SET-INVARIANT ESTIMATORS FOR LINEAR SYSTEMS SUBJECT TO DISTURBANCES AND MEASUREMENT NOISE

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Abstract: The concept of set-invariance is applied to the design of full-order state observers with limitation of the estimation error, for discrete-time linear systems subject to unknown but bounded persistent disturbances and measurement noise. It is shown that if the initial error belongs to a $\mathcal{D}$-(C, A)-invariant set, then it can be kept in this set by means of a suitable output injection, for all admissible disturbances and noise. Polyhedral $\mathcal{D}$-(C, A)-invariant sets are explicitly characterized by necessary and sufficient conditions. Based on such conditions, algorithms are proposed for the computation of a $\mathcal{D}$-(C, A)-invariant polyhedron containing the one to which the initial error belongs. The results are illustrated and compared to other approaches by means of a numerical example.

Keywords: Linear systems, state estimation, invariance, constraints, geometric approaches.

1. INTRODUCTION

Several problems of control systems subject to constraints on their state, control or output variables have been solved in the last years through the so-called set-invariance approach, mainly when such constraints are linear, corresponding, hence, to polyhedral sets defined in the state space (see e.g. (Blanchini, 1999) for a survey). In particular, for linear discrete-time systems, it is possible to construct the largest (A, B)-invariant set contained in the polyhedron defined by the constraints ((Blanchini, 1994; Dórea and Hennet, 1999)). This means that if the initial state belongs to this largest set, then a suitable sequence of control inputs can be computed so as to enforce the constraints along the state trajectory.

An important limitation of such techniques is the fact that most of the proposed solutions assume the use of state feedback control laws, requiring the full measurement of the state, which is not always possible due to physical or economical reasons. Very often, this difficulty can be circumvented by building an observer which estimates the unaccessible states.

This is the case of the approaches based on the so-called *set-valued observers* (Shamma and Tu, 1998; Shamma, 1999). Roughly speaking, for each time instant, the (polyhedral) set of states which could generate the measured output is computed and a point-wise optimal state is selected. Since such a set have to be computed on-line,

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the computational burden can be excessive, and its practical implementation in fast systems can become infeasible.

This work analyses another class of observers, based on the concept of (C, A)-invariant sets (Wonham, 1985; Basile and Marro, 1992). The (C, A)-invariance of polyhedral sets was analyzed in (Pimenta and Dórea, 2004) for deterministic systems. Here, this concept is extended to systems subject to unknown but bounded exogenous disturbances and measurement noise and applied to the design of full-order asymptotic state observers with estimation error limitation, for discrete-time single output systems. Firstly, the $\mathcal{D}$-(C, A)-invariance is defined as the possibility of keeping the estimation error in a given set in spite of the action of disturbances and noise belonging to polyhedral sets. Then, from the derivation of necessary and sufficient conditions, a complete characterization of $\mathcal{D}$-(C, A)-invariance for compact convex polyhedra is proposed. In the case for which the polyhedron defined by the uncertainty on the initial state is not $\mathcal{D}$-(C, A)-invariant, numerical algorithms are proposed to compute another polyhedron which satisfies this property and bounds the estimation error as much as possible. It is shown that when the initial polyhedron is symmetrical, under some conditions, the best error limitation can be achieved by the computation of the smallest $\mathcal{D}$-(C, A)-invariant set containing it. The proposed results are illustrated by means of a numerical example.

Notation: In mathematical expressions, the symbol ":" stands for "such that". $\mathbf{1}$ represents a vector of appropriate dimensions whose components are all equal to 1. M_i represents the i-th row of matrix M. $\mathrm{Conv}(\Omega)$ represents the convex hull of the set Ω, i.e., the smallest convex set which contains Ω.

2. $\mathcal{D}$-(C, A)-INVARIANT SETS

Consider the linear, time-invariant, discrete-time, single-output system, described by:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 d(k), \\ y(k) &= Cx(k) + \eta(k), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^r$ is the disturbance, $y \in \mathbb{R}$ is the output, $\eta \in \mathbb{R}$ is the measurement noise and k is the sampling time, with $k \in \mathbb{N}$. The pair (C, A) is supposed to be detectable.

An estimation of the state can be obtained by means of the following full-order observer:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) - v(z(k)), \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned} \tag{2}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimated state, $\hat{y} \in \mathbb{R}$ is the estimated output and $v(.)$ is the output injection.

The estimation error and the difference between the measured output and the estimated output are respectively defined as:

$$\begin{aligned} e(k) &= x(k) - \hat{x}(k), \\ z(k) &= y(k) - \hat{y}(k). \end{aligned}$$

Then, the error dynamics is given by:

$$\begin{aligned} e(k+1) &= Ae(k) + B_1 d(k) + v(z(k)), \\ z(k) &= Ce(k) + \eta(k), \end{aligned} \tag{3}$$

The disturbance d is assumed to be unknown but bounded to a compact (closed and limited) set $\mathcal{D} \subset \mathbb{R}^r$. The measurement noise is assumed to belong to the set $\mathcal{N} = \{\eta \ : \ |\eta| \leq \bar{\eta}\}$.

Consider now a compact set Ω whose interior contains the origin, defined on the estimation error space. The following set of *admissible outputs* is associated to Ω:

$$\mathcal{Z}(\Omega) = \{z : \ z = Ce + \eta \text{ for some } e \in \Omega, \ \eta \in \mathcal{N}\}.$$

$\mathcal{Z}(\Omega)$ is the set, also compact, of all values of z which can be generated by $e \in \Omega$ and $\eta \in \mathcal{N}\}$. Therefore, if $e(k) \in \Omega$, then $z(k) \in \mathcal{Z}(\Omega)$.

Definition 2.1. The set $\Omega \subset \mathbb{R}^n$ *is said to be* $\mathcal{D}$-(C, A)-invariant *with respect to system (3) if* $\forall z \in \mathcal{Z}(\Omega)$, $\exists v : Ae + B_1 d + v \in \Omega, \forall d \in \mathcal{D}, \forall e \in \Omega : z = Ce + \eta,$ *for some* $\eta \in \mathcal{N}$.

After the application of the output injection, Ω is simply said to be positively $\mathcal{D}$-invariant.

Definition 2.2. Given $0 < \lambda < 1$, *the set* $\Omega \subset \mathbb{R}^n$ *is said to be* $\mathcal{D}$-(C, A)-invariant λ-contractive (or simply $\mathcal{D}$-(C, A)-λ-contractive) *with respect to system (3) if* $\forall z \in \mathcal{Z}(\Omega)$, $\exists v : Ae + B_1 d + v \in \lambda\Omega, \forall d \in \mathcal{D}, \forall e \in \Omega : z = Ce + \eta,$ *for some* $\eta \in \mathcal{N}$.

In words, if the observation error at time k belongs to Ω, with Ω $\mathcal{D}$-(C, A)-λ-contractive, then, the knowledge of only $z(k)$ is sufficient to enforce e(k+1) $\in \lambda\Omega$ through the computation of $v(z(k))$, in spite of the disturbance and the noise. As a consequence, if the initial observation error $e(0)$ is known to belong to Ω then, by means of a suitable output injection $v(z(k))$, it is possible to keep it always limited to this set.

One should notice that a necessary condition for $\mathcal{D}$-(C, A)-λ contractivity is that $\lambda\Omega$ contains the disturbance set $\mathcal{D}$.

3. $\mathcal{D}$-(C, A)-INVARIANCE OF CONVEX POLYHEDRA

Assume now that Ω and $\mathcal{D}$ are compact, convex polyhedra containing the origin, defined by:

$$\Omega = \{e : Ge \le \mathbf{1}\},\ \mathcal{D} = \{d : Sd \le \mathbf{1}\},$$

with $G \in \mathbb{R}^{g\times n}$, $S \in \mathbb{R}^{s\times n}$.

The set of related outputs is also a compact and convex polyhedron defined by:

$$\begin{aligned}\mathcal{Z}(\Omega) = \{z : z = Ce + \eta \text{ for some}\\ e:\ Ge \le \mathbf{1} \text{ and } \eta : |\eta| \le \bar{\eta}\}.\end{aligned}$$

In the case of single-output systems, $\mathcal{Z}(\Omega)$ is a line segment in $\mathbb{R}$.

Considering Definition 2.2, it is clear that Ω is $\mathcal{D}$-(C, A)-λ-contractive if and only if, $\forall z \in \mathcal{Z}(\Omega)$:

$$\begin{gathered}\exists v(z) : G(Ae + B_1 d + v(z)) \le \lambda\mathbf{1},\\ \forall e, \eta : z = Ce + \eta,\ Ge \le \mathbf{1},\ |\eta| \le \bar{\eta},\\ \forall d : Sd \le \mathbf{1}\end{gathered} \tag{4}$$

Since the same $v(z)$ must work for all $d \in \mathcal{D}$, then the effect of disturbances can be taken into account by considering their worst case row by row. Let the elements of vector $\delta \in \mathbb{R}^g$ be defined by the following linear programming problems (LP):

$$\begin{aligned}\delta_i = \ & \max_d G_i B_1 d\\ & \text{under: } Sd \le \mathbf{1}\end{aligned}.$$

Then, condition (4) becomes:

$$\begin{gathered}\exists v(z) : G(Ae + v(z)) \le \lambda\mathbf{1} - \delta,\\ \forall e, \eta : z = Ce + \eta,\ Ge \le \mathbf{1},\ |\eta| \le \bar{\eta}.\end{gathered}$$

Let now $\phi(z)$ be the vector whose components are given by the solution of the following LP:

$$\begin{gathered}\phi_i(z) = \max_{e,\eta} G_i Ae\\ \text{under: } Ge \le \mathbf{1},\ |\eta| \le \bar{\eta},\ Ce + \eta = z.\end{gathered} \tag{5}$$

which can be rewritten as:

$$\begin{gathered}\phi_i(z) = \max_{e} G_i Ae\\ \text{under: } Ge \le \mathbf{1},\ |Ce - z| \le \bar{\eta}.\end{gathered} \tag{6}$$

Since the same $v(z)$ must work for all $e \in \Omega$ which could have generated the output z, then the worst case e can be computed row by row. Hence, condition (4) is equivalent to:

$$\exists v(z) : \phi(z) + Gv(z) \le \lambda\mathbf{1} - \delta \tag{7}$$

From the numerical point of view, the treatment of this condition is difficult, because the functions $\phi_i(z)$ are concave, piece-wise linear and continuous with respect to z (Sakarovitch, 1983). Hence the computation of their break points (for which the linear function defining $\phi_i(z)$ changes) would be necessary.

Consider now the external representation of the compact polyhedron Ω in terms of its vertices e^j, $j = 1, ..., n_v$. For each e^j, two outputs are associated: $z^j_- = Ce^j - \bar{\eta}$ and $z^j_+ = Ce^j + \bar{\eta}$. Let the discrete set $\mathcal{Z}_d(\Omega) \subset \mathcal{Z}(\Omega)$ be composed by all such outputs as follows:

$$\mathcal{Z}_d(\Omega) = \{z\ :\ z = z^j_-,\ z = z^j_+,\ j = 1 \cdots, n_v\},$$

and let n_z be the cardinality of $\mathcal{Z}_d(\Omega)$.

It is assumed that the elements z^l of $\mathcal{Z}_d(\Omega)$ are organized in increasing order, i.e., $z^1 \le z^2 \le ... \le z^{n_z}$.

The following necessary and sufficient conditions can be established:

Theorem 3.1. The polyhedron $\Omega = \{Ge \le \mathbf{1}\}$ is $\mathcal{D}$-(C, A)-λ-contractive if and only if:

$$\forall l = 1, ..., n_z,\ \exists v(z^l) : \phi(z^l) + Gv(z^l) \le \lambda\mathbf{1} - \delta. \tag{8}$$

Proof: The necessity is obvious, since all z^l, $l = 1, ..., n_z$ must satisfy (7). For the sufficiency, consider the set of constraints of the LP (6):

$$Ge \le \mathbf{1},\ Ce \le z + \bar{\eta},\ -Ce \le -z + \bar{\eta}. \tag{9}$$

The optimal solution of (7) is a vertex of this set, i.e. a point e for which n inequalities are active (the equality holds). A break in $\phi_i(z)$ corresponds to a point e for which an active constraint becomes inactive and an inactive one becomes active, that is to a point for which at least $n + 1$ inequalities are active (Luenberger, 1989; Sakarovitch, 1983).

For a given $z \in \mathcal{Z}(\Omega)$, two situations must be considered:

- if none of the two inequalities $Ce \le z + \bar{\eta}$ or $-Ce \le -z + \bar{\eta}$ is active, then the optimal e is a vertex of Ω. Hence, a break in $\phi_i(z)$ can only occur when one of the two inequalities becomes active, i.e. when one of the two hyperplanes $Ce = z + \bar{\eta}$ or $-Ce = -z + \bar{\eta}$ reaches a vertex of Ω, that is, for $z \in \mathcal{Z}_d(\Omega)$;
- if one of the two inequalities $Ce \le z + \bar{\eta}$ or $-Ce \le -z + \bar{\eta}$ is active, then the break will occur when either this inequality or an inequality of Ω becomes inactive. It can be verified that, in both cases, one of the two hyperplanes $Ce = z + \bar{\eta}$ or $-Ce = -z + \bar{\eta}$ reaches a vertex of Ω.

As a consequence, for $z \in \mathcal{Z}(\Omega)$ between two consecutive $z \in \mathcal{Z}_d(\Omega)$ the functions $\phi_i(z)$ are linear.

Consider now the following function:

$$\begin{gathered}\varepsilon(z) = \min_{\varepsilon,v} \varepsilon\\ \text{under: } \phi(z) + Gv \le \varepsilon\mathbf{1} - \delta.\end{gathered} \tag{10}$$

With $\phi(z)$ linear in the considered interval, it can be proved (Sakarovitch, 1983) that $\varepsilon(z)$ is a continuous, piece-wise linear and convex function. Thus, for $z^l \leq z \leq z^{l+1}$, the maximum value of $\varepsilon(z)$ is obtained for either z^l or z^{l+1}. Considering all intervals of $\mathcal{Z}(\Omega)$, one can conclude that the maximum value of $\varepsilon(z)$ corresponds to one of the $z^l \in \mathcal{Z}_d(\Omega)$. Therefore, if $\max_l \varepsilon(z^l) \leq \lambda$, then Ω is $\mathcal{D}$-(C, A)-λ-contractive. $\square$

From this Theorem, in order to check for the $\mathcal{D}$-(C, A)-λ-contractivity of Ω it is enough to solve the LP (10), for all $z^l \in \mathcal{Z}_d(\Omega)$, which are associated to the vertices of Ω.

Assume now that Ω and $\mathcal{D}$ are symmetrical with respect to the origin. Hence, they can be represented as:

$$\Omega = \{e : |Qe| \leq \mathbf{1}\}, \ \mathcal{D} = \{d : |Ed| \leq \mathbf{1}\}.$$

Ω (and $\mathcal{D}$ accordingly) can be written in the standard form $Ge \leq \mathbf{1}$, with $G = \begin{bmatrix} Q \\ -Q \end{bmatrix}$.

Let also the elements of the vector of worst case disturbances be now defined as:

$$\begin{aligned} \xi_i = & \max_d Q_i B_1 d \\ & \text{under: } |Ed| \leq \mathbf{1} \end{aligned}.$$

In this case, considering $Q \in \mathbb{R}^{q \times n}$, for $i \leq q$, $\phi_i(z) = \max_e Q_i Ae$ under $|Qe| \leq \mathbf{1}$, $|Ce - z| \leq \bar{\eta}$. Therefore, $\phi_{i+q}(z) = \max_e -Q_i Ae$ under the same constraints.

One can then conclude that, for $z = 0$, $\phi_{i+q}(0) = \phi_i(0)$. Hence, $\phi(0) = \begin{bmatrix} \phi_q(0) \\ \phi_q(0) \end{bmatrix}$, where $\phi_q(z)$ corresponds to the q firsts rows of $\phi(z)$. Thus condition (7) becomes, for $z = 0$:

$$\exists v(0) : \begin{bmatrix} \phi_q(0) \\ \phi_q(0) \end{bmatrix} + \begin{bmatrix} Q \\ -Q \end{bmatrix} v(0) \leq \lambda \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} - \begin{bmatrix} \xi \\ \xi \end{bmatrix}$$

As an immediate consequence, the following necessary condition can be stated:

Lemma 3.1. $\Omega = \{e : |Qe| \leq \mathbf{1}\}$ *is* $\mathcal{D}$-(C, A)-λ-*contractive only if:*

$$\phi_q(0) \leq \lambda \mathbf{1} - \xi \qquad (11)$$

One should notice that this necessary condition is much easier to be verified than the necessary and sufficient ones of Theorem 3.1, insofar as the computation of vertices is no more required. Moreover, in many systems tested along the development of this research it turned out to be sufficient as well. Condition (11) is also very useful to compute a $\mathcal{D}$-(C, A)-λ-contractive polyhedron, as described in next section.

4. COMPUTATION OF A $\mathcal{D}$-(C, A)-INVARIANT POLYHEDRON

In a typical state observer design problem, the initial state of the system is not known, but it is possible to define a region to which it belongs. Assume that this region is defined by linear inequalities which generate a compact symmetrical polyhedron $\Omega_x = \{x : \ |Qx| \leq \mathbf{1}\}$. Then, initializing the observer with $\hat{x}(0) = 0$, the initial error $e(0)$ is such that $|Qe(0)| \leq \mathbf{1}$, thus $e(0) \in \Omega = \{e : |Qe| \leq \mathbf{1}\}$.

The goal now is to compute an output injection $v(z(k))$ such that $e(k)$ is limited as much as possible. This objective would be satisfied if Ω were $\mathcal{D}$-(C, A)-λ-contractive. Indeed, in this case, from the definition of $\mathcal{D}$-(C, A)-λ-contractivity, there would be an output injection $v(z(k))$ such that $e(k) \in \lambda\Omega \ \forall k$ and $\forall d \in \mathcal{D}$, $\eta \in \mathcal{N}$. Therefore the estimation error would not exceed the known limits of the initial error, the set Ω.

However, it is quite rare that the polyhedron defined by the uncertainty on the initial state is $\mathcal{D}$-(C, A)-λ-contractive. Thus, it is necessary to construct a polyhedron which satisfies this propriety, the smallest possible one containing Ω.

For a nonsymmetrical polyhedra, such a smallest set may not exist. This happens because the intersection of two $\mathcal{D}$-(C, A)-invariant polyhedra is not necessarily $\mathcal{D}$-(C, A)-invariant. For symmetrical polyhedra, it is not clear whether this property holds or not. In our experiments to date we could not find a counter-example either.

Nevertheless, a polyhedron which is $\mathcal{D}$-(C, A)-λ-contractive and results in a suitable limitation of the observation error can be computed. To this end, the following algorithm is proposed (**Algorithm I**):

Given: $\Omega = \{e : \ |Qe| \leq \mathbf{1}\}$, the initial set of estimation errors; λ, the desired contraction rate.

(1) Define the tolerance $\Delta\delta$. Initialize $i = 0$, $Q^0 = Q$, $Q^0 \in \mathbb{R}^{q^0 \times n}$, $\mathcal{C}^0 = \{e : \ Q^0 e \leq \mathbf{1}\}$
(2) Compute the vertices of $\mathcal{C}^i$, e^{i^j};
(3) Compute the set $\mathcal{Z}_d(\mathcal{C}^i)$; Set n_{z_i} equal to the cardinality of $\mathcal{Z}_d(\mathcal{C}^i)$.
(4) For $l = 1, ..., n_{z_i}$:
    (a) Compute $\phi^{i^l}(Ce^l)$ from (6);
    (b) Compute $\varepsilon(Ce^l)$ from (10);
(5) Set $\varepsilon^i = \max_l \varepsilon^l$; Set v^i and e^i as the optimal values of v and e in (10), associated to ε^i;
(6) If $\varepsilon^i \leq \lambda(1 + \Delta\delta)$, STOP! $\mathcal{C}^i = \{e : \ |Q^i e| \leq \mathbf{1}\}$ is $\mathcal{D}$-(C, A)-λ-contractive;
(7) Compute the set:

$$\begin{aligned} \mathcal{Q}^i = \{ & x : x = Ae + B_1 d + v^i, \text{ for some } e, d \\ & \text{such that } |Q^i e| \leq \mathbf{1}, |Ce - Ce^i| \leq \bar{\eta}, \\ & |Ed| \leq \mathbf{1}\}. \end{aligned}$$

(8) Compute $\mathcal{C}^{i+1} = \mathrm{Conv}(\mathcal{C}^i \cup \frac{1}{\lambda}\mathcal{Q}^i)$
(9) Do $i = i+1$ and return to step 2.

The key point of this algorithm is step 8. It picks up the worst case ε, computes the corresponding optimal v which tries to place the set $\mathcal{Q}^i$ (the one-step propagation of all possible points e associated to the output z^i) inside $\mathcal{C}^i$. If it succeeds, then $\mathcal{C}^i$ is $\mathcal{D}$-(C,A)-λ-contractive. Otherwise, another candidate set is computed through the convex hull of the union of $\mathcal{C}^i$ and $\frac{1}{\lambda}\mathcal{Q}^i$. Even though the convergence of this algorithm has not been proved, no example for which it does not converge has been found.

As mentioned before, in general, this algorithm does not generate the smallest $\mathcal{D}$-(C,A)-λ-contractive polyhedron containing Ω. For symmetrical polyhedra, however, it is possible to guarantee, in some situations, the existence of this smallest polyhedron. Consider then the following class of polyhedra which satisfy the necessary condition (11):

$$\mathcal{K}(\Omega,\mathcal{D},\lambda) = \{\text{set of symmetrical polyhedra containing } \Omega \text{ such that } \phi_q(0) \le \lambda\mathbf{1} - \xi\}.$$

Lemma 4.1. The intersection of two polyhedra belonging to $\mathcal{K}(\Omega,\mathcal{D},\lambda)$ also belongs to $\mathcal{K}(\Omega,\mathcal{D},\lambda)$.

Proof: the condition $\phi_q(0) \le \lambda\mathbf{1} - \xi$ is equivalent to $|Q(Ae + B_1 d)| \le \lambda\mathbf{1}, \forall e : |Qe| \le \mathbf{1}, |Ce| \le \bar{\eta}$, $\forall d : |Ed| \le \mathbf{1}$. Then, it is quite straightforward to verify that the condition $|Q(Ae + B_1 d)| \le \lambda\mathbf{1}$ will remain satisfied if another set of constrains, say, $|Q_n e| \le \mathbf{1}$ is added to $|Qe| \le \mathbf{1}$ (corresponding to the intersection of $\{|Qe| \le \mathbf{1}\}$ and $\{|Q_n e| \le \mathbf{1}\}$.□

This Lemma assures the existence of the set:

$$\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},\lambda) = \text{ infimal set in } \mathcal{K}(\Omega,\mathcal{D},\lambda),$$

which is the smallest set containing Ω satisfying the necessary condition $\phi_q(0) \le \lambda\mathbf{1} - \xi$. Therefore, if $\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},\lambda)$ satisfies the sufficient condition too, it can be assured that it is the smallest $\mathcal{D}$-(C,A)-λ-contractive set containing Ω.

$\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},\lambda)$ can be computed by means of a simplified algorithm, which will be called **Algorithm II**, with the following main modifications with respect to Algorithm I:

- replace steps 2 to 5 by *compute* $\phi_q(0)$ *and* $\varepsilon = \max_i \phi_{q_i}$;
- $\mathcal{Q}^i = \{x : x = Ae + B_1 d, \text{ for some } e, d \text{ such that } |Q^i e| \le \mathbf{1}, |Ce| \le \bar{\eta}, |Ed| \le \mathbf{1}\}$.

The computational burden associated to Algorithm II is much smaller than the one of Algorithm I, especially because it is not necessary to compute $\phi(.)$ in all the vertices. The following procedure can then be proposed for symmetrical polyhedra: compute $\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},\lambda)$ through algorithm II; if $\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},\lambda)$ is $\mathcal{D}$-(C,A)-λ-contractive, STOP: it is also the smallest $\mathcal{D}$-(C,A)-λ-contractive set containing Ω. Otherwise, compute another $\mathcal{D}$-(C,A)-λ-contractive polyhedron containing $\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\lambda)$ from algorithm I.

Concerning the implementation of algorithms I and II, it is necessary to solve linear programming problems and to manipulate polyhedra (to compute vertices and convex hulls). Several methods are available for such computations (see, e.g. (Schrijver, 1987; Avis and Fukuda, 1992)).

5. NUMERICAL EXAMPLE

The proposed method is now applied to a system for which a set-valued observer was synthesized in (Shamma, 1999). The system matrices are:

$$A = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The symmetrical polyhedron $\Omega_x = \{x : |Qx| \le \mathbf{1}\}$ which represents the uncertainty about the initial state $x(0)$ is represented by: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The disturbance and noise sets are given respectively by: $\mathcal{D} = \{d : |d| \le 1\}$ and $\mathcal{N} = \{\eta : |\eta| \le 1\}$.

The goal is to design a dynamic state observer (2) such that the estimation error is limited as much as possible.

Initializing the observer with $\hat{x}(0) = 0$, the initial estimation error $e(0)$ belongs to the polyhedron $\Omega = \{e : |Qe| \le \mathbf{1}\}$. The necessary condition (11) is not verified, thus Ω is not $\mathcal{D}$-(C,A)-invariant. Then, a $\mathcal{D}$-(C,A)-λ-invariant polyhedron containing Ω must be computed. Through the application of Algorithm II for $\lambda = 0,9$ and $\Delta\delta = 10^{-5}$, after 7 iterations, the polyhedron $\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},\lambda) = \{e : |Q^7 e| \le \mathbf{1}\}$ is obtained, with
$$Q^7 = \begin{bmatrix} 0 & 0.2944 \\ 0.5294 & 0 \\ -0.3403 & -0.3403 \end{bmatrix}.$$

$\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},0.9)$ satisfies the sufficient condition (8), therefore it is $\mathcal{D}$-(C,A)-λ-contractive and it is not necessary to run Algorithm I. Hence, the optimal error limitation is achieved.

In figure 1, the polyhedron $\mathcal{C}^\infty_{\mathcal{K}}(\Omega,\mathcal{D},0.9)$ and the initial polyhedron Ω are shown, together with two trajectories: one computed by the proposed $\mathcal{D}$-(C,A)-invariance approach and the other generated by the set-valued observer proposed in (Shamma, 1999), for $x(0) = 0$ and the following disturbance and noise sequences:

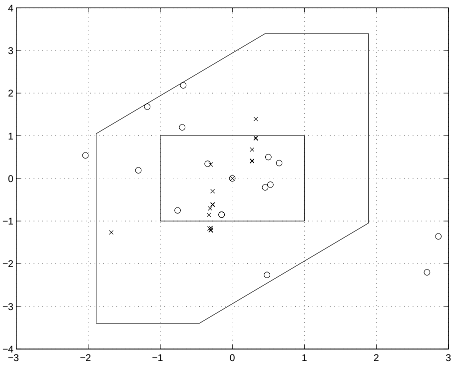

Fig. 1. Ω (.) and $\mathcal{C}_{\mathcal{K}}^{\infty}(\Omega, \mathcal{D}, 0.9)$. Trajectories for the $\mathcal{D}$-(C, A)-invariant (x) and the set-valued (o) estimators.

$$\{d\} = \{1, -1, -1, 1, 1, -1, 1, -1, -1, \\ -1, 1, -1, 1, -1, -1, -1, 1\},$$

$$\{\eta\} = \{1, -1, 1, -1, 1, -1, 1, -1, -1, \\ -1, 1, -1, 1, -1, -1, -1, 1\}.$$

The output injection was computed "on-line" from the following LP:

$$\min_{\varepsilon, v} \ \varepsilon$$
$$\text{under: } \phi(z) + Gv \leq \varepsilon \mathbf{1} - \delta.$$

We believe however that the piece-wise affine output injection law proposed in (Pimenta and Dórea, 2004), which can be computed off-line, can be easily extended to the system dealt with here.

As depicted in Figure 1, the set-valued observer generated errors further from zero than the proposed $\mathcal{D}$-(C, A)-invariant estimator,

6. CONCLUSIONS

In this work a new approach for the design of full-order state estimators for discrete-time linear systems subject to persistent disturbances and measurement noise was presented. Based on the concepts of set-invariance, it has been shown that the estimation error can be forced to remain inside a polyhedral set by means of a suitable output injection. Two algorithms were proposed with the objective of limiting as much as possible the error. In a particular case, it has been shown that the optimal limitation can be achieved, that is, the error can be confined to the smallest $\mathcal{D}$-(C, A)-invariant polyhedron which contains the polyhedron the initial error is known to belong to.

Compared to set-valued observers, we believe that the set-invariant estimator has some important advantages:

- it is able to impose a limitation to the estimation error. In many cases the optimal limitation can be achieved;
- the $\mathcal{D}$-(C, A)-invariant set is computed off-line, resulting in less on-line calculation;
- in the case of systems without disturbances and noise, the contraction rate λ can be used to accelerate the convergence of the error to zero.

Some important points, which are presently under investigation, still have to be clarified such as: the convergence of Algorithm I, the study of conditions under which the smallest $\mathcal{D}$-(C, A)-invariant set can be computed and the extension of the results to multiple-output systems.

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