NONLINEAR $\mathcal{H}_\infty$ CONTROLLERS FOR UNDERACTUATED COOPERATIVE MANIPULATORS

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Abstract: In this paper, two nonlinear $\mathcal{H}_\infty$ control techniques are used to solve the position control problem of underactuated cooperative manipulators. The first technique consists in representing the nonlinear system in a quasi-Linear Parameter Varying form. In the second technique, the game theory gives an explicit solution of the $\mathcal{H}_\infty$ control of manipulators. The control of the squeeze force between the manipulator end-effectors and the object is also evaluated. Results obtained from an actual cooperative manipulator, that it is able to work as a fully actuated and an underactuated manipulator, are presented.

Keywords: Underactuated cooperative manipulators, nonlinear $\mathcal{H}_\infty$ control.

1. INTRODUCTION

Robotic systems consisting of two or more manipulators transporting an object are denominated cooperative manipulators. In (Wen and Kreutz-Delgado, 1992), it is proposed a control paradigm for cooperative manipulators, where the position and force controls are decomposed and can be designed independently. The applied force between the manipulator end-effectors and the object are decomposed in motion force (force generated by the system movement) and squeeze force. It is shown that only the squeeze force must be controlled, since the motion force goes to zero if the position control is stable, (Wen and Kreutz-Delgado, 1992).

Cooperative manipulators, as well as individual manipulators, are subject to parametric uncertainties and external disturbances. In (Lian *et al.*, 2002), an adaptive Fuzzy controller with $\mathcal{H}_\infty$ performance is developed for cooperative manipulators, the dynamic equation used in this reference is derived from the order reduction procedure proposed in (Mcclamroch and Wang, 1988) for constraint manipulators.

This paper deals with robust control of underactuated cooperative manipulators. The underactuation can be a result of failures on the actuators, taking the manipulator to a free-joint configuration. Two nonlinear $\mathcal{H}_\infty$ control techniques are used to attenuate the effects of the parametric uncertainties and external disturbances in the object: the $\mathcal{H}_\infty$ control for linear parameter varying (LPV) systems (Wu *et al.*, 1996), Section 3, and the $\mathcal{H}_\infty$ control based on the game theory (Chen *et al.*, 1994), Section 5. See details of these controllers, applied to the individual underactuated manipulator UArm II, in (Siqueira and Terra, 2004). Here, two manipulators UArm II are connected to an object to form an underactuated cooperative manipulator. The experimental results obtained with the nonlinear $\mathcal{H}_\infty$ control techniques are compared based on performance indexes, Section 6.

The robust control strategies proposed in this paper are also compared with the controller presented in (Tinós and Terra, 2002), where the hybrid position force controller proposed in (Wen and Kreutz-Delgado, 1992) is extended to underactuated cooperative manipulators.

2. COOPERATIVE MANIPULATORS

2.1 Fully actuated cooperative manipulators

Consider a cooperative manipulator consisting of m manipulators, each one with n degrees of freedom. Let $q_i \in \Re^n$ be the vector of generalized coordinates of manipulator i and $x_o \in \Re^n$ the vector of Cartesian coordinates of the object. The geometric constraints are given by $\varphi_i(x_o, q_i) = 0$ for $i = 1, 2, \cdots, m$. Denote $J_{o_i}(x_o, q_i)$ and $J_i(x_o, q_i)$ the Jacobian matrices of $\varphi_i(x_o, q_i)$ with relation to x_o and q_i, that is, $J_{o_i}(x_o, q_i) = \partial\varphi_i/\partial x_o$ and $J_i(x_o, q_i) = \partial\varphi_i/\partial q_i$, respectively. Hence, the velocity constraint are given by $\dot{\varphi}_i(x_o, q_i) = J_{o_i}(x_o, q_i)\dot{x}_o + J_i(x_o, q_i)\dot{q}_i = 0$, $i = 1, 2, \cdots, m$. Assume that the relation

$$\dot{q}_i = -J_i^{-1}(x_o, q_i)J_{o_i}(x_o, q_i)\dot{x}_o,$$

for $i = 1, 2, \cdots, m$, can always be computed. Then, the kinematic constraints are expressed by

$$\dot{\theta} = \begin{bmatrix} I_n \\ -J^{-1}(x_o)J_o(x_o) \end{bmatrix} \dot{x}_o \equiv B(x_o)\dot{x}_o, \quad (1)$$

where $\theta = [x_o^T \; q_1^T \; \cdots \; q_m^T]^T$, $J(x_o) = diag[J_1(x_o, q_1), \cdots, J_m(x_o, q_m)]$, $diag[A, \cdots, Z]$ is a block-diagonal matrix, and $J_o(x_o) = [J_{o_1}^T(x_o, q_1) \; \cdots \; J_{o_m}^T(x_o, q_m)]^T$. The dynamic equation of the object is given by

$$M_o(x_o)\ddot{x}_o + C_o(x_o, \dot{x}_o)\dot{x}_o + g_o(x_o) = J_o^T(x_o)h, \quad (2)$$

where $M_o(x_o)$ is the inertial matrix, $C_o(x_o, \dot{x}_o)$ is the Coriolis and centripetal matrix, $g_o(x_o)$ is the gravitational torque vector and $h = [h_i^T \cdots h_m^T]^T$, with $h_i \in \Re^n$, the applied force by the manipulator i in the object. The dynamic equation of the manipulator i is given by

$$M_i(q_i)\ddot{q}_i + C_i(q_i, \dot{q}_i)\dot{q}_i + g_i(q_i) = \tau_i + J_i^T(x_o, \Omega_i(x_o))h_i, \quad (3)$$

where $M_i(q_i)$ is the inertial matrix, $C_i(q_i, \dot{q}_i)$ is the Coriolis and centripetal matrix, $g_i(q_i)$ is the gravitational torque vector, and τ_i is the applied torque, of manipulator i. Then, the dynamic equation of the cooperative manipulator can be represented as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = \begin{bmatrix} 0 \\ \tau \end{bmatrix} + \begin{bmatrix} J_o^T(x_o) \\ J^T(x_o) \end{bmatrix} h, \quad (4)$$

where $M(\theta) = diag[M_o(x_o), M_1(q_1), \cdots, M_m(q_m)]$, $C(\theta, \dot{\theta}) = diag[C_o(x_o, \dot{x}_o), C_1(q_1, \dot{q}_1), \cdots, C_m(q_m, \dot{q}_m)]$, $g(\theta) = [g_o(x_o)^T \; g_1(q_1)^T \; \cdots \; g_m^T(q_m)]^T$ and $\tau = [\tau_1^T \; \cdots \; \tau_m^T]^T$.

Let h_o be the projection of h on the object frame C (frame fixed on the center of mass of the object), $h_o = J_{oq}^T(x_o)h$, with $J_{oq}(x_o) = diag[J_{o_1}(x_o, q_1), \cdots, J_{o_m}(x_o, q_m)]$. The resulting force on the object, $h_{ro} = J_o^T(x_o)h$, can be rewritten as

$$h_{ro} = A^T J_{oq}^T(x_o)h = A^T h_o,$$

where $A^T = [I_n \; I_n \; \cdots \; I_n] \in \Re^{n \times (nm)}$. Since A^T is a nonsquare and full row-rank matrix, there exists a nontrivial null space, denoted squeeze subspace X_S, given by $X_S = \{h_{oS} \in \Re^{nm} | A^T h_{oS} = 0\}$. If h_o belongs to the null space X_S, the resulting force has no contribution to the object movement. Define the following orthogonal decomposition of the projection of the applied force: $h_o = h_{oS} + h_{oM}$, where h_{oS} is the projection of h_o in X_S, named squeeze force, and h_{oM} the forces induced by the system movement, named motion force. Considering this decomposition, the dynamic equation of the cooperative manipulator, (4), can be represented as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = \tau_v + \overline{A}^T(x_o)h_{oS}, \quad (5)$$

where τ_v is an auxiliary control input,

$$\tau_v = \begin{bmatrix} A^T h_{oM} \\ \tau + J^T(x_o)J_{oq}^{-T}(x_o)h_{oM} \end{bmatrix},$$

and $\overline{A}(x_o) = [A \; J_{oq}^{-1}(x_o)J(x_o)]$ a Jacobian matrix.

If the auxiliary control input is partitioned in two vectors, $\tau_{v1} = A^T h_{oM}$ and $\tau_{v2} = \tau + J^T(x_o)J_{oq}^{-T}(x_o)h_{oM}$, the applied torque can be computed by

$$\tau = \tau_{v2} - J^T(x_o)J_{oq}^{-T}(x_o)(A^T)^+\tau_{v1}, \quad (6)$$

where $(A^T)^+ = A(A^T A)^{-1}$ is the pseudo-inverse of A^T. The motion forces are given by $h_{oM} = (A^T)^+\tau_{v1}$. Hence, the control problem is to find an auxiliary control in order to guarantee stability and robustness against disturbances.

Considering the kinematic constraints (1) and multiplying the dynamic equation of the cooperative manipulator (5) by $B^T(x_o)$, ones obtains

$$\overline{M}(x_o)\ddot{x}_o + \overline{C}(x_o, \dot{x}_o)\dot{x}_o + \overline{g}(x_o) = \overline{\tau}_v, \quad (7)$$

where $\overline{M}(x_o) = B^T(x_o)M(x_o)B(x_o)$, $\overline{C}(x_o, \dot{x}_o) = B^T(x_o)M(x_o)\dot{B}(x_o) + B^T(x_o)C(x_o, \dot{x}_o)B(x_o)$, $\overline{g}(x_o) = B^T(x_o)g(x_o)$ and $\overline{\tau}_v = B^T(x_o)\tau_v$.

2.2 Underactuated cooperative manipulators

Consider that the joints of the cooperative system are formed by n_a are active joints (with actuators) and n_p passive joins (without actuators). The kinematic constraints (1) can be rewritten as

$$\dot{\widetilde{\theta}} = \begin{bmatrix} I_n \\ -J_{AP}^{-1}(x_o)J_o(x_o) \end{bmatrix} \dot{x}_o \equiv \widetilde{B}(x_o)\dot{x}_o, \quad (8)$$

where $\widetilde{\theta} = [x_o^T \; q_a^T \; q_p^T]^T$, $q_a \in \Re^{n_a}$ is the position vector of active joints, $q_p \in Re^{n_p}$ is the position vector of passive joints and $J_{AP}(x_o)$ is a Jacobian matrix generated from the orthogonal permutation matrix P_{AP}, (Tinós and Terra, 2002). Therefore, if $\widetilde{q} = [q_a^T \; q_p^T]^T = P_{AP}[q_1^T \; q_2^T \; \cdots \; q_m^T]^T$, then $J_{AP}(x_o) = [J_a(x_o) \; J_p(x_o)] = J(x_o)P_{AP}$.

The dynamic equation of underactuated cooperative manipulators can be given by

$$\widetilde{M}(\widetilde{\theta})\ddot{\widetilde{\theta}} + \widetilde{C}(\widetilde{\theta}, \dot{\widetilde{\theta}})\dot{\widetilde{\theta}} + \widetilde{g}(\widetilde{\theta}) = \begin{bmatrix} 0 \\ \tau_a \\ 0 \end{bmatrix} + \begin{bmatrix} J_o^T(x_o) \\ J_a^T(x_o) \\ J_p^T(x_o) \end{bmatrix} h, \quad (9)$$

where $\widetilde{M}(\widetilde{\theta}) = diag[M_o(x_o), M_{AP}(\widetilde{q})]$, $M_{AP}(\widetilde{q}) = P_{AP} diag[M_1(q_1), \cdots, M_m(q_m)]\ P_{AP}^T$, $\widetilde{C}(\widetilde{\theta}, \dot{\widetilde{\theta}}) = diag[C_o(x_o, \dot{x}_o), C_{AP}(\widetilde{q}), \dot{\widetilde{q}}]$, $C_{AP}(\widetilde{q}, \dot{\widetilde{q}}) = P_{AP} diag[C_1(q_1, \dot{q}_1), \cdots, C_m\ (q_m, \dot{q}_m)] P_{AP}^T$, $\widetilde{g}(\widetilde{\theta}) = [g_o(x_o)^T\ g_{AP}(\widetilde{q})^T)]^T$ and $g_{AP} = P_{AP}[g_1^T(q_1)\ \cdots\ g_2^T(q_2)]^T$.

Consider the decomposition of the projection of the applied force on the squeeze and motion subspaces

$$\widetilde{M}(\widetilde{\theta})\ddot{\widetilde{\theta}} + \widetilde{C}(\widetilde{\theta}, \dot{\widetilde{\theta}})\dot{\widetilde{\theta}} + \widetilde{g}(\widetilde{\theta}) = \tau_v + \widetilde{A}^T(x_o) h_{oS}, \quad (10)$$

where τ_v is an auxiliary control input,

$$\tau_v = \begin{bmatrix} A^T h_{oM} \\ \tau_a + J_a^T(x_o) J_{oq}^{-T}(x_o) h_{oM} \\ J_p^T(x_o) J_{oq}^{-T}(x_o) h_{oM} \end{bmatrix},$$

and $\widetilde{A}(x_o) = [A\ J_{oq}^{-1}(x_o) J_a(x_o)\ J_{oq}^{-1}(x_o) J_p(x_o)]$ a Jacobian matrix. If the auxiliary control input is partitioned in three vectors, $\tau_{v1} = \widetilde{A}^T(x_o) h_{oM}$, $\tau_{v2} = \tau_a + J_a^T(x_o) J_{oq}^{-T}(x_o) h_{oM}$ and $\tau_{v3} = \tau_a + J_a^T(x_o) J_{oq}^{-T}(x_o) h_{oM}$, the applied torques in the active joints can be computed as

$$\tau_a = \tau_{v2} - J_a^T(x_o) J_{oq}^{-T}(x_o) \begin{bmatrix} A^T \\ J_p^T(x_o) J_{oq}^{-T}(x_o) \end{bmatrix}^+ \begin{bmatrix} \tau_{v1} \\ \tau_{v3} \end{bmatrix}. \quad (11)$$

Considering the kinematic constraints (8) and multiplying (10) by $\widetilde{B}^T(x_o)$, the dynamic equation of the underactuated cooperative manipulator is given by

$$\widetilde{M}(x_o)\ddot{x}_o + \widetilde{C}(x_o, \dot{x}_o)\dot{x}_o + \widetilde{g}(x_o) = \widetilde{\tau}_v, \quad (12)$$

where $\widetilde{M}(x_o) = \widetilde{B}^T(x_o)\widetilde{M}(\widetilde{\theta})\widetilde{B}(x_o)$, $\widetilde{C}(x_o, \dot{x}_o) = \widetilde{B}^T(x_o)\left(\widetilde{M}(\widetilde{\theta})\dot{\widetilde{B}}(x_o) + \widetilde{C}(\widetilde{\theta}, \dot{\widetilde{\theta}})\widetilde{B}(x_o)\right)$, $\widetilde{g}(x_o) = \widetilde{B}^T(x_o)\widetilde{g}(\widetilde{\theta})$, $\widetilde{\tau}_v = \widetilde{B}^T(x_o)\tau_v$.

From the control paradigm introduced in (Wen and Kreutz-Delgado, 1992) for cooperative manipulators, the position and squeeze force control problems can be decomposed and solved independently. In this case, the applied torques can be computed by

$$\tau = \tau_P + \tau_S,$$

where τ_P are the torques generated by the position control and τ_S are torques generated by the squeeze force control. In this paper, τ_P are given by (6) and (11) for fully actuated and underactuated manipulators, respectively. In Sections 3 and 5, the dynamic equations (7) and (12) are used to design robust controllers for position control of cooperative manipulators, considering parametric uncertainties and external disturbances in the manipulator and in the object.

2.3 Squeeze force control

For the squeeze force control, (Wen and Kreutz-Delgado, 1992) proposed the utilization of an integral controller. For fully actuated manipulators, the applied torques related with the squeeze force control are given by

$$\tau_S = D^T(x_o)\left[h_{oS}^d + K_i \int (h_{oS}^d - h_{oS}) dt\right], \quad (13)$$

where $D(x_o) = diag[J_{o_1}^{-1}(x_o, \Omega_1(x_o)) J_1(x_o, \Omega_1(x_o)), \cdots, J_{o_m}^{-1}(x_o, \Omega_m(x_o)) J_m(x_o, \Omega_m(x_o))]$, h_{oS}^d are the desired squeeze force and K_i is a positive definite matrix.

The dimension of h_{oS} is nm and, since the dimension of X_S is equal to $n(m-1)$, it is possible to write $h_{oS} = \widehat{A}^T \lambda_S$. Hence, the $n(m-1)$-dimensional vector λ_S are now the variables to be controlled. For the underactuated cooperative manipulator, n_p constraints are imposed in the components of λ_S. As the manipulators considered here are nonredundant ones, not all the components of λ_S can be independently controlled, (see (Tinós and Terra, 2002) for more details).

If the vector λ_S is partitioned in λ_{Sc}, the independently controlled components, and λ_{Sn}, the not controlled components, the squeeze force controller is given by

$$\lambda_{Sc} = \left[\lambda_{Sc}^d + K_{i_S} \int (\lambda_{Sc}^d - \lambda_{Sc}) dt\right] \quad (14)$$

where λ_{Sc}^d is the desired value for λ_{Sc} and K_{i_S} is a positive definite matrix. λ_{Sn} is computed from the constraints as function of λ_{Sc}. The applied torques in the active joints related with the squeeze force control are given by

$$\tau_{Sa} = D_a^T(x_o)\widehat{A}^T \lambda_{Sc}. \quad (15)$$

3. $\mathcal{H}_\infty$ CONTROL OF LPV SYSTEMS

Consider the following state feedback control problem for Linear Parameter Varying (LPV) systems

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & 0 & 0 \\ C_2(\rho) & 0 & I \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \quad (16)$$

where x is the state, u is the control input, w is the external input, and z_1 and z_2 are the output variables.

Assume that the underlying parameter $\rho(x)$ varies in the allowable set

$$F_P^\nu = \left\{\rho \in \mathcal{C}^1(\Re^+, \Re^k) : \rho(t) \in P,\ |\dot{\rho}_i| \le \nu_i\right\}$$

for $i = 1, \ldots, k$, where $P \subset \Re^k$ is a compact set, and $\nu = [\nu_1 \cdots \nu_k]^T$ with $\nu_i \ge 0$. The system (16) has $\mathcal{L}_2$ gain $\le \gamma$ if

$$\int_0^T \|z(t)\|^2 dt \le \gamma^2 \int_0^T \|w(t)\|^2 dt \quad (17)$$

for all $T \ge 0$ and all $w \in \mathcal{L}_2(0, T)$, with $x(0) = 0$.

We want to find a continuous function $F(\rho)$ such that the closed loop system has an $\mathcal{L}_2$ gain less than or equal to γ under a state feedback law $u = F(\rho)x$.

Lemma 1. **(Wu *et al.*, 1996)** If there exists a continuously differentiable matrix function $X(\rho(x)) > 0$ that satisfy

$$\begin{bmatrix} E(\rho) & X(\rho)C_1^T(\rho) & B_1(\rho) \\ C_1(\rho)X(\rho) & -I & 0 \\ B_1^T(\rho) & 0 & -\gamma^2 I \end{bmatrix} < 0 \quad (18)$$

where

$$E(\rho) = -\sum_{i=1}^{m} \pm \left(\nu_i \frac{\partial X}{\partial \rho_i} \right) - B_2(\rho)B_2^T(\rho) + \widehat{A}(\rho)X(\rho) + X(\rho)\widehat{A}(\rho)^T$$

and $\widehat{A}(\rho) = A(\rho) - B_2(\rho)C_2(\rho)$, then the closed loop system has $\mathcal{L}_2$ gain $\leq \gamma$ under the state feedback law

$$u(t) = -(B_2(\rho(t))X^{-1}(\rho(t)) + C_2(\rho(t)))x(t).$$

Hence, we have to solve a set of parametric Linear Matrix Inequalities (LMI), (18), that is an infinitesimal convex optimization problem. Fortunately, a practical scheme using basis functions for $X(\rho)$ and griding the parameter set P was developed in (Wu *et al.*, 1996) to solve this problem.

4. QUASI-LPV REPRESENTATION OF COOPERATIVE MANIPULATORS

Nonlinear systems can always be represented as LPV systems. However, in this case, the parameter ρ in (16) is not only function of the time, but also of the system states. This kind of systems is denominated quasi-LPV. The $\mathcal{H}_\infty$ control of LPV systems presented in Section 3 can be applied to a nonlinear system represented in a quasi-LPV form. In this section, we develops quasi-LPV representations of fully actuated and underactuated cooperative manipulators, based on the the following dynamic equations

$$\widehat{M}_0(x_o)\ddot{x}_o + \widehat{C}_0(x_o,\dot{x}_o)\dot{x}_o + \widehat{g}_0(x_o) + \widehat{\tau}_d = \widehat{\tau}_v \quad (19)$$

where $\widehat{M}_0(x_o) = \overline{M}_0(x_o)$, $\widehat{C}_0(x_o,\dot{x}_o) = \overline{C}_0(x_o,\dot{x}_o)$, $\widehat{g}_0(x_o) = \overline{g}_0(x_o)$ and $\widehat{\tau}_v = \overline{\tau}_v$ if the manipulators are fully actuated, (7); or $\widehat{M}_0(x_o) = \widetilde{M}_0(x_o)$, $\widehat{C}_0(x_o,\dot{x}_o) = \widetilde{C}_0(x_o,\dot{x}_o)$, $\widehat{g}_0(x_o) = \widetilde{g}_0(x_o)$ and $\widehat{\tau}_v = \widetilde{\tau}_v$ if any of the manipulators is underactuated, (12). The index 0 indicates nominal values for the matrices and vectors. $\widehat{\tau}_d$ are parametric uncertainties and external disturbances.

The state tracking error is defined as

$$\widetilde{x} = \begin{bmatrix} \dot{x}_o - \dot{x}_o^d \\ x_o - x_o^d \end{bmatrix} = \begin{bmatrix} \dot{\widetilde{x}}_o \\ \widetilde{x}_o \end{bmatrix} \quad (20)$$

where x_o^d and $\dot{x}_o^d \in \Re^n$ are the desired reference trajectory and the desired velocity, respectively. The quasi-LPV representation of cooperative manipulators is found using (19) and (20)

$$\dot{\widetilde{x}} = A(x_o,\dot{x}_o)\widetilde{x} + Bu + Bw \quad (21)$$

with $w = \widehat{M}_0^{-1}(x_o)\widehat{\tau}_d$, $B = [I_n^T \;\; 0^T]^T$, and

$$A(x_o,\dot{x}_o) = \begin{bmatrix} -\widehat{M}_0^{-1}(x_o)\widehat{C}_0(x_o,\dot{x}_o) & 0 \\ I_n & 0 \end{bmatrix}.$$

From the above equation, the variable $\widehat{\tau}_v$ can be represented as

$$\widehat{\tau}_v = \widehat{M}_0(x_o)(\ddot{x}_o^d + u) + \widehat{C}_0(x_o,\dot{x}_o)\dot{x}_o^d + \widehat{g}_0(x_o).$$

Although the matrix $\widehat{M}_0(x_o)$ explicitly depends on the object position, x_o, we can consider it as function of the position error, $\widetilde{x}_o$. The same can be observed for $\widehat{C}_0(x_o,\dot{x}_o)$. Hence, (21) is a quasi-LPV representation of fully actuated and underactuated cooperative manipulators.

5. NONLINEAR $\mathcal{H}_\infty$ CONTROL VIA GAME THEORY

In this section, we use game theory to solve the $\mathcal{H}_\infty$ control problem of cooperative manipulators. This solution is based on the results presented in (Chen *et al.*, 1994). From (20), after the state transformation given by

$$\widetilde{z} = \begin{bmatrix} \widetilde{z}_1 \\ \widetilde{z}_2 \end{bmatrix} = T_0\widetilde{x} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\widetilde{x}}_o \\ \widetilde{x}_o \end{bmatrix} \quad (22)$$

where $T_1 = [T_{11} \;\; T_{12}]$ with T_{11}, $T_{12} \in \Re^{n\times n}$ constant matrices to be determined, the dynamic equation of the state tracking error becomes

$$\dot{\widetilde{x}} = A_T(\widetilde{x},t)\widetilde{x} + B_T(\widetilde{x},t)u + B_T(\widetilde{x},t)w \quad (23)$$

with $w = \widehat{M}_0(x_o)T_{11}\widehat{M}_0^{-1}(x_o)\widehat{\tau}_d$,

$$A_T(\widetilde{x},t) = T_0^{-1}\begin{bmatrix} -\widehat{M}_0^{-1}(x_o)\widehat{C}_0(x_o,\dot{x}_o) & 0 \\ T_{11}^{-1} & -T_{11}^{-1}T_{12} \end{bmatrix} T_0,$$

$$B_T(\widetilde{x},t) = T_0^{-1}\begin{bmatrix} \widehat{M}_0^{-1}(x_o) \\ 0 \end{bmatrix}.$$

The relationship between the auxiliary control input, $\widehat{\tau}_v$, and the control input, u, is given by

$$\widehat{\tau}_v = \widehat{M}_0(x_o)\ddot{x}_o^c + \widehat{C}_0(x_o,\dot{x}_o)\dot{x}_o + \widehat{g}_0(x_o) \quad (24)$$

with $\ddot{x}_o^c = \ddot{x}_o^d - T_{11}^{-1}T_{12}\dot{\widetilde{x}}_o - T_{11}^{-1}\widehat{M}_0^{-1}(x_o)(\widehat{C}_0(x_o,\dot{x}_o) B^T T_0\widetilde{x} - u)$.

The $\mathcal{H}_\infty$ control problem for a manipulator seeks the attenuation of the effects of the disturbance w in the system by a state feedback control strategy of the form $u = F(\tilde{x})\tilde{x}$. With this intention, and subject to the tracking error dynamics, the following performance criterion, including a desired disturbance attenuation level γ, is proposed in (Chen *et al.*, 1994)

$$\min_{u\in\mathcal{L}_2} \max_{0\neq w\in\mathcal{L}_2} \frac{\int_0^\infty \left(\frac{1}{2}\widetilde{x}^T Q\widetilde{x} + \frac{1}{2}u^T Ru\right) dt}{\int_0^\infty \left(\frac{1}{2}w^T w\right) dt} \leq \gamma^2 \quad (25)$$

where Q and R are positive definite symmetric weighting matrices and $\widetilde{x}(0) = 0$. According to game theory, the solution of this *minimax* problem is found if there exist matrices T_0 and K satisfying the following algebraic matrix equation

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} - T_0^T B \left(R^{-1} - \frac{1}{\gamma^2} I \right) B^T T_0 + Q = 0. \quad (26)$$

The solution of this equation is given in (Chen *et al.*, 1994). The optimal control is given by

$$u^* = -R^{-1} B^T T_0 \tilde{x}. \quad (27)$$

6. EXPERIMENTAL RESULTS

To validate the proposed $\mathcal{H}_\infty$ control solutions, we apply the techniques displayed in previous sections to our underactuated cooperative manipulator, composed of two underactuated manipulators UArm II (Underactuated Arm II), designed and built by H. Ben Brown, Jr. of Pittsburgh, PA, USA. The kinematic and dynamic parameters of the manipulator UArm II can be found in (Siqueira and Terra, 2004) . The parameters of the object can be found in (Tinós and Terra, 2002).

The following desired trajectory is defined to the object center of mass: to move along the line in the plane X-Y from $x_o(0) = [0.20m \;\; 0.35m \;\; 0°]^T$ to $x_o^d(T) = [0.25m \;\; 0.40m \;\; 0°]^T$, where $T = 5.0s$ is the trajectory duration time. The reference trajectory, $x_o^d(t)$, is a fifth degree polynomial. External disturbances were introduced to verify the robustness of the proposed controllers. The desired values for the squeeze force are $h_{oS}^d = 0$ and $\lambda_{Sc}^d = 0$. The gains of the integral controllers for the squeeze force control are $K_i = 0.9$ and $K_{i_S} = 0.9$.

6.1 Fully Actuated Configuration

To apply the algorithm described in Section 3, the manipulator needs to be represented as in (16). The parameters $\rho(\tilde{x})$ chosen are the state representing the object position and orientation errors, that is, $k = 3$ and $\rho(\widetilde{x}) = \widetilde{x}_o$.

The compact set P is defined as $\rho \in [-0.1, 0.1]m \times [-0.1, 0.1]m \times [-9, 9]°$. The parameter variation rate is bounded by $|\dot{\rho}| \leq [0.06m/s \; 0.06m/s \; 6°/s]$. The basis functions for $X(\rho)$ selected are: $f_1(\rho(\widetilde{x})) = 1$, $f_2(\rho(\widetilde{x})) = \widetilde{x}_{o_X}$, $f_3(\rho(\widetilde{x})) = \widetilde{x}_{o_Y}$ and $f_4(\rho(\widetilde{x})) = cos(\widetilde{x}_{o_\phi})$, where $\widetilde{x}_o = [\widetilde{x}_{o_X} \; \widetilde{x}_{o_Y} \; \widetilde{x}_{o_\phi}]$, $\widetilde{x}_{o_X}$ e $\widetilde{x}_{o_Y}$ are the X and Y coordinate errors of the object center of mass, respectively, and $\widetilde{x}_{o_\phi}$ is the orientation error. The parameter space was divided in $(L = 3)$. The best attenuation level found was $\gamma = 1.25$.

For the nonlinear $\mathcal{H}_\infty$ control designed via game theory, described in Section 5, the attenuation level found was $\gamma = 4.0$. The weighting matrices used were: $Q_1 = I_3$, $Q_2 = 10I_3$, $Q_{12} = 0$ e $R = I_3$.

Three performance indexes are used to compare the nonlinear $\mathcal{H}_\infty$ controllers: the $\mathcal{L}_2$ norm of the state vector,

$$\mathcal{L}_2[\widetilde{x}] = \left(\frac{1}{(t_r)} \int_0^{t_r} \|\widetilde{x}(t)\|_2^2 dt \right)^{\frac{1}{2}},$$

where $\|\cdot\|_2$ is the Euclidean norm, the sum of the applied torque for both manipulators,

$$E[\tau] = \sum_{j=0}^{m} \left(\sum_{i=0}^{n} \left(\int_0^{t_r} |\tau_i(t)| dt \right) \right),$$

and the sum of the squeeze force,

$$E[h_{oS}] = \sum_{i=0}^{nm} \left(\int_0^{t_r} |h_{oE_i}(t)| dt \right),$$

where t_r is spent time for the object to reach the desired position.

Table 1 shows the values of $\mathcal{L}_2[\widetilde{x}]$, $E[\tau]$ and $E[h_{oS}]$ computed with the results obtained from the implementation of the nonlinear $\mathcal{H}_\infty$ controllers, considering the fully actuated configuration.

Table 1. Fully actuated configuration.

Nonlinear $\mathcal{H}_\infty$	$\mathcal{L}_2[\widetilde{x}]$	$E[\tau]$ (Nms)	$E[h_{oS}]$ (Ns)
Quasi-LPV	0.01815	0.8318	0.2193
Game theory	0.01158	1.1200	0.3875

Note that the nonlinear $\mathcal{H}_\infty$ control via game theory presented the lowest trajectory tracking error, $\mathcal{L}_2[\widetilde{x}]$, although the spent energy, reflect in the calculation of $E[\tau]$, and the squeeze force are bigger with this controller in comparison with the nonlinear $\mathcal{H}_\infty$ control via quasi-LPV representation.

6.2 Underactuated configuration

In this section, it is considered that the joint 1 of the manipulator 1 is passive. In this case, only two components of the squeeze force can be controlled independently, (Tinós and Terra, 2002). It is defined here that the component of the squeeze force referring to the moment applied to the object will not be controlled.

The parameters $\rho(\widetilde{x})$, the variation rate bounds, and the functions used as base for $X(\rho)$ are the same used for the fully actuated case. The parameter space was divided in $L = 3$. The best level of attenuation found was $\gamma = 1.25$. The weighting matrices for the nonlinear $\mathcal{H}_\infty$ control via game theory were also the same defined to the fully actuated case. The level of attenuation used was $\gamma = 4.0$.

The experimental results are shown in Figs. 1 and 2. The performance indexes are shown in Table 2. Note that, in this case, the nonlinear $\mathcal{H}_\infty$ controller via game theory presented the lowest values of the trajectory tracking error and squeeze force control. The best value for the spent energy is given by the nonlinear $\mathcal{H}_\infty$ controller via quasi-LPV representation.

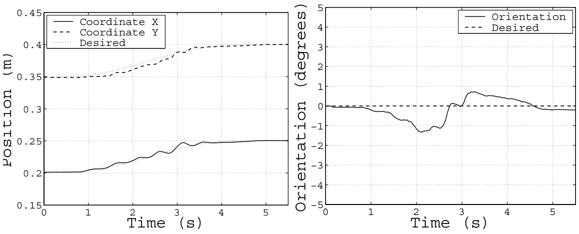

Figure 1. Underactuated configuration, control via quasi-LPV representation.

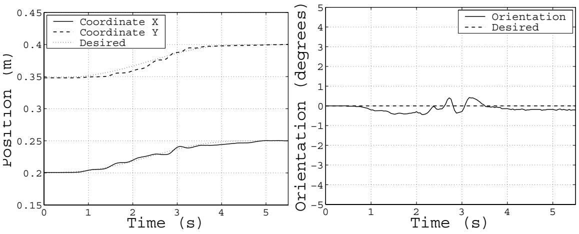

Figure 2. Underactuated configuration, control via game theory.

Table 2. Underactuated configuration.

Nonlinear $\mathcal{H}_\infty$	$\mathcal{L}_2[\widetilde{x}]$	$E[\tau]$ (Nms)	$E[h_{oS}]$ (Ns)
Quasi-LPV	0.0154	0.9976	0.4477
Game theory	0.0103	1.0609	0.3973

In Fig. 3 are presented the squeeze force components when the squeeze force control is applied (continuous line) and when it is not applied (dashed line) for the control via quasi-LPV representation. It can be observed that only the two components of the squeeze force related to the linear coordinates are controlled when the squeeze force control is applied, the are close to the desired values $\lambda_{Sc}^d = 0$. The component of the squeeze force related to the moment is not controlled in both cases, as described above.

For the case where the squeeze force is not controlled, the values of $\mathcal{L}_2[\widetilde{x}]$ and $E[\tau]$ are close to the values of Table 2. However, the values of $E[h_{oS}]$, given by 1.6319 Ns and 0.9250 Ns, for the controllers via quasi-LPV representation and via game theory, respectively, are, in average, three times bigger that the values of $E[h_{oS}]$ for the case where the squeeze force is controlled.

The same experiment was also implemented using the hybrid position force control for underactuated manipulators proposed in (Tinós and Terra, 2002). The performance indexes computed from the results are given by: $\mathcal{L}_2[\widetilde{x}] = 0.0128$, $E[\tau] = 1.7781$ and $E[h_{oS}] = 0.5741$. It can be observed that, although the value of $\mathcal{L}_2[\widetilde{x}]$ is lower than the obtained with the controller via quasi-LPV representation, the values of $E[\tau]$ and $E[h_{oS}]$ are approximately 70% and 40%, respectively, bigger than the values obtanied via nonlinear $\mathcal{H}_\infty$ controllers.

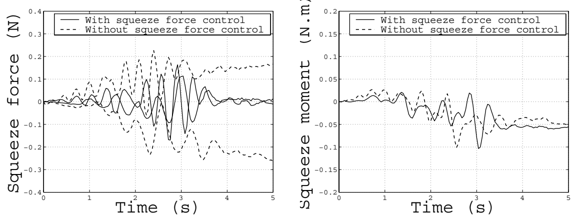

Figure 3. Squeeze force control.

7. CONCLUSIONS

In this work, experimental results obtained from the application of nonlinear $\mathcal{H}_\infty$ controls in an actual underactuated cooperative manipulator, subject to parametric uncertainties and external disturbances, are presented. From the computed performance indexes, the nonlinear $\mathcal{H}_\infty$ controllers presented better energy consumption and squeeze force control than the hybrid position force controller proposed in (Tinós and Terra, 2002) for underactuated manipulators.

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