DYNAMIC PROGRAMMING SOLUTION OF STATE ESTIMATION PROBLEMS WITH CONSTRAINED DISTURBANCES

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Abstract: This paper is concerned with maximum a posteriori state estimation of linear systems in the presence of constrained scalar process noise. The goal of this work is to investigate closed form solutions aimed at reducing the online computations required by the estimation problem. Dynamic programming is used to derive a closed form solution that can be precomputed offline. The optimal solution is given by a piece-wise affine function of the data (the mean value of the initial state and the sequence of measurement data). The data space is partitioned into a number of polyhedral regions, inside each of which a unique affine function is applied.

Keywords: Constraint satisfaction problems, Dynamic programming, Estimators

1. INTRODUCTION

The problem of estimating the internal states of a dynamical system based on the knowledge of some measurable output data is of importance to many areas of engineering. Such problems can be formulated in various ways; namely, minimum variance, maximum likelihood, maximum a posteriori, conditional expected value, etc. It is well known that, in the case of a *linear* system and *Gaussian* noise, the optimal solution to all the above problems is provided by the Kalman filter. One of the desirable properties of the Kalman filter is that it can be pre-computed in closed form and then used online in very simple schemes. However, as one steers away from the ideal setup of conditions for the Kalman filter, the situation changes substantially and closed form solutions are no longer available. For example, most variables in real systems are bound to be limited to certain ranges (either due to physical constraints or due to safety requirements). If one wishes to take these constraints into account in the formulation of the problem, then the Kalman filter does not provide the optimal estimates.

Several approaches have been proposed for the estimation problem in the case of linear systems in the presence of constraints. Of particular relevance to the present paper is a strategy that formulates the problem as a quadratic program; i.e., as the minimization of a quadratic objective function subject to linear constraints. For example, it has been shown (see, e.g., Goodwin *et al.*, 2004) that such formulation yields the estimate that maximizes the joint a posteriori probability distribution. However, the presence of inequality constraints precludes general recursive solutions and makes these problems computationally intractable as the problem size grows with new incoming data. Hence, a standard approach is to consider a fixed horizon problem of length N at each time and to implement it in a *moving horizon* fashion; see, e.g., Muske *et al.* (1993), Rao *et al.* (2001), Robertson *et al.* (1996) and Rao *et al.* (2003).

One problem that needs to be addressed in the implementation of moving horizon estimators [MHE], in the presence of constraints, is the computational time required to solve online the underlying quadratic program since the computations required can limit applicability to relatively slow processes. Hence, it would be of interest to count with closed form solutions that can be precomputed offline so as to speed up online computations. To the best of our knowledge, this issue has not been addressed in the literature. Here, we derive a closed form solution to the fixed horizon estimation problem that needs to be solved at each sampling time in a moving horizon implementation. The case considered is that of linear systems with constrained process noise and the solution is obtained using dynamic programming (see Bellman, 1957; Cox, 1964). The employed methodology is analogous to the one used in Mare and De Doná (2004) by the authors of this paper to derive a closed form solution to the input constrained LQR problem. The optimal solution to the estimation problem is given by a piecewise affine function of the data for this particular problem (i.e., the mean value of the initial state and the sequence of measurement data). The data space is partitioned into a number of polyhedral regions defined by linear inequalities such that, inside each region, a unique affine function is valid. Thus, the online implementation problem reduces to: given the data, find the corresponding region and, via a simple affine function evaluation, obtain the optimal estimate. The main online computational requirement is that of determining the region to which the data vector belongs to (this requires the evaluation of a potentially large number of linear inequalities). Although not explored in this paper, we mention that there exist algorithms, e.g., binary tree search algorithms, that can be employed to perform this task in a very efficient way; see, e.g., Tondel *et al.* (2003).

2. LINEAR STATE ESTIMATION WITH CONSTRAINTS

Consider the discrete-time linear state-space model

$$x_{k+1} = Ax_k + Bw_k, \tag{1}$$

$$y_k = Cx_k + v_k, \tag{2}$$

where $x_k \in I\!R^n$, $w_k \in I\!R^m$, $y_k \in I\!R^r$ and $v_k \in I\!R^r$. Suppose that x_0, $\{w_k\}$, $\{v_k\}$ are i.i.d. sequences with truncated Gaussian distributions. That is, for $Q > 0$, $R > 0$, $P_0 > 0$, $\beta_w \triangleq (2\pi)^{-\frac{m}{2}}(\det Q)^{-\frac{1}{2}}$, $\beta_v \triangleq (2\pi)^{-\frac{r}{2}}(\det R)^{-\frac{1}{2}}$, $\beta_{x_0} \triangleq (2\pi)^{-\frac{n}{2}}(\det P_0)^{-\frac{1}{2}}$, and for some nonempty sets $\Omega_1 \subseteq I\!R^m$, $\Omega_2 \subseteq I\!R^r$ and $\Omega_3 \subseteq I\!R^n$, the probability distributions are given by

$$p_w(w_k) = \frac{\beta_w \mathrm{e}^{-\frac{1}{2}w_k^{\mathrm{T}}Q^{-1}w_k}}{\beta_w \int_{\Omega_1} \mathrm{e}^{-\frac{1}{2}\nu^{\mathrm{T}}Q^{-1}\nu}d\nu} \tag{3}$$

for $w_k \in \Omega_1$ and 0 otherwise,

$$p_v(v_k) = \frac{\beta_v \mathrm{e}^{-\frac{1}{2}v_k^{\mathrm{T}}R^{-1}v_k}}{\beta_v \int_{\Omega_2} \mathrm{e}^{-\frac{1}{2}\nu^{\mathrm{T}}R^{-1}\nu}d\nu} \tag{4}$$

for $v_k \in \Omega_2$ and 0 otherwise, and

$$p_{x_0}(x_0) = \frac{\beta_{x_0} \mathrm{e}^{-\frac{1}{2}(x_0-\mu_0)^{\mathrm{T}}P_0^{-1}(x_0-\mu_0)}}{\beta_{x_0} \int_{\Omega_3} \mathrm{e}^{-\frac{1}{2}(\nu-\mu_0)^{\mathrm{T}}P_0^{-1}(\nu-\mu_0)}d\nu} \tag{5}$$

for $x_0 \in \Omega_3$ and 0 otherwise. Consider the following vector definitions

$$\mathbf{y}_N = \begin{bmatrix} y_1^{\mathrm{T}} & \dots & y_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{6}$$

$$\mathbf{y}_N^d = \begin{bmatrix} {y_1^d}^{\mathrm{T}} & \dots & {y_N^d}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{7}$$

$$\mathbf{x}_N = \begin{bmatrix} x_0^{\mathrm{T}} & \dots & x_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{8}$$

$$\hat{\mathbf{x}}_N = \begin{bmatrix} \hat{x}_0^{\mathrm{T}} & \dots & \hat{x}_N^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{9}$$

Given the observations $\mathbf{y}_N^d = [{y_1^d}^{\mathrm{T}} \dots {y_N^d}^{\mathrm{T}}]^{\mathrm{T}}$ and the "unconstrained" mean value of x_0, denoted by μ_0, the aim is to obtain the *joint a posteriori most probable* (JAPMP) state estimates $\hat{\mathbf{x}}_N = [\hat{x}_0^{\mathrm{T}} \dots \hat{x}_N^{\mathrm{T}}]^{\mathrm{T}}$. That is, based on the knowledge of the *a posteriori* distribution of $\mathbf{x}_N$ given $\mathbf{y}_N$, denoted $p_{\mathbf{x}_N|\mathbf{y}_N}$, and on the observations $\mathbf{y}_N^d$, we want to determine the vector $\hat{\mathbf{x}}_N$ that solves the following optimization problem

$$\hat{\mathbf{x}}_N^{\mathrm{OPT}} \triangleq \arg\max_{\hat{\mathbf{x}}_N} p_{\mathbf{x}_N|\mathbf{y}_N}(\hat{\mathbf{x}}_N|\mathbf{y}_N^d) \tag{10}$$

In the sequel we will formulate this problem as a quadratic program. First, we present an expression for the joint probability density function for $\mathbf{y}_N$ and $\mathbf{x}_N$, which can be obtained using Bayes' rule and the Markovian structure of the model (for the details, see Goodwin *et al.*, 2004). In the following, c denotes a generic constant. The joint pdf can be expressed as

$$\begin{aligned} &p_{\mathbf{y}_N,\mathbf{x}_N}(\mathbf{y}_N = \mathbf{y}_N^d, \mathbf{x}_N = \hat{\mathbf{x}}_N) \\ &\quad = c \times \mathrm{e}^{\left(-\frac{1}{2}\sum_{k=0}^{N-1}\hat{w}_k^{\mathrm{T}}Q^{-1}\hat{w}_k\right)} \\ &\quad\quad \times \mathrm{e}^{\left(-\frac{1}{2}\sum_{k=1}^{N}\hat{v}_k^{\mathrm{T}}R^{-1}\hat{v}_k\right)} \\ &\quad\quad \times \mathrm{e}^{\left(-\frac{1}{2}(\hat{x}_0-\mu_0)^{\mathrm{T}}P_0^{-1}(\hat{x}_0-\mu_0)\right)}, \end{aligned} \tag{11}$$

whenever

$$\begin{aligned} &\hat{w}_k \in \Omega_1 \quad \text{for } k = 0, \dots, N-1, \\ &\hat{v}_k \in \Omega_2 \quad \text{for } k = 1, \dots, N, \\ &\hat{x}_0 \in \Omega_3, \end{aligned}$$

where

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + B\hat{w}_k \quad \text{for } k = 0, \dots, N-1, \\ \hat{v}_k &= y_k^d - C\hat{x}_k \quad \text{for } k = 1, \dots, N. \end{aligned}$$

From the joint probability density function above, the *a posteriori distribution* of $\mathbf{x}_N$ given $\mathbf{y}_N$ can be expressed as

$$p_{\mathbf{x}_N|\mathbf{y}_N}(\hat{\mathbf{x}}_N|\mathbf{y}_N^d) = \frac{p_{\mathbf{y}_N,\mathbf{x}_N}(\mathbf{y}_N^d,\hat{\mathbf{x}}_N)}{p_{\mathbf{y}_N}(\mathbf{y}_N^d)}. \tag{12}$$

As $p_{\mathbf{y}_N}(\mathbf{y}_N^d)$ is independent of $\mathbf{x}_N$, the solution of the estimation problem (10) is obtained by maximising the numerator in (12), that is,

$$\begin{aligned}\hat{\mathbf{x}}_N^{\mathrm{OPT}} &= \arg\max_{\hat{\mathbf{x}}_N} p_{\mathbf{y}_N,\mathbf{x}_N}(\mathbf{y}_N^d,\hat{\mathbf{x}}_N)\\ &= \arg\min_{\hat{\mathbf{x}}_N}\ -\ln p_{\mathbf{y}_N,\mathbf{x}_N}(\mathbf{y}_N^d,\hat{\mathbf{x}}_N),\end{aligned} \tag{13}$$

(since $\ln(\cdot)$ is a monotonically increasing function). Substituting (11) into (13), the estimation problem can be stated as the following optimisation problem:

Given the data $\mathcal{Y} \triangleq \left[\mu_0^{\mathrm{T}}, y_1^{d\,\mathrm{T}}, \ldots, y_N^{d\,\mathrm{T}}\right]^{\mathrm{T}}$, solve

$$\mathcal{P}_e: \quad V_N^{\mathrm{OPT}}(\mathcal{Y}) \triangleq \min V_N(\{\hat{x}_k\},\{\hat{v}_k\},\{\hat{w}_k\}), \tag{14}$$

subject to:

$$\hat{x}_{k+1} = A\hat{x}_k + B\hat{w}_k \quad \text{for } k=0,\ldots,N-1, \tag{15}$$

$$\hat{v}_k = y_k^d - C\hat{x}_k \quad \text{for } k=1,\ldots,N, \tag{16}$$

$$\hat{w}_k \in \Omega_1 \quad \text{for } k=0,\ldots,N-1, \tag{17}$$

$$\hat{v}_k \in \Omega_2 \quad \text{for } k=1,\ldots,N, \tag{18}$$

$$\hat{x}_0 \in \Omega_3, \tag{19}$$

where

$$\begin{aligned}V_N(\{\hat{x}_k\},\{\hat{v}_k\},\{\hat{w}_k\}) &\triangleq \frac{1}{2}\sum_{k=0}^{N-1}\hat{w}_k^{\mathrm{T}}Q^{-1}\hat{w}_k\\ &+\frac{1}{2}\sum_{k=1}^{N}\hat{v}_k^{\mathrm{T}}R^{-1}\hat{v}_k + \frac{1}{2}(\hat{x}_0-\mu_0)^{\mathrm{T}}P_0^{-1}(\hat{x}_0-\mu_0).\end{aligned} \tag{20}$$

Note that, in the case when the sets Ω_1, Ω_2 and Ω_3 are defined by linear inequalities (i.e., they are polyhedral sets) the optimization problem (14)–(20) is a quadratic program.

3. DYNAMIC PROGRAMMING

To solve problem $\mathcal{P}_e$ defined in (14)–(20), dynamic programming (see Bellman, 1957) can be used. In the case of estimation problems it is more convenient to use *forward* dynamic programming (see Cox, 1964).

Assuming that A is nonsingular, the sequence $\{\hat{x}_k\}$ can be expressed in terms of $\hat{x}_N$ and $\{\hat{w}_k\}$, since

$$\hat{x}_k = A^{-1}(\hat{x}_{k+1} - B\hat{w}_k) \quad \text{for } k=0,\ldots,N-1, \tag{21}$$

In addition, the sequence $\{\hat{v}_k\}$ can be expressed in terms of $y_1^d,\ldots,y_N^d$, $\hat{x}_N$ and $\{\hat{w}_k\}$, since $\hat{v}_k = y_k^d - C\hat{x}_k$ for $k=1,\ldots,N$.

From the above discussion, it follows that $V_N(\{\hat{x}_k\},\{\hat{v}_k\},\{\hat{w}_k\})$ in (20) can be written as $V_N(\hat{x}_N,\{\hat{w}_k\})$, and then $\mathcal{P}_e$ can be stated as

$$V_N^{\mathrm{OPT}}(\mathcal{Y}) = \min V_N(\hat{x}_N,\{\hat{w}_k\}). \tag{22}$$

We next define, for $\hat{x}_0 \in \Omega_3$, the partial value function at time 0 as

$$V_0^{\mathrm{OPT}}(\hat{x}_0,\mu_0) \triangleq \frac{1}{2}(\hat{x}_0-\mu_0)^{\mathrm{T}}P_0^{-1}(\hat{x}_0-\mu_0), \tag{23}$$

and the partial value function at time k as

$$\begin{aligned}&V_k^{\mathrm{OPT}}(\hat{x}_k,\mu_0,y_1^d,\ldots,y_k^d) \triangleq\\ &\min_{\hat{w}_0,\ldots,\hat{w}_{k-1}}\Bigg\{\frac{1}{2}(\hat{x}_0-\mu_0)^{\mathrm{T}}P_0^{-1}(\hat{x}_0-\mu_0)\\ &+\frac{1}{2}\sum_{j=0}^{k-1}\hat{w}_j^{\mathrm{T}}Q^{-1}\hat{w}_j\\ &+\frac{1}{2}\sum_{j=1}^{k}(y_j^d-C\hat{x}_j)^{\mathrm{T}}R^{-1}(y_j^d-C\hat{x}_j)\Bigg\},\end{aligned} \tag{24}$$

subject to:

$$\hat{x}_j = A^{-1}(\hat{x}_{j+1} - B\hat{w}_j) \quad \text{for } j=0,\ldots,k-1, \tag{25}$$

$$\hat{w}_j \in \Omega_1 \quad \text{for } j=0,\ldots,k-1, \tag{26}$$

$$y_j^d - C\hat{x}_j \in \Omega_2 \quad \text{for } j=1,\ldots,k, \tag{27}$$

$$\hat{x}_0 \in \Omega_3. \tag{28}$$

By the Principle of Optimality, for $k \geq 0$, and $\hat{x}_{k+1} \in I\!\!R^n$ such that $y_{k+1}^d - C\hat{x}_{k+1} \in \Omega_2$, we have that

$$\begin{aligned}&V_{k+1}^{\mathrm{OPT}}(\hat{x}_{k+1},\mu_0,y_1^d,\ldots,y_{k+1}^d) =\\ &\min_{\hat{w}_k}\Bigg\{V_k^{\mathrm{OPT}}(A^{-1}\hat{x}_{k+1}-A^{-1}B\hat{w}_k,\mu_0,y_1^d,\ldots,y_k^d)\\ &+\frac{1}{2}\hat{w}_k^{\mathrm{T}}Q^{-1}\hat{w}_k\\ &+\frac{1}{2}(y_{k+1}^d-C\hat{x}_{k+1})^{\mathrm{T}}R^{-1}(y_{k+1}^d-C\hat{x}_{k+1})\Bigg\},\end{aligned} \tag{29}$$

subject to:

$$\hat{w}_k \in \Omega_1, \tag{30}$$

$$y_k^d - C(A^{-1}\hat{x}_{k+1} - A^{-1}B\hat{w}_k) \in \Omega_2. \tag{31}$$

In the absence of constraints (i.e., when $\Omega_1 = I\!\!R^m, \Omega_2 = I\!\!R^r, \Omega_3 = I\!\!R^n$) the above dynamic programming algorithm leads to the Kalman filter (for details, see §9.6 in Goodwin *et al.*, 2004).

4. ANALYTICAL SOLUTION

In this section, the case of a scalar process noise constrained to be in the interval $\Omega_1 = [\Delta_1,\ \Delta_2]$

is considered (i.e., constraints on the states and on the measurement noise are not considered and will be the subject of future work). As explained in Section 2, the estimation problem (10) can be equivalently formulated as problem $\mathcal{P}_e$ in (14)–(20). Moreover, the latter problem can be solved by the dynamic programming technique, explained in Section 3. The partial value function at time 0 is considered first, which was defined in (23) as

$$V_0^{\mathrm{OPT}}(\hat{x}_0, \mu_0) \triangleq \frac{1}{2}(\hat{x}_0 - \mu_0)^{\mathrm{T}} P_0^{-1}(\hat{x}_0 - \mu_0).$$

Then, the partial value function at time 1 is expressed, using the Principle of Optimality (29), as

$$\begin{aligned} V_1^{\mathrm{OPT}}(\hat{x}_1, \mu_0, y_1^d) = \min_{\hat{w}_0 \in \Omega_1} \bigg\{ & \frac{1}{2}\hat{w}_0^{\mathrm{T}} Q^{-1} \hat{w}_0 \\ & + V_0^{\mathrm{OPT}}(A^{-1}\hat{x}_1 - A^{-1}B\hat{w}_0, \mu_0) \\ & + \frac{1}{2}(y_1^d - C\hat{x}_1)^{\mathrm{T}} R^{-1}(y_1^d - C\hat{x}_1) \bigg\}, \end{aligned} \tag{32}$$

Substituting V_0^{OPT} into V_1^{OPT}, taking derivatives respect to $\hat{w}_0$ and setting to zero, the expression of the *unconstrained* $\hat{w}_0^{\mathrm{unc}}$ minimising V_1^{OPT} is obtained:

$$\begin{aligned} \hat{w}_0^{\mathrm{unc}} = & \left[(A^{-1}B)^{\mathrm{T}} P_0^{-1}(A^{-1}B) + Q^{-1}\right]^{-1} \times \\ & (A^{-1}B)^{\mathrm{T}} P_0^{-1}(A^{-1}\hat{x}_1 - \mu_0) \end{aligned}$$

In the following, I denotes the identity matrix with the same number of columns and rows as A, and we define $\hat{x}_0^* \triangleq \mu_0$. Notice that the objective function in (32) is a quadratic function of $\hat{w}_0$ whose *unconstrained* minimum is achieved at $\hat{w}_0^{\mathrm{unc}}$ computed above. From the convexity of the objective function it follows that the *constrained* optimum, $\hat{w}_0^{\mathrm{OPT}}$, is given by the point in the allowed interval $[\Delta_1, \Delta_2]$ that is closest in distance to the unconstrained optimum $\hat{w}_0^{\mathrm{unc}}$. Hence, three different cases arise, depending on whether $\hat{w}_0^{\mathrm{unc}} < \Delta_1$, $\Delta_1 \leq \hat{w}_0^{\mathrm{unc}} \leq \Delta_2$, or $\hat{w}_0^{\mathrm{unc}} > \Delta_2$. It follows that the optimal constrained solution can be written as

$$\hat{w}_0^{\mathrm{OPT}} = Z_1 \left[L_1 \; K_1\right] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_0^* \end{bmatrix} + h_1, \tag{33}$$

where

$$Z_1 = \left[(A^{-1}B)^{\mathrm{T}} P_0^{-1}(A^{-1}B) + Q^{-1}\right]^{-1} (A^{-1}B)^{\mathrm{T}} P_0^{-1},$$

$L_1 = A^{-1}$ if $\Delta_1 \leq Z_1(A^{-1}\hat{x}_1 - \mu_0) \leq \Delta_2$, and 0 otherwise,

$K_1 = -I$ if $\Delta_1 \leq Z_1(A^{-1}\hat{x}_1 - \mu_0) \leq \Delta_2$, and 0 otherwise, and

$$h_1 = \begin{cases} 0 & \text{if} \quad \Delta_1 \leq Z_1(A^{-1}\hat{x}_1 - \mu_0) \leq \Delta_2, \\ \Delta_1 & \text{if} \quad Z_1(A^{-1}\hat{x}_1 - \mu_0) < \Delta_1 \\ \Delta_2 & \text{if} \quad Z_1(A^{-1}\hat{x}_1 - \mu_0) > \Delta_2. \end{cases}$$

Substituting (33) into (32) and completing squares, the minimum attained in (32) can be written as

$$V_1^{\mathrm{OPT}}(\hat{x}_1, \hat{x}_1^*) = \frac{1}{2}(\hat{x}_1 - \hat{x}_1^*)^{\mathrm{T}} P_1^{-1}(\hat{x}_1 - \hat{x}_1^*) + c, \tag{34}$$

where

$$\hat{x}_1^* = P_1 \left[S_1 \; T\right] \begin{bmatrix} \mu_0 \\ y_1^d \end{bmatrix} + P_1 U_1 h_1, \tag{35}$$

$$\begin{aligned} P_1^{-1} = & (A^{-1} - A^{-1}BZ_1L_1)^{\mathrm{T}} P_0^{-1}(A^{-1} - A^{-1}BZ_1L_1) \\ & + (Z_1L_1)^{\mathrm{T}} Q^{-1}(Z_1L_1) + C^{\mathrm{T}} R^{-1} C \end{aligned}$$

$$\begin{aligned} S_1 = & (A^{-1} - A^{-1}BZ_1L_1)^{\mathrm{T}} P_0^{-1}(A^{-1}BZ_1K_1 + I) \\ & - (Z_1L_1)^{\mathrm{T}} Q^{-1}(Z_1K_1) \end{aligned}$$

$$T = C^{\mathrm{T}} R^{-1}$$

$$U_1 = (A^{-1} - A^{-1}BZ_1L_1)^{\mathrm{T}} P_0^{-1}(A^{-1}B) - (Z_1L_1)^{\mathrm{T}} Q^{-1}$$

Motivated by the previous expressions, we now introduce the induction hypothesis that V_{k-1}^{OPT} has the form

$$\begin{aligned} & V_{k-1}^{\mathrm{OPT}}(\hat{x}_{k-1}, \hat{x}_{k-1}^*) = \\ & \frac{1}{2}(\hat{x}_{k-1} - \hat{x}_{k-1}^*)^{\mathrm{T}} P_{k-1}^{-1}(\hat{x}_{k-1} - \hat{x}_{k-1}^*) + c, \end{aligned} \tag{36}$$

Now, using the Principle of Optimality,

$$\begin{aligned} & V_k^{\mathrm{OPT}}(\hat{x}_k, \mu_0, y_1^d, \ldots, y_k^d) = \\ & \min_{\hat{w}_{k-1} \in \Omega_1} \bigg\{ \frac{1}{2}\hat{w}_{k-1}^{\mathrm{T}} Q^{-1} \hat{w}_{k-1} \\ & \quad + V_{k-1}^{\mathrm{OPT}}(A^{-1}\hat{x}_k - A^{-1}B\hat{w}_{k-1}, \hat{x}_{k-1}^*) \\ & \quad + \frac{1}{2}(y_k^d - C\hat{x}_k)^{\mathrm{T}} R^{-1}(y_k^d - C\hat{x}_k) \bigg\}. \end{aligned} \tag{37}$$

and substituting (36) into (37) and taking derivatives as before, the expression for the *unconstrained* $\hat{w}_{k-1}^{\mathrm{unc}}$ minimising V_k^{OPT} is obtained

$$\begin{aligned} \hat{w}_{k-1}^{\mathrm{unc}} = & \left[(A^{-1}B)^{\mathrm{T}} P_{k-1}^{-1}(A^{-1}B) + Q^{-1}\right]^{-1} \times \\ & (A^{-1}B)^{\mathrm{T}} P_{k-1}^{-1}(A^{-1}\hat{x}_k - \hat{x}_{k-1}^*). \end{aligned}$$

Again, the optimal *constrained* solution of (37), $\hat{w}_{k-1}^{\mathrm{OPT}}$, can be written as:

$$\hat{w}_{k-1}^{\mathrm{OPT}} = Z_k \left[L_k \; K_k\right] \begin{bmatrix} \hat{x}_k \\ \hat{x}_{k-1}^* \end{bmatrix} + h_k, \tag{38}$$

where

$$Z_k = \left[(A^{-1}B)^{\mathrm{T}} P_{k-1}^{-1}(A^{-1}B) + Q^{-1}\right]^{-1} (A^{-1}B)^{\mathrm{T}} P_{k-1}^{-1},$$

$L_k = A^{-1}$ if $\Delta_1 \leq Z_k(A^{-1}\hat{x}_k - \hat{x}_{k-1}^*) \leq \Delta_2$, and 0 otherwise,

$K_k = -I$ if $\Delta_1 \leq Z_k(A^{-1}\hat{x}_k - \hat{x}_{k-1}^*) \leq \Delta_2$, and 0 otherwise, and

$$h_k = \begin{cases} 0 & \text{if} \quad \Delta_1 \le Z_k(A^{-1}\hat{x}_k - \hat{x}^*_{k-1}) \le \Delta_2, \\ \Delta_1 & \text{if} \quad Z_k(A^{-1}\hat{x}_k - \hat{x}^*_{k-1}) < \Delta_1 \\ \Delta_2 & \text{if} \quad Z_k(A^{-1}\hat{x}_k - \hat{x}^*_{k-1}) > \Delta_2. \end{cases}$$

Substituting (38) into (37) and completing squares, the minimum attained in (37) can be written as

$$V_k^{\mathrm{OPT}}(\hat{x}_k, \hat{x}_k^*) = \frac{1}{2}(\hat{x}_k - \hat{x}_k^*)^{\mathrm{T}} P_k^{-1}(\hat{x}_k - \hat{x}_k^*) + c, \tag{39}$$

where

$$\hat{x}_k^* = P_k \left[S_k \; T\right] \begin{bmatrix} \hat{x}^*_{k-1} \\ y_k^d \end{bmatrix} + P_k U_k h_k, \tag{40}$$

$$P_k^{-1} = (A^{-1} - A^{-1}BZ_kL_k)^{\mathrm{T}} P_{k-1}^{-1}(A^{-1} - A^{-1}BZ_kL_k) + (Z_kL_k)^{\mathrm{T}} Q^{-1}(Z_kL_k) + C^{\mathrm{T}} R^{-1} C$$

$$S_k = (A^{-1} - A^{-1}BZ_kL_k)^{\mathrm{T}} P_{k-1}^{-1}(A^{-1}BZ_kK_k + I) - (Z_kL_k)^{\mathrm{T}} Q^{-1}(Z_kK_k)$$

$$T = C^{\mathrm{T}} R^{-1}$$

$$U_k = (A^{-1} - A^{-1}BZ_kL_k)^{\mathrm{T}} P_{k-1}^{-1}(A^{-1}B) - (Z_kL_k)^{\mathrm{T}} Q^{-1}$$

We can see that (39) is of the same form of the induction hypothesis, and hence, by induction the expressions are valid for $k = 1, \cdots, N$. Therefore, the procedure can be repeated recursively until time N, at which point, upon minimizing the right hand side of (39) (for the case $k = N$) it is straightforward to see that the optimal estimate for $\hat{x}_N$ is given by $\hat{x}_N^*$. It follows that substituting recursively $\hat{x}^*_{k-1}$ in (40) for $k = N, N-1, \ldots, 1$, then the optimal estimate for $\hat{x}_N$ can be written as $\hat{x}_N^{\mathrm{OPT}} = \hat{x}_N^* = \alpha_N \mathcal{Y} + \beta_N$, where

$$\begin{aligned} \alpha_N = \Big[& \prod_{i=1}^{N} P_{N-i+1} S_{N-i+1}, \\ & \Big(\prod_{i=1}^{N-1} P_{N-i+1} S_{N-i+1}\Big) P_1 T, \\ & \Big(\prod_{i=1}^{N-2} P_{N-i+1} S_{N-i+1}\Big) P_2 T, \quad \cdots \;, \\ & P_N S_N P_{N-1} T, \quad P_N T \quad \Big] \end{aligned} \tag{41}$$

and

$$\beta_N = \sum_{i=1}^{N} \left[\prod_{j=1}^{N} P_{N-j+1} S_{N-j+1}\right] P_i U_i h_i \tag{42}$$

The previous results are summarized in the following theorem.

Theorem 1. Consider the linear system (1)–(2) where $\{w_k\}$ is an i.i.d. process noise sequence having a truncated Gaussian distribution given by (3) with $\Omega_1 = [\Delta_1 \; \Delta_2]$, $\{v_k\}$ is an i.i.d. measurement noise sequence with normal distribution $N(0, R)$ and x_0 is a normal random vector with distribution $N(\mu_0, P_0)$, independent of $\{w_k\}$ and $\{v_k\}$. Consider the joint a posteriori most probable state estimation problem stated in (10). Denote the optimal state estimated sequence by

$$\hat{\mathbf{x}}_N^{\mathrm{OPT}} = \left\{\hat{x}_0^{\mathrm{OPT}}, \ldots, \hat{x}_N^{\mathrm{OPT}}\right\}. \tag{43}$$

Then, given the data $\mathcal{Y} \triangleq \left[\mu_0^{\mathrm{T}}, {y_1^d}^{\mathrm{T}}, \ldots, {y_N^d}^{\mathrm{T}}\right]^{\mathrm{T}}$, the last element of the optimal state estimated sequence is given by:

$$\hat{x}_N^{\mathrm{OPT}} = \alpha_N \mathcal{Y} + \beta_N \tag{44}$$

with α_N and β_N defined as in (41) and (42) respectively.

Remark 2. Notice that, as each triplet $\{L_k, K_k, h_k\}$ for $k = 1, \ldots, N$ used in the calculations can take 3 different sets of values corresponding to $\hat{w}_{k-1}$ saturating or not, there are 3^N possible values for α_N and β_N. The methodology we are presenting consists on calculating the 3^N possibilities, and determining in which region each of these possibilities is valid (see Mare and De Doná, 2004).

Remark 3. In many applications, the filtered estimate $\hat{x}_N^{\mathrm{OPT}}$ is of interest (that is, the last element of the optimal sequence $\hat{\mathbf{x}}_N^{\mathrm{OPT}}$ which is based on data up to time N). In such case, the optimal value is given directly in Theorem 1 by the affine function of the data (44). However, if smoothed estimates $\hat{\mathbf{x}}_N^{\mathrm{OPT}} = \left\{\hat{x}_0^{\mathrm{OPT}}, \ldots, \hat{x}_N^{\mathrm{OPT}}\right\}$ are of interest, these can be easily computed by using the inverse dynamics (21) and the optimal estimated process noise sequence (a by product of Theorem 1) given by equations (33),(38).

Corollary 4. The region of the data-space where the estimated state $\hat{x}_N^{\mathrm{OPT}} = \alpha_N \mathcal{Y} + \beta_N$ is optimal is given by the set of inequalities whose elements are, for each $k = 1, \ldots, N$, the inequalities corresponding to one of the following cases

$$\Delta_1 \le Z_k(A^{-1}\hat{x}_k - \hat{x}^*_{k-1}) \le \Delta_2, \text{or} \tag{45}$$

$$Z_k(A^{-1}\hat{x}_k - \hat{x}^*_{k-1}) < \Delta_1, \text{or} \tag{46}$$

$$Z_k(A^{-1}\hat{x}_k - \hat{x}^*_{k-1}) > \Delta_2. \tag{47}$$

As both $\hat{x}^*_{k-1}$ (given by (40) recursively) and $\hat{x}_k$ (given by the inverse dynamic (21)) depend linearly on the data $\mathcal{Y}$, these inequalities can be posed as linear inequalities defining polyhedral regions in the data-space. Indeed, the regions defined constitute a polyhedral partition of the data-space.

5. EXAMPLE

The model considered and the parameters in the cost function for this example are taken from Rao *et al.* (2003). Consider a discrete time system given by (1)–(2), with matrices

$$A = \begin{bmatrix} 0.99 & 0.2 \\ -0.1 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -3 \end{bmatrix}.$$

The constrained estimation problem is formulated assuming that $\{v_k\}$ is a sequence of independent, zero mean, normally distributed random variables with covariance 0.01 and $\{w_k\}$ is a sequence of independent random variables, having a truncated Gaussian distribution in the interval $[-1, 1]$, with zero mean and unit covariance. The initial state is also assumed to have a normal distribution, with mean equal to μ_0 and covariance equal to the identity. For the fixed horizon cost function, the values $N = 2$, $Q = 1$, $R = 0.01$ and $P_0 = I_{2\times 2}$ are taken. The process noise $\{w_k\}$ is constrained to lie in the interval $[-1, 1]$.

Figure 1 shows a projection of the data-space partition obtained, and the values for the affine function corresponding to each region are given on Table 1.

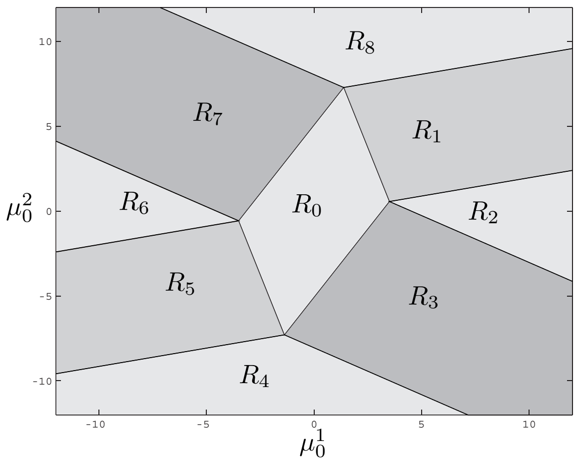

Fig. 1. Projection of the data-space partition onto the plane $\mu_0 = (\mu_0^1, \mu_0^2)$ (cut corresponding to $y_1^d = y_2^d = 0$).

6. CONCLUSIONS

An analytical solution for a constrained estimation problem for linear systems was derived using dynamic programming. The optimal solution is given by a piece-wise affine function of the data of this particular problem (the mean value of the initial state and the sequence of measurement data), and consists of a partition of the data space into a number of polyhedral regions where a unique affine function is valid. An example was provided to illustrate the structure of the optimal piece-wise affine solution.

REFERENCES

Bellman, R. (1957). *Dynamic Programming.* Princeton University Press. Princeton, New Jersey, USA.

Table 1. Optimal Estimator $\hat{x}_N^{\mathrm{OPT}} = \alpha_N \mathcal{Y} + \beta_N$

R_i	α_N	β_N
R_0	$\begin{bmatrix} 0.8157 & 0.2578 & 0.0067 & 0.1008 \\ 0.2716 & 0.0859 & 0.0021 & -0.2994 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
R_1	$\begin{bmatrix} 0.0095 & 0.0033 & -0.2961 & 0.9892 \\ 0.0002 & 0.0002 & -0.0998 & -0.0003 \end{bmatrix}$	$\begin{bmatrix} 2.9676 \\ 0.9990 \end{bmatrix}$
R_2	$\begin{bmatrix} 0.0067 & 0.0162 & -0.2785 & 0.9745 \\ -0.0001 & 0.0013 & -0.0983 & -0.0016 \end{bmatrix}$	$\begin{bmatrix} 2.9701 \\ 0.9992 \end{bmatrix}$
R_3	$\begin{bmatrix} 0.3130 & 0.5703 & 0.4496 & 0.0498 \\ 0.1042 & 0.1899 & 0.1496 & -0.3164 \end{bmatrix}$	$\begin{bmatrix} 1.5837 \\ 0.5273 \end{bmatrix}$
R_4	$\begin{bmatrix} 0.0067 & 0.0162 & -0.2785 & 0.9745 \\ -0.0001 & 0.0013 & -0.0983 & -0.0016 \end{bmatrix}$	$\begin{bmatrix} -2.8770 \\ -0.9910 \end{bmatrix}$
R_5	$\begin{bmatrix} 0.0095 & 0.0033 & -0.2961 & 0.9892 \\ 0.0002 & 0.0002 & -0.0998 & -0.0003 \end{bmatrix}$	$\begin{bmatrix} -2.9676 \\ -0.9990 \end{bmatrix}$
R_6	$\begin{bmatrix} 0.0067 & 0.0162 & -0.2785 & 0.9745 \\ -0.0001 & 0.0013 & -0.0983 & -0.0016 \end{bmatrix}$	$\begin{bmatrix} -2.9701 \\ -0.9992 \end{bmatrix}$
R_7	$\begin{bmatrix} 0.3130 & 0.5703 & 0.4496 & 0.0498 \\ 0.1042 & 0.1899 & 0.1496 & -0.3164 \end{bmatrix}$	$\begin{bmatrix} -1.5837 \\ -0.5273 \end{bmatrix}$
R_8	$\begin{bmatrix} 0.0067 & 0.0162 & -0.2785 & 0.9745 \\ -0.0001 & 0.0013 & -0.0983 & -0.0016 \end{bmatrix}$	$\begin{bmatrix} 2.8770 \\ 0.9910 \end{bmatrix}$

Cox, H. (1964). On the estimation of state variables and parameters for noisy dynamic systems. *IEEE Transactions on Automatic Control* pp. 5–12.

Goodwin, G.C., M.M. Seron and J.A. De Doná (2004). *Constrained Control and Estimation. An Optimisation Approach.* Springer-Verlag. London.

Mare, J.B. and J.A. De Doná (2004). Use of dynamic programming for the analytical solution of input-constrained LQR problems. *5th Asian Control Conference* pp. 441–447.

Muske, K.R., J.B. Rawlings and J.H. Lee (1993). Receding horizon recursive state estimation. *American Control Conference* pp. 900–904.

Rao, C.V., J.B. Rawlings and D.Q. Mayne (2003). Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations. *IEEE Transactions on Automatic Control* **48**, 246–258.

Rao, C.V., J.B. Rawlings and J.H. Lee (2001). Constrained linear estimation - a moving horizon approach. *Automatica* **37**, 1619–1628.

Robertson, D.G., J.H. Lee and J.B. Rawlings (1996). Moving horizon-based approach for least squares estimation. *AIChE Journal* **42(8)**, 2209–2224.

Tondel, P., T.A. Johansen and A. Bemporad (2003). Evaluation of piecewise affine control via binary search tree. *Automatica* **39**, 945–950.