R(s)$ of Theorem 11, note that its number of rows is equal to the rank of any minimal kernel representation of $\tilde{\mathcal{B}}$, as stated by Theorem 7.

5. THE GENERAL CASE

As we already showed in Section 2, we can use a similar notation for discrete and continuous-time systems. In order to prove Theorem 11 also in the continuous case, we develop this similarity further by introducing the dual module of a behavior.

Any element r of the ring $\mathcal{R} = \mathbb{K}[s]$ of polynomials in n indeterminates can be seen both as a linear operator on discrete and on continuous-time trajectories. So, if $w \in \mathbb{K}^{\mathbb{Z}^n}$, rw is the trajectory $r(\sigma)w(t)$, $\forall t \in \mathbb{Z}^n$; if $w \in C^\infty(\mathbb{R}^n, \mathbb{K})$ then rw is the trajectory $r(\partial)w(t)$, $\forall t \in \mathbb{R}^n$.

Therefore, if we just denote by $\mathcal{A}$ the function space $\mathbb{K}^{\mathbb{Z}}$ or $C^\infty(\mathbb{R}, \mathbb{K})$, from an algebraic point of view, $\mathcal{A}$ is an $\mathcal{R}$-module. Only this property will be used in the rest of the section.

If for any $A \in \mathcal{A}^q$ we define

$$A^\perp = \{r \in \mathcal{R}^q : ra = 0, \forall a \in \mathcal{A}^q\},$$

(where we assume that trajectories in $\mathcal{A}^q$ are column vectors and operators in $\mathcal{R}^q$ are row vectors) then one of the most important results proved in (Oberst, 1990), the "fundamental principle", states the existence of a duality between any linear, complete and shift-invariant behavior $\mathcal{B} \in \mathcal{A}^q$ and the quotient module

$$\mathcal{M}(\mathcal{B}) = \frac{\mathcal{R}^q}{\mathcal{B}^\perp},$$

for many function spaces $\mathcal{A}$ including, among others, discrete sequences, smooth function and distributions.

Our aim is to prove that Theorem 11 is algebraic in nature, relating properties of an $\mathcal{R}$-module and of a matrix with entries in $\mathcal{R}$. This fact immediately extends Theorem 11 to any $\mathcal{R}$-module $\mathcal{A}$.

First of all, we show that a behavior $\mathcal{B}$ is ρ-symmetric if and only if its dual module $\mathcal{M}(\mathcal{B})$ shares this property too, i.e., that (3) holds true if and only if

$$\mathcal{M}(\mathcal{B})\rho \subseteq \mathcal{M}(\mathcal{B}).$$

Lemma 13. Given a behavior $\mathcal{B} \subseteq \mathcal{A}^q$ and a nonsingular constant matrix T with entries in $\mathbb{K}$ and suitable size, then $\mathcal{M}(\mathcal{B}) = \mathcal{M}(T\mathcal{B})T$.

PROOF. We first prove that $(T\mathcal{B})^{\perp} = \mathcal{B}^{\perp}T^{-1}$. Indeed,

$$\begin{aligned}(T\mathcal{B})^{\perp} &= \{r \in \mathcal{R}^q : rTw = 0, \forall w \in \mathcal{B}\} \\ &= \{rT \in \mathcal{R}^q : rTw = 0, \forall w \in \mathcal{B}\}T^{-1} \\ &= \{r \in \mathcal{R}^q : rw = 0, \forall w \in \mathcal{B}\}T^{-1} \\ &= \mathcal{B}^{\perp}T^{-1}.\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{M}(T\mathcal{B})T &= \frac{\mathcal{R}^q}{(T\mathcal{B})^{\perp}}T \\ &= \frac{\mathcal{R}^q T}{(T\mathcal{B})^{\perp}T} \\ &= \frac{\mathcal{R}^q}{\mathcal{B}^{\perp}},\end{aligned}$$

thus proving the statement. □

Proposition 14. Given a representation ρ of G on $\mathbb{K}^q$, the behavior $\mathcal{B} \subseteq \mathcal{A}^q$ is ρ-symmetric if and only if $\mathcal{M}(\mathcal{B})$ is ρ-symmetric.

PROOF. As we already mentioned, the proposition is proved once we show that $\rho\mathcal{B} = \mathcal{B} \iff \mathcal{M}(\mathcal{B})\rho = \mathcal{M}(\mathcal{B})$. Actually, by Lemma 13, the equality

$$\mathcal{M}(\mathcal{B}) = \mathcal{M}(\rho\mathcal{B})\rho = \mathcal{M}(\rho\mathcal{B})$$

is equivalent both to the first and to the second condition. □

Before proving the main theorem, note that if $\mathcal{B} = \ker R$, with $R \in \mathcal{R}^{p\times q}$, then $\mathcal{B}^{\perp} = \mathcal{R}^p R$, which is the $\mathcal{R}$-module generated by the rows of the matrix R – see, for instance, (Wood, 2000).

Therefore, the following equivalence is obtained,

$$\mathcal{B} = \ker R \iff \mathcal{M}(\mathcal{B}) = \operatorname{coker} R, \qquad (7)$$

where the cokernel of R is defined by $\operatorname{coker} R = \mathcal{R}^q/\mathcal{R}^p R$.

Theorem 15. Given a representation ρ of G on $\mathbb{K}^q$, the $\mathbb{K}[s]$-module $\mathcal{M}$ is ρ-symmetric if and only if there exist a matrix R with entries in $\mathbb{K}[s]$ and a representation ρ' of G such that $\mathcal{M} = \operatorname{coker} R$ and

$$R(s)\rho = \rho' R(s). \qquad (8)$$

Moreover, if $\rho = m_1\eta^1 \oplus \cdots \oplus m_r\eta^r$, then $\rho' = m_1'\eta^1 \oplus \cdots \oplus m_r'\eta^r$ and there exists $M \in \mathbb{N}$ such that

$$0 \leq m_i' \leq Mm_i \text{ for every } i = 1, \ldots, r. \qquad (9)$$

PROOF. We only have to join the results we already obtained. Let $\mathcal{B}$ be the discrete behavior which is the dual of $\mathcal{M}$. By Proposition 14, it is ρ-symmetric if and only if $\mathcal{M}$ is ρ-symmetric. At the same time, by Theorem 11, this is true if and only if there exist R and ρ' such that equations (8) and (9) are satisfied and $\mathcal{B} = \ker R$. Since, by (7), this last condition is equivalent to $\mathcal{M} = \operatorname{coker} R$, the theorem is proved. □

Corollary 16. The statement of Theorem 11 remains true if, instead of a discrete behavior $\mathcal{B} \subseteq W^{\mathbb{Z}^n}$, we consider a continuous (linear, complete and shift-invariant) behavior $\mathcal{B} \subseteq C^{\infty}(\mathbb{R}^n, W)$.

6. FINAL REMARKS

The method we used in Section 5 to state for continuous systems a result that was obtained only for discrete dynamical systems in (Vettori, 2004), is a rather straightforward application of the duality between behaviors and modules, which was proved for the first time in (Oberst, 1990). However, to our knowledge, this simple idea has never been exploited till now and we think that it could have many fruitful consequences.

For example, let us have a look at some facts that were stated about $\mathcal{B}^i$ in Section 4.

The analog of $\mathcal{B}^i$ in the continuous case, is the set that contains trajectories

$$w^i(t) = \begin{bmatrix} w(t,0) \\ w'(t,0) \\ \vdots \\ w^{(i-1)}(t,0) \end{bmatrix}, \ \forall t \in \mathbb{R}^{n-1},$$

where $w^{(k)}(t,0) = \partial_n^k w(t,0)$ are the derivatives of $w \in \mathcal{B}$ with respect to the n-th variable.

Thus, $\mathcal{B}^i$ represents a sort of set of "initial conditions" of a generalized Cauchy problem and Theorem 9 states that for some i it contains all the information which is necessary to reconstruct exactly the whole behavior $\mathcal{B}$.

As another example, consider the case $i = 1$. The set B^1 is the restriction of the trajectories of $\mathcal{B}$ to the subspace $\mathbb{R}^{n-1} \times \{0\}$. The aforementioned Theorem of (Komorník *et al.*, 1991) states that, in the discrete case, $\mathcal{B}^i$ always admits kernel representation and so, passing to the continuous case, we obtain the following interesting proposition: given an nD continuous behavior, the trajectories obtained by restricting the domain to a subspace of $\mathbb{R}^n$ constitute, again, a linear, complete and shift-invariant behavior.

Note that it is not easy to prove this result directly: the proof, in the discrete case, is based on properties of the topology of pointwise convergence which do not hold in the Fréchet space of smooth functions.

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