TRUE CONCURRENT STOCHASTIC PROCESSES

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Abstract: In this work we present an algebraic framework for true concurrent stochastic processes. Concurrency is modelled using partial order relations. Markov processes are considered using an abstraction given by their excessive functions. Applications include embedded systems, as cardiac stimulators, encephalogram analysers and air traffic control systems.

Keywords: stochastic processes, true concurrency, reactive embedded systems.

1. INTRODUCTION

Embedded systems work in a real life environment, whose behaviour is highly unpredictable. In many situations, these behaviours are governed by (partial) differential equations that can be changed by discrete events (triggers). These behaviours are difficult to study by classical mathematical tools: solutions of partial equations are partial system evolutions, thus we can not derive conclusions on the global evolutions. A natural model of these systems constitutes *stochastic hybrid systems* (SHSs). The uncertainty is modelled using stochastic analysis and the alternate of continuous (partial) evolutions by the *hybrid system* model. We model hybrid systems as partial orders (modelling the changes in time) on an abstract set of behaviours (modelling both deterministic and stochastic continuous evolutions).

Practical motivations for this work include cardiac stimulators, encephalogram analysers and air traffic control systems. A cardiac stimulator record series of cardiac potentials and reacts when "dangerous" potentials appear. A encephalogram analyser has a similar behaviour, except that neuralgic potentials are involved. In (Bujorianu, M.C. and Bujorianu, M.L., 2002), all these biological potentials have characterised as mathematical structures, denoted as basic processes in Section 3. Air traffic management systems are inherently distributed and stochastic analysis has been recently applied in their study (Bujorianu, M.L., 2004), (Pola, G. *et al.*, 2003), (Bujorianu, M.L. and Lygeros, J., 2004). Distribution (or concurrency) is modelled following two different philosophies: *interleaving* (i.e. simultaneous executions are modelled by any arbitrary sequence of evolutions) and *true concurrency* (Best E. and Fernandez, C., 1990) (i.e. simultaneous executions are modelled by partial orders).

In this paper we present an algebraic framework for *true concurrent Markov processes*. The basic ingredients of this framework are the *causality relation*, modeled as partial order relations ($a \prec b$ means the event a is the cause of b) and an algebraic structure (called here *extended* *processes* - see Section 3) that can associated to Markov process in a standard way (see Example 1). Markov processes are abstracted using tools specific to stochastic analysis, like excessive functions (Boboc N. *et al.*, 1981) and Dirichlet forms (Fukushima, M., 1980). Two system evolutions a, b that are causal independent (i.e. $a \not\prec b$ nor $b \not\prec a$) can take place simultaneously (*true concurrency*).

The paper is structured as follows. In the next section we introduce the main algebraic notations and concepts. In Section 3 we define and investigate the basic interplay between multiform time and the abstract trajectories of a Markov process. Some tools of stochastic analysis, as the energy integral, are added to the framework in the following section. In the final section we discuss related and future work.

2. PRELIMINARIES

This section presents the main concepts regarding causal orders, i.e. partial order relations modelling the evolutions of elements of a set (that could be the support of an abstract mathematical structure, modelling the nature of real life events - see next Section) through time.

Let $\prec$ be an order relation on the set B. We shall use the notations

i) $\preccurlyeq = \prec \cup id_{|B}$;

ii) $\succ = \prec^{-1}$;

iii) $\lessdot = \prec - \prec^2$;

Let us define the following notations:

• $li = \prec \cup \succ \cup id_{|B}$

• $co = \overline{li} \cup id_{|B}$

• for any $b \in B$: ${}^{\blacklozenge}b = \{a \in B; a \prec b\}$, $b^{\blacklozenge} = \{a \in B; a \succ b\}$

• $l \subseteq B$ is a *li-set* iff $(\forall a, b \in l) : (a, b) \in li$

• $l \subseteq B$ is a *line* iff l is maximal wrt li: $(\forall a \in b - l)$, $(\exists b \in l) : (a, b) \in (B \times B) - li.$

Let $L = L(B)$ be the set of lines of B.

• $c \subseteq B$ is a *co-set* iff $(\forall a, b \in co) : (a, b) \in co$

• $c \subseteq B$ is a *cut* iff c is maximal wrt co : $(\forall a \in B - c)$, $(\exists b \in l) : (a, b) \in (B \times B) - co.$

Let $C = C(B)$ be the set of cuts of B

Remark 1. i) $(x\ li\ y)$ or $(x\ co\ y)$;

ii) $(x\ li\ y$ and $x\ co\ y) \Leftrightarrow x = y$;

iii) A is a line iff

(a) $(\forall x, y \in A) : x < y$ or $y < x$ or $x = y$;

(b) $(\forall\ x \in M - A)$, $(\exists y \in A) : \neg(x \prec y$ or $y \prec x)$;

iv) A is a cut iff

(a) $(\forall x, y \in A) : \neg(x \prec y$ or $y \prec x)$;

(b) $(\forall x \in M - A)$, $(\exists y \in A) : (x \prec y$ or $y \prec x)$.

A *Dedekind cut* (D-cut for short) is a partition $(A, \overline{A})$ for which $(\forall a \in A \forall b \in \overline{A}) : \neg(b \prec a)$. For $A \subset B$, define $M(A) = max(A) \cup \min(\overline{A})$.

If $(A, \overline{A})$ is a D-cut then define

• $Obmax(A)$ =: $\{\ a \in Max(A)\ ;\ \forall A' \in D(B)$ $\forall l \in L : a \in Max(A' \cap l) \Rightarrow a \in Max(A')\}$;

• $Obmin(A)$ =: $\{\ a \in Min(A)\ ;\ \forall A' \in D(B)$ $\forall l \in L : a \in Min(A' \cap l) \Rightarrow a \in Min(A')\}$;

• $c(A) = Obmax(A) \cup Ob\min(A)$.

Proposition 1. Let $A \in D(B)$ and $a \in Max(A)$, $b \in Min(\overline{A})$

• $a \notin Ob\max(A) \Leftrightarrow \exists c \in B\ \exists l \in L : a \prec c$ and $l \cap [a, c] = \{a\}$;

• $b \notin Ob\min(A) \Leftrightarrow \exists c \in B\ \exists l \in L : c \prec b$ and $l \cap [c, b] = \{b\}$.

A *complete lattice* is a partially ordered set in which every subset has a least upper bound and a greatest lower bound. A *conditionally complete lattice* is a lattice which have the property that every non-void bounded subset has a least upper bound and a greatest lower bound.

3. EXTENDED PROCESSES

In this section we present the mathematical model of true concurrent stochastic processes, namely the extended processes. We define first real spaces, the mathematical model of dynamics of the environment recorded by an embedded system. The elements of a real space are then decorated with elements of a basic space, a mathematical frame in which many biological potentials and dynamical systems can be defined.

Definition 1. A real space is defined as being a structure $< \mathbb{M}, \prec >$ such that

($\mathbf{M}_1$) $< \mathbb{M}, \prec >$ *is a lower complete semi-lattice. The order* $\prec$ *will be called the causal order. We shall note by* $\curlywedge$ (resp. $\curlyvee$) *the infimum (resp. supremum if exists) of this semi-lattice* and

($\mathbf{M}_2$) if $(\alpha_i)_{i\in I}$ is increasing and dominated in $\mathbb{M}$ by α, $\alpha \in \mathbb{M}$, then there exists $\underset{i\in I}{\curlyvee} \alpha_i$.

Definition 2. Let $\mathbb{D} \subseteq \mathbb{M}$.We call $\mathbb{D}$

- *dense in order from below* (in $\mathbb{M}$) if for any $\alpha \in \mathbb{M}$ we have $\alpha = \curlyvee\{\gamma \in \mathbb{D}; \gamma \preccurlyeq \alpha\}$;
- *increasingly dense* if the set $\{\gamma \in \mathbb{D}; \gamma \preccurlyeq \alpha\}$ is increasing to α for any $\alpha \in \mathbb{M}$;

Definition 3. *A basic space* is defined as being a structure $< \mathbb{S}, \leq, \perp, \top, \odot >$ where:

($\mathbf{S}_1$) $< \mathbb{S}, \leq, \perp, \top >$ *is a lattice for which:*

• $\perp$ *the minimal element and* $\top$ *the greatest element ;*

• *the lattice* $(\mathbb{S}\backslash\{\top\}, \leq_{|\mathbb{S}\backslash\{\top\}}, \perp)$ *is lower complete and upper conditionally complete ;*

• $\leq$ *will be called the essential order;*

we shall note by $\vee$ resp. $\wedge$ t*he supremum resp. infimum of this lattice;*

•$\perp$ *will be called the nil action*; $\top$ *will be called deadlock*;

($\mathbf{S}_2$) $(\mathbb{S}, \odot, \perp)$ is a monoid;

($\mathbf{S}_3$) $s = \perp$ if $s \odot s = \perp$ $(\forall s \in \mathbb{S})$;

($\mathbf{S}_4$) $s \odot \top = \top$ $(\forall s \in \mathbb{S})$;

($\mathbf{S}_5$) $s \odot (a \vee b) = (s \odot a) \vee (s \odot b)$ $(\forall a, b, s \in \mathbb{S})$;

($\mathbf{S}_6$) $a \odot b = (a \wedge b) \odot (a \vee b)$ $(\forall a, b \in \mathbb{S})$;

Definition 4. Two elements $a, b \in \mathbb{S}$ are called *strongly dual* if $a \wedge b = \perp$.

We note $a \in b^{\perp}$ if a and b are orthogonal and $a^{\perp} =: \{s \in \mathbb{S}; a\perp s\}$.

Let $\mathbb{S}$ be an basic space. *The specific order* $\leq_{\odot}$ is defined by $a \leq_{\odot} b$ iff $(\exists c \in \mathbb{S}) : b = a \odot c$.

We shall note by $\bigvee_{\odot}$ resp. $\bigwedge_{\odot}$ the supremum resp. infimum in this order (if they exists).

Definition 5. $a : b$ is called the residuu of a by b and it is the greatest element (if exists) which holds $b \odot (a : b) \leq a$.

Definition 6. A basic space $\mathbb{S}$ has the *decomposition property* if for any $s, s_1, s_2 \in \mathbb{S}$ such that $s \leq s_1 \odot s_2$ there exists $t_1, t_2 \in \mathbb{S}$ such that $t_1 \leq s_1$, $t_2 \leq s_2$, $s = t_1 \odot t_2$.

Proposition 2. Every basic space has the decomposition property.

Lemma 3. Let $\mathbb{S}$ be an basic space and $s, a, b \in \mathbb{S}$ .Then

i) $a \odot b \geq a \vee b$

ii) if $a \leq b$ then $s \odot a \leq s \odot b$

iii) $(a \bigwedge_{\odot} b) \odot (a \bigvee_{\odot} b) = a \odot b$

iv) if $a, b \leq s$ and $a\perp b$ then $a \odot b \leq s$

Proposition 4. $\leq_{\odot} \subseteq \leq$.

Proposition 5. Any basic space is a distributive lattice.

A subset A of a basic space is called linearisable if

Definition 7. **(L)** $s \odot a \leq s \odot b$ implies $a \leq b$ $(\forall a, b \in A)$.

We define the order topology $\tau_{\leq}$ on $< \mathbb{S}, \leq >$ by putting $(a_i)_{i\in I} \underset{\tau_{\leq}}{\rightarrow} a$ iff ($(a_i)_{i\in I}$ is *increasing and dominated and* $\bigvee\limits_{i\in I} a_i = a$) or ($(a_i)_{i\in I}$ is *decreasing and* $\bigwedge\limits_{i\in I} a_i = a$). Analogously can be defined the specific order topology $\tau_{\leq_{\odot}}$on $< \mathbb{S}, \leq_{\odot} >$

Proposition 6. The superposition is continuous in the order topology.

Proof. We prove that the followings relations holds in any basic space:

($\mathbf{ID_1}$) for any increasing and dominated net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigvee_{i\in I}(s \odot s_i) = s \odot (\bigvee_{i\in I} s_i)$

($\mathbf{ID_2}$) for any net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigwedge_{i\in I}(s \odot s_i) = s \odot (\bigwedge_{i\in I} s_i)$

We prove first **($\mathbf{ID_2}$).** We set $a =: \bigwedge_{i\in I} s_i$ and $b =: \bigwedge_{i\in I}(s \odot s_i)$. Observe that $s \odot a \leq b$. From $b \leq s \odot s_i$ we obtain $b : s \leq s_i$ $(\forall i \in I)$. Therefore

$b : s \leq a \Leftrightarrow b \leq s \odot a \Leftrightarrow \bigwedge_{i\in I}(s \odot s_i) = s \odot (\bigwedge_{i\in I} s_i)$.

We prove now **($\mathbf{ID_1}$).** We set $a =: \bigvee_{i\in I} s_i$ and $b =: \bigvee_{i\in I}(s \odot s_i)$. Observe that $s \odot a \geq b$. From $b \geq s \odot s_i$ we obtain $b : s \geq s_i$ $(\forall i \in I)$. Therefore

$b : s \geq a \Leftrightarrow b \geq s \odot a \Leftrightarrow \bigvee_{i\in I}(s \odot s_i) = s \odot (\bigvee_{i\in I} s_i)$.$\square$

Remark 2. The laticial operations $\vee$ and $\wedge$ are continuous in the order topology.

Lemma 7. The followings relations holds in any basic space:

($\mathbf{GD_1}$) for any increasing and dominated net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigvee_{i\in I}(s \bigwedge s_i) = s \bigwedge(\bigvee_{i\in I} s_i)$;

($\mathbf{GD_2}$) *f*or any net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigwedge_{i\in I}(s \bigvee s_i) = s \bigvee(\bigwedge_{i\in I} s_i)$.

Proof. We prove first **($\mathbf{GD_2}$).** We set $a =: \bigwedge_{i\in I} s_i$ and $a_i =: s_i : a$, $(\forall i \in I)$. Obviously $\bigwedge_{i\in I} a_i =\perp$. We have $s \wedge a \leq (s \odot a_i) \vee (a \odot a_i) = (s \vee a) \odot a_i$. Thus $\bigwedge_{i\in I}(s \vee s_i) \leq \bigwedge_{i\in I}((s \vee a) \odot a_i) = s \vee a$. The converse inequality is immediate.

We prove now **($\mathbf{GD_1}$).** We set $a =: \bigvee_{i\in I} s_i$ and $a_i =: a : s_i$, $(\forall i \in I)$. Obviously $\bigwedge_{i\in I} a_i =\perp$. We have $s \vee s_i \leq (s \odot a_i) \wedge (s_i \odot a_i) = (s \wedge s_i) \odot a_i \leq \bigvee_{i\in I}((s \wedge s_i) \odot a_i)$. Thus $s \wedge a \leq \bigvee_{i\in I}(s \wedge s_i)$. The converse inequality is immediate.□

Definition 8. An *extended process* is a three-tuple $< \mathbb{M}, \mathbb{S}, \ell >$, where $< \mathbb{M}, \prec >$ is a real space, $< \mathbb{S}, \leq, \perp, \top, \odot >$ is a basic space and $\ell : \mathbb{M} \to \mathbb{S}$ is an injective isotone labelling function such that, if $\mathbb{B} = \ell(\mathbb{M})$ then:

($\mathbf{P}_1$) $\ell(\alpha \curlyvee \beta) \geq_\odot \ell(\alpha) \vee \ell(\beta)$ if $\alpha \curlyvee \beta$ exists

($\mathbf{P}_2$) if $\ell(\alpha \curlyvee \beta) = \top$ and $\gamma \succ \alpha \curlyvee \beta$ then $\ell(\gamma) = \top$

($\mathbf{P}_3$) $\perp \in \mathbb{B}$

($\mathbf{P}_4$) $< \mathbb{B}, \leq_{|\mathbb{B}}, \wedge >$ is a lower complete semilattice of $< \mathbb{S}, \leq >$

($\mathbf{P}_5$) $\mathbb{B}$ is linearisable ;

($\mathbf{P}_6$) $(\mathbb{B}, \odot, \perp)$ is a monoid;

($\mathbf{P}_7$) The superposition is continuous in the order topology on $\mathbb{B}$;

($\mathbf{P}_8$) $\mathbb{B}$ has the *decomposition property.*

Remark 3. The elements of an extended process will be called *basic occurences* and will be denoted by greek letters: α, β,etc. Their labels $\ell(\alpha), \ell(\beta)$ will be called *elementary processes.* In the next we shall identify these concepts.

Definition 9. An extended process is *called*

- *dense* iff $\lessdot = \varnothing \iff \forall \alpha, \beta \in \mathbb{B} : \alpha \prec \beta \Rightarrow \exists \gamma \in \mathbb{B} : \alpha \prec \gamma \prec \beta$;
- *combinatorial* iff $\preccurlyeq = (\lessdot)^+$;
- *K-dense* iff $(\forall l \in L)$ $(\forall c \in C)$ $l \cap c \neq \emptyset$;
- *N-dense* iff $(\forall \alpha, \beta, \gamma, \delta \in \mathbb{B}) : (\gamma\ co\ \beta\ \&\ \beta\ co\ \alpha\ \&\ \alpha\ co\ \delta\ \&\ \alpha\ li\ \gamma\ \&\ \gamma\ li\ \delta\ \&\ \delta\ li\ \beta) \Rightarrow$

  $(\exists e \in \mathbb{B} : e\ co\ \alpha\ \&\ e\ co\ \beta\ \&\ e\ li\ \gamma\ \&\ e\ li\ \delta)$.;
- *of finite degree* iff $\forall \beta \in \mathbb{B} : |^\blacklozenge\beta| < \infty$ and $|\beta^\blacklozenge| < \infty$;
- with *finite intervals* iff $(\forall \alpha, \beta \in \mathbb{B}) : |[\alpha, \beta]| < \infty$;
- *boundedly discrete* iff $(\forall \alpha, \beta \in \mathbb{B})$ $(\exists n \in \omega)$ $(\forall l \in L) : |[\alpha, \beta] \cap l| < n$.

Definition 10. A *discrete observer* is a function $dob : \mathbb{B} \to \omega : \alpha \prec \beta \Rightarrow dob(\alpha) \prec dob(\beta)$

Definition 11. An extended process is called *discrete observable* if admits a discrete observer.

Definition 12. An extended process is *injectively observable* iff there exists an injective discrete observer.

Definition 13. A *continuous observer* is a function $cob : \mathbb{B} \to \overline{\mathbb{R}}_+$ with the following properties:

($\mathbf{CO_1}$) $\alpha \prec \beta \Rightarrow cob(\alpha) \leq cob(\beta)$, $(\forall \alpha, \beta \in \mathbb{B})$;

($\mathbf{CO_2}$) $cob(\beta) = \sup_{i\in I}(cob(\beta_i))$ if $(\beta_i)_{i\in I} \uparrow \beta$;

($\mathbf{CO_3}$) $(\forall \beta \in \mathbb{B})$ $(\exists (\beta_i)_{i\in I} \uparrow \beta) : cob(\beta_i) < \infty$.

Definition 14. A continuous observer *cob* is called *nondeterministic* iff $cob(\alpha \odot \beta) = \max(cob(\alpha), cob(\beta))$

Definition 15. An extended process is called *continuous observable* if admits a continuous observer

We shall state without proof the followings connections between observability and discreteness

Proposition 8. If an extended process is discrete observable then it is boundedly discrete. If the extended process is countable then the converse also holds.

Proposition 9. An extended process is *injectively observable* iff the extended process has finite intervals and it is countable.

Definition 16. An extended process $\mathbb{B}$ is called *continuous* if for any Dedekind-cut $(\mathbb{A}, \overline{\mathbb{A}})$ of $\mathbb{B}$ and any line $l : |M(\mathbb{A}) \cap l| = 1$.

Proposition 10. If the extended process $\mathbb{B}$ is continuous then $\mathbb{B}$ is dense.

Definition 17. An extended process $\mathbb{B}$ is called

- *gap-free* iff $\forall \mathbb{A} \in D(\mathbb{B})$ $\forall l \in L : |c(\mathbb{A}) \cap l| \neq 0$;
- *jump-free* iff $\forall \mathbb{A} \in D(\mathbb{B})$ $\forall l \in L : |c(\mathbb{A}) \cap l| \neq 2$.

Definition 18. An extended process is called *D-continuous* if for any Dedekind-cut $(\mathbb{A}, \overline{\mathbb{A}})$ of $\mathbb{B}$ and any line $l : |c(\mathbb{A}) \cap l| = 1$

Remark 4. If the extended process $\mathbb{B}$ is combinatorial then

$\bullet Obmax(\mathbb{A}) = \{\alpha \in Max(\mathbb{A}) / |\alpha^{\blacklozenge}| \leq 1\}$;

$\bullet Obmin(\mathbb{A}) = \{\alpha \in Min(\mathbb{A}) / |{}^{\blacklozenge}\alpha| \leq 1\}$.

Proposition 11. Let $\mathbb{A} \subset \mathbb{B}$ be specifically decreasing. Then we have $\bigwedge_{\odot} \mathbb{A} = \bigwedge \mathbb{A}$.

Proposition 12. Let $\mathbb{A} \subset \mathbb{B}$ be specifically increasing and dominated. Then we have $\bigvee_{\odot} \mathbb{A} = \bigvee \mathbb{A}$.

Corollary 13. The order topology $\tau_{\leq}$ is finer than the specific order topology $\tau_{\leq_{\odot}}$.

4. ENERGY

In mathematical physics, the energy integral plays a very important role. In this section we formalise this approach as energetic spaces.

Definition 19. The *mutual energy* $\mathcal{E}[a, b]$ of two elements a,b is a map $\mathcal{E} : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ with the following properties:

$\mathbf{(EN_1)}$ $\mathcal{E}[a \odot b, s] = \mathcal{E}[a, s] + \mathcal{E}[b, s]$ (the *superposition principle*)

$\mathbf{(EN_2)}$ $\mathcal{E}[a, b] = \mathcal{E}[b, a]$ (the *symmetry condition*)

$\mathbf{(EN_3)}$ $\mathcal{E}[s] = \mathcal{E}[s, s]$ (the *energy* of the element s)

$\mathbf{(EN_4)}$ $\mathcal{E}[s] > 0$ if $s \neq \perp$ ($\mathcal{E}$ is *positive definite*)

$\mathbf{(EN_5)}$ $|\mathcal{E}[a, b]|^2 \leq \mathcal{E}[a, b] \cdot \mathcal{E}[a, b]$ (the *weak sector condition*)

Remark 5. We can extend the energy to $[\mathbb{S}] \times [\mathbb{S}]$ by

$$\mathcal{E}[a : b, c : d] = \mathcal{E}[a, c] + \mathcal{E}[b, d] - \mathcal{E}[a, d] - \mathcal{E}[b, c].$$

Definition 20. The elements $a, b \in \mathbb{S}$ are called dual in energy (noted $a \in b^{\perp}_{\mathcal{E}}$)if $\mathcal{E}[a, b] = 0$

Lemma 14. For any $a, b \in [\mathbb{S}]$

i) $\mathcal{E}[\perp] = 0$; ii) $\mathcal{E}[a, \perp] = 0$;iii) $\mathcal{E}[a] > 0$ if $a \neq \perp$;

iv) $\mathcal{E}[a^*] = \mathcal{E}[a]$; v) $\mathcal{E}^{\frac{1}{2}}[a \odot b] \leq \mathcal{E}^{\frac{1}{2}}[a] + \mathcal{E}^{\frac{1}{2}}[b]$;

vi) $\mathcal{E}[a \odot b] + \mathcal{E}[a : b] = 2(\mathcal{E}[a] + \mathcal{E}[b])$;

Definition 21. An energetic space is a structure $< [\mathbb{S}], \mathcal{E} >$ such that $[\mathbb{S}]$ is an extended space, $\mathcal{E} : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ is an energy and

$\mathbf{(ES}_1\mathbf{)}$ $[\mathbb{S}] = \overline{[\mathbb{S}]}$;

$\mathbf{(ES}_2\mathbf{)}$ $a \in b^{\perp} \Rightarrow a \in b^{\perp}_{\mathcal{E}}$, $(\forall a, b \in [\mathbb{S}])$.

Remark 6. The terms energy and energetic space have been inspired by their use in the mathematical modelling world.

Example 1. Let $\overline{[\mathbb{S}]}$ be the class of all the spaces of excessive functions $\xi_{\mathcal{V}}$ of all sub-Markovian resolvents $\mathcal{V}$ which are in duality (with respect to a finite measure μ) and for which the initial kernels are proper. For any $\xi_{\mathcal{V}}$, $\xi_{\mathcal{W}} \subseteq \overline{[\mathbb{S}]}$ and $a \in \xi_{\mathcal{V}}$, $b \in \xi_{\mathcal{W}}$ define the mutual energy $\mathcal{E}[a, b]$ of a and b by

$$\mathcal{E}[a, b] =: \sup\{\int f\, W g\, d\mu\, ;\, a, b \in \mathfrak{F},\, V f \leq s, \quad (1)$$
$$W g \leq t\}$$

where V is the initial kernel for $\mathcal{V}$, W is the initial kernel for $\mathcal{W}$ and $\mathfrak{F}$ denotes the set of all $\mathfrak{B}$-measurable positive numerical functions on X, $(X, \mathfrak{B}, \mu)$ being the measurable space.

Theorem 15. The structure $< [\mathbb{S}], \mathcal{E} >$ is an energetic space iff $[\mathbb{S}]$ is closed in the energy topology and the energy $\mathcal{E}$ is a laticial valuation.

Definition 22. An extended process $\mathbb{B}$ is called *W-like process* if there exists a map $\urcorner : \mathbb{B} \to \mathrm{Im}\,\mathbb{B}$ such that :

$\mathbf{(W_1)}$ $\urcorner[\alpha \odot \beta] = \urcorner[\alpha] + \urcorner[\beta]$, and

$\alpha \leq \beta \Leftrightarrow \urcorner[\alpha] \leq \urcorner[\beta]$, $(\forall \alpha, \beta \in \mathbb{B})$;

$\mathbf{(W_2)}$ $\urcorner[\mathbb{B}]$ is solid and increasingly dense in $Im\mathbb{B}$;

$\mathbf{(W_3)}$ $\urcorner[R(\alpha)] = \tilde{R}(\urcorner[\alpha])$, $(\forall \alpha \in \mathbb{B})$;

$\mathbf{(W_4)}$ for any two sweepings S and T on $\mathbb{B}$ such that $S \vee T = id_{\mathbb{B}}$ we have $S \circ T = T \circ S$.

A basic intuition behind a W-like process is that its labels could be interpreted as the weak solutions (i.e. solutions in the sense distributions theory) of a very general classes of stochastic differential operators.
In the remaining of this section we show that one can associate an energetic space to a W-like process. In this way, many important properties of physical events (like cardiac potentials or weather turbulence) can be formulated algebraically.

Let $\mathcal{C} : \mathbb{B} \times \mathbb{B} \to \overline{\mathbb{R}}_+$ defined by $\mathcal{C}[\alpha, \beta] = \urcorner[\beta](\alpha)$, $(\forall \alpha, \beta \in \mathbb{B})$. For any W-like process $\mathbb{B}$ define $\mathbb{B}^f =: \{\beta \in \mathbb{B}; \mathcal{C}[\beta, \beta] < \infty\}$

Lemma 16. The couple of observers $\mathcal{C}$ has the followings properties:

*)For any $\beta \in \mathbb{B} : \alpha \leq \beta \Rightarrow \mathcal{C}[\sigma, \alpha] \leq \mathcal{C}[\sigma, \beta]$

**) If $(\alpha_i)_{i \in I} \uparrow \alpha$ then $\bigvee_{i \in I} \mathcal{C}[\sigma, \alpha_i] = \mathcal{C}[\sigma, \alpha]$.

***) For any $cob \in \mathrm{Im}\,\mathbb{B}$ there exists $\beta \in \mathbb{B}$: $cob(\alpha) = \mathcal{C}[\beta, \alpha]$, $(\forall \alpha \in \mathbb{B})$.

Corollary 17. The axioms $\mathbf{W_1}$) $\mathbf{W_2}$) are logical equivalent with the properties *), **), ***). The axiom $\mathbf{W_3}$) is logical equivalent with the following property for any sweeping S on $\mathbb{B}$: $\mathcal{C}[S\alpha, \beta] = \mathcal{C}[\alpha, S\beta]$, $(\forall \alpha, \beta \in \mathbb{B})$.

For any $\beta \in \mathbb{B}$ define

$$\mathbb{B}_\beta =: \{\alpha \in \mathbb{B}^f; \exists m, n \in \mathbb{N}, \alpha^{(m)} \leq \beta^{(n)}\}$$

Remark 7. $\mathbb{B}^f = \bigcup_{\beta \in \mathbb{B}^f} \mathbb{B}_\beta$.

Proposition 18. $\mathbb{B}^f$ is solid and increasingly dense in $\mathbb{B}$.

Lemma 19. $\mathbb{B}^f$ is a basic space if $\mathcal{C}[\beta^f, \beta^f] \geq 0$ for any $\beta^f \in [\mathbb{B}^f_\alpha]$ and $\alpha \in \mathbb{B}^f$.

Corollary 20. For any $\alpha, \beta \in \mathbb{B}$

$$\mathcal{C}[\alpha, \beta] + \mathcal{C}[\beta, \alpha] \leq \mathcal{C}[\alpha, \alpha] + \mathcal{C}[\beta, \beta]$$

and $\mathcal{C}[\alpha, \alpha] = 0 \Rightarrow \alpha = \perp$.

Lemma 21. Let $\beta \in [\mathbb{B}^{'}]$, $\mathbb{B}^{'} \subseteq \mathbb{B}$ be solid in $\mathbb{B}$ with respect to the specific order and such that $\mathcal{C}[\beta] < \infty$, $\beta = \alpha : \alpha^{'}$, $\alpha : \alpha^{'} \in \mathbb{B}$ and $(\beta_n)_{n \in N}$ be the sequence defined by $\beta_1 = \beta$, $\beta_{n+1} = \overline{\beta}_n : \beta_n$. Then $\mathcal{C}[\beta] = \sum_{n=1}^{\infty} \mathcal{C}[\overline{\beta}_n]$.

Lemma 22. Let $\mathbb{A} \subset \mathbb{B}$ a inferior semilattice, solid with respect the specific order and $\mathcal{C}[\alpha] < +\infty$, $(\forall \alpha \in \mathbb{A})$. If the couple of observers $\mathcal{C}$ is regular, then

$$\mathcal{C}[S_\alpha \sigma, \sigma^{'}] = \mathcal{C}[\sigma, S_\alpha \sigma^{'}] \ , \ (\forall \alpha \in [\mathbb{A}]_\uparrow \ , \ \forall \sigma, \sigma^{'} \in \mathbb{A}).$$

Proposition 23. Let $\mathbb{B}$ be a W-like process. Then $< [\mathbb{B}^f_\alpha], \mathcal{E}_\mathcal{C} >$ is an energetic space, $(\forall \alpha \in [\mathbb{B}])$.

The map $\mathcal{E}_\mathcal{C} : [\mathbb{S}] \times [\mathbb{S}] \rightarrow R$ defined by $\mathcal{E}_\mathcal{C}[\alpha, \beta] =: \frac{\mathcal{C}[\alpha, \beta] + \mathcal{C}[\beta, \alpha]}{2}$ is an energy which will be called the energy associated to the W-like process $\mathbb{B}$.

5. FINAL REMARKS

Many detailed examples, the omitted (because of space limit) proofs and bibliographic discussions are presented in (Bujorianu, M.C. and Bujorianu, M.L., 2002). In particular, it is presented the origin of mathematical concepts investigated in this paper, like extended processes, energetic spaces and observers and basic space. Also it is proved that mathematical models of cardiac potentials (and, by extension, of most of biological potentials) satisfy the axioms of basic spaces.

A different model of Markov processes with multiform time is presented in (Benveniste A. *et al.*, 1995). Computer networks inspire the model and therefore it is developed based on different guiding principles. The main difference relies on the system/environment emphasis. The system-oriented approach in (Benveniste A. *et al.*, 1995) considers a richer concurrency structure, for e.g. conflict relations. The classes of reactive systems we consider have behaviour driven by the environment and therefore axiomatic modelling of real life environments plays a dominant role.

In a future work, we will investigate more concrete temporal structures, like those arising from event structures (Best E. and Fernandez, C., 1990). A case study from air traffic control is under development.

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