# SLIDING MODE CONTROL OF UNCERTAIN LINEAR SYSTEMS BASED ON A HIGH GAIN OBSERVER FREE OF PEAKING

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Abstract: A model-reference variable structure controller based on a high gain observer (HGO) is proposed and analyzed. For single-input-single-output (SISO) linear plants with relative degree larger than one, the switching law is generated using the HGO state while the modulation function in the control law is generated using signals from state variable filters free of high gains. This scheme achieves global exponential stability with respect to a small residual set and does not generate the peaking phenomena.



Keywords: Variable-structure control, high gain observer, model reference control.

## 1. INTRODUCTION

This paper presents a model-reference variable structure controller (VSC) that uses simultaneously a high gain observer (HGO) and state variable filters to implement the control law. A HGO is employed motivated by its robustness to uncertainties and disturbances (Oh and Khalil, 1997). The *peaking phenomena*, an usual problem in HGOs which can degrade the system performance (Sussmann and Kokotović, 1991), is avoided by means of the following strategy: **(i)** The estimated state is used only in the definition of the sliding surface. This way, the peaking is confined to the observer. The sign function blocks the transmission of the peaking to the plant. **(ii)** The modulation function in the control law is generated using signals from state variable filters without high gains and thus peaking free. **(iii)** The peaking can be eliminated from the HGO through an appropriate state scaling transformation.

The resulting closed-loop control system is shown to be globally exponentially stable with respect to a small residual set and the tracking error can be kept arbitrarily small.

The $\mathscr{L}_{\infty e}$ norm of the signal $x(t) \in \mathbb{R}^n$ is defined as $\|x_{t,t_0}\|_\infty := \sup_{t_0 \leq \tau \leq t} \|x(\tau)\|$. A mixed time-domain and Laplace transform domain is adopted. The output signal $y$ of a linear time-invariant system with transfer function matrix $H(s)$ and input $u$ is denoted by $H(s)u$. Pure convolution $h(t) * u(t)$, where $h(t)$ is the impulse response of $H(s)$, is denoted by $H(s) * u$.

## 2. PROBLEM STATEMENT

Let a linear, time-invariant, observable and controllable plant be described by

$$\dot{x}_p = A_p x_p + B_p[u + d(t)], \qquad y = C_p x_p, \qquad (1)$$

where $x_p \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output and $d \in \mathbb{R}$ is an unmeasured input disturbance. The corresponding input-output model is

$$G(s) = C_p(sI - A_p)^{-1} B_p = K_p \frac{N_p(s)}{D_p(s)}, \qquad (2)$$

where $K_p \in \mathbb{R}$ is the high frequency gain, $N_p(s)$ and $D_p(s)$ are monic polynomials. The parameters of

[1] Partially supported by CNPq and FAPERJ, Brazil.

the plant are uncertain, i.e., only known within finite bounds. The following assumptions regarding the plant are taken as granted: **(A1)** $G(s)$ is minimum phase; **(A2)** $G(s)$ is strictly proper; **(A3)** the order of the system ($n$) is known; **(A4)** the relative degree $n^*$ of $G(s)$ is known; **(A5)** the sign of $K_p$ is known and assumed positive for simplicity; **(A6)** the disturbance $d(t)$ is piecewise continuous and a bound $\bar{d}(t)$ is known such that $|d(t)| \leq \bar{d}(t) \leq \bar{d}_{\sup} < +\infty$, $\forall t \geq 0$.

*The reference model* is defined by (Ioannou and Sun, 1996)

$$y_M = W_M(s)\, r\,, \quad r, y_M \in \mathbb{R}\,. \tag{3}$$

where $r(t)$ is a piecewise continuous and uniformly bounded reference signal. For plants with relative degree one, $W_M(s)$ must be *strictly positive real* (SPR) (Hsu and Costa, 1989). For plants with relative degree $n^* > 1$, (Hsu *et al.*, 1994; Hsu *et al.*, 1997) a monic Hurwitz polynomial $L(s)$ of degree $N := n^* - 1$ should be chosen such that the transfer function $W_M(s)L(s)$ be SPR. Here the choice of the reference model is more restrictive since it is required that $W_M(s)L(s) = K_M/(s+\gamma)$, imposed by the HGO structure and the closed-loop control system design. Hence, the reference model transfer function is given by

$$W_M(s) = \frac{K_M}{L(s)(s+\gamma)}\,, \tag{4}$$

where $K_M > 0$ is the high frequency gain of the reference model, $\gamma > 0$, and $L(s)(s+\gamma) = s^{n^*} + a_{n^*-1}s^{n^*-1} + \cdots + a_1 s + a_0$. It should be noted that the zeros of $L(s)$ are not required to be real. In this aspect the design of the proposed scheme has more freedom than the variable structure model-reference adaptive controller (VS-MRAC) (Hsu *et al.*, 1994; Hsu *et al.*, 1997), which requires that all the zeros of $L(s)$ be real, since the operator $L(s)$ is realized *loc. cit.* through the cascade connection of $N$ variable structure lead filters with single real zeros.

*The control objective* is to achieve asymptotic convergence of the output error $e(t) := y(t) - y_M(t)$ to some small residual neighborhood of zero.

## 3. CONTROL PARAMETERIZATION

If the plant and the disturbance $d(t)$ are perfectly known, then a control law which achieves matching between the closed-loop transfer function and $W_M(s)$ is given by

$$u^* = \theta^{*T}\omega - W_d(s) * d(t)\,, \tag{5}$$

$$W_d(s) = 1 - \theta_1^{*T}\frac{A(s)}{\Lambda(s)}\,, \tag{6}$$

where the parameter vector is given by

$$\theta^{*T} = \left[\theta_1^{*T}\ \theta_2^{*T}\ \theta_3^*\ \theta_4^*\right]\,, \tag{7}$$

with $\theta_1^*, \theta_2^* \in \mathbb{R}^{(n-1)}$, $\theta_3^*, \theta_4^* \in \mathbb{R}$ and the regressor vector

$$\omega = [\omega_1^T\ \omega_2^T\ y\ r]^T\,. \tag{8}$$

The state variable filters are given by

$$\omega_1 = \frac{A(s)}{\Lambda(s)}u\,, \quad \omega_2 = \frac{A(s)}{\Lambda(s)}y\,, \tag{9}$$

where $A(s) = [s^{n-2}, s^{n-3}, \ldots, s, 1]^T$ and $\Lambda(s)$ is an arbitrary monic Hurwitz polynomial of degree $n-1$.

The matching conditions require that $\theta_4^* = K_p^{-1}K_M$ (Ioannou and Sun, 1996). The signal $W_d(s) * d(t)$ cancels the input disturbance $d(t)$ (Cunha *et al.*, 2003).

## 4. ERROR EQUATIONS

Let $X = [x_p^T, \omega_1^T, \omega_2^T]^T$. Consider a non minimal realization $\{A_c, B_c, C_o\}$ of $W_M(s)$ with state vector $X_M$. Then, the state error ($X_e := X - X_M$) and the output error satisfy (Hsu *et al.*, 1994)

$$\dot{X}_e = A_c X_e + B_c K\left[u - \theta^{*T}\omega + W_d(s) * d(t)\right]\,, \tag{10}$$

$$e = C_o X_e\,, \tag{11}$$

where $K := (\theta_4^*)^{-1} = K_M^{-1}K_p$. From (10)–(11), the output error can be expressed as

$$e = W_M(s)K\left[u - \theta^{*T}\omega + W_d(s) * d(t)\right]\,. \tag{12}$$

For the control parameterization described above, it is assumed that

**(A7)** The control law satisfies the inequality $\|u_{t,0}\|_\infty \leq k_\omega\|\omega_{t,0}\|_\infty + k_{rd}$, where $k_\omega,\ k_{rd} > 0$.

This assumption assures that no finite time escape occurs in the system signals (Sastry and Bodson, 1989).

## 5. VARIABLE STRUCTURE CONTROL

For plants of relative degree $n^* = 1$, the VS-MRAC algorithm proposed by (Hsu and Costa, 1989) based on output-feedback requires no state observer.

If $n^* > 1$, then the output-feedback VSC cannot be directly applied, since the reference model is not SPR. A solution based on the use of prediction error and variable structure lead filters was presented in (Hsu *et al.*, 1994; Hsu *et al.*, 1997). In the present paper an alternative control strategy based on a HGO is proposed. A natural idea would be to estimate the plant state, as in the robust stabilization controller of (Oh and Khalil, 1995). Here, however, the error state is estimated instead. This way, the reference model (stable and perfectly known) is used instead of the plant model (uncertain and possibly unstable) for the estimator. In (Chang and Lee, 1996), it is recognized that the design of an observer for the reference model is easier than the design of an observer for the plant. A similar approach was proposed by (Oh and Khalil, 1997) for trajectory tracking in nonlinear systems. In (Oh and Khalil, 1995) and (Oh and Khalil, 1997), VSC laws are applied where the modulation functions are globally bounded in order to avoid the *peaking* *phenomena*. However, as a result, only semi-global stability properties can be guaranteed.

Here, an HGO is employed only to generate the switching law. The modulation function in the control law is synthesized using signals from the state filters. This will lead us to obtain global stability without peaking phenomena in the plant and control signals.

Consider the minimal order observer canonical form realization $\{A_M, B_M, C_M\}$ of the model $W_M(s)$. Then, the error equations (10)–(11) can be rewritten as

$$\dot{x}_e = A_M x_e + B_M K\left[u - \theta^{*T}\omega + W_d(s)*d(t) + \pi_e\right], \quad e = C_M x_e, \tag{13}$$

where the initial condition $x_e(0)$ and the exponentially decreasing scalar signal $\pi_e(t)$ are adequate for representing the initial condition of the observable but uncontrollable modes in the error equations (10)–(11). This signal satisfies $|\pi_e(t)| \leq k_e \exp(-\lambda_e t)\|X_e(0)\|$ with appropriate constants $k_e, \lambda_e > 0$.

It is possible to design a matrix $S \in \mathbb{R}^{1\times n^*}$, which defines the ideal sliding surface $\sigma(x_e) = Sx_e = 0$, such that $\{A_M, B_M, S\}$ be a realization of the SPR transfer function $W_M(s)L(s)$. Assuming that an estimate $\hat{x}_e$ for $x_e$ is available, then a control law is given by

$$u = u^{\text{nom}} + U, \qquad U = -\rho\,\text{sgn}(S\hat{x}_e), \tag{14}$$

$$u^{\text{nom}} = \theta^{\text{nom}T}\omega, \tag{15}$$

where $\theta^{\text{nom}}$ is the nominal value of $\theta^*$. The nominal control $u^{nom}$ allows the reduction of the modulation function amplitude if the parameter uncertainty $\|\theta^* - \theta^{\text{nom}}\|$ is small. If $\hat{x}_e(t) \equiv x_e(t)$ and the modulation function $\rho$ satisfies the inequality

$$\rho(t) \geq \left\|(\theta^{\text{nom}} - \theta^*)^T\omega + W_d(s)*d(t)\right\|, \tag{16}$$

then the output error $e(t)$ converges exponentially to zero, as can be concluded by applying Lemma 1 of (Hsu *et al.*, 1997).

## 6. HIGH GAIN OBSERVER

Since the state $x_e$ is not measured, the control law will use the state ($\hat{x}_e$) estimated by the HGO

$$\dot{\hat{x}}_e = A_M\hat{x}_e + B_M K^{\text{nom}}U - \left[\alpha(\varepsilon^{-1}) - a_M\right]\tilde{e}, \tag{17}$$

$$\tilde{e} = C_M\hat{x}_e - e, \tag{18}$$

with $\tilde{e}$ being the observer output error, $K^{\text{nom}}$ being the nominal value of the gain $K$ and, $a_M = [a_{n^*-1}, \ldots, a_1, a_0]^T$. The coefficients $\alpha_i$ in the observer feedback vector (Lu and Spurgeon, 1998)

$$\alpha(\varepsilon^{-1}) = \left[\frac{\alpha_{n^*-1}}{\varepsilon} \;\cdots\; \frac{\alpha_1}{\varepsilon^{n^*-1}} \;\; \frac{\alpha_0}{\varepsilon^{n^*}}\right]^T \tag{19}$$

must be chosen such that the characteristic polynomial of the closed-loop observer is Hurwitz, which is achieved if $N_\alpha(s) = s^{n^*} + \alpha_{n^*-1}s^{n^*-1} + \cdots + \alpha_0$ is Hurwitz and $\varepsilon > 0$. Since it is desired that the uncertainties and disturbances have negligible effects in the estimated state $\hat{x}_e$, the norm of the observer feedback vector ($\|\alpha(\varepsilon^{-1})\|$) shall be large, which imposes that $\varepsilon$ be small.

### *6.1 Upper bound for the estimation error*

Theorem 1 below gives upper bounds for the state estimation error $\tilde{x}_e(t) := \hat{x}_e(t) - x_e(t)$ of the HGO. These bounds are required in the stability proof of the closed-loop control system. In order to simplify Theorem 1, a scalar $\lambda_\alpha$ which satisfies the inequality $0 < \lambda_e < \lambda_\alpha < \bar{\lambda}_\alpha$ is introduced, where $\bar{\lambda}_\alpha$ is the stability margin of the polynomial $N_\alpha(s)$ (i.e., $\bar{\lambda}_\alpha := \min_j\{-\Re(z_j)\}$, where $\{z_j\}$ are the roots of $N_\alpha(z_j) = 0$).

*Theorem 1.* Consider the observer (17)–(19) and the error equations (10)–(11). If assumption (A7) is satisfied, the signals $r(t)$ and $d(t)$ are uniformly bounded, the polynomial $N_\alpha(s)$ is Hurwitz and the parameter $\varepsilon \in (0,1]$, then $\exists k_1, \ldots, k_6 > 0$ such that the state estimation error ($\tilde{x}_e$) satisfies the inequalities

$$\|\tilde{x}_e(t)\| \leq \frac{k_1}{\varepsilon^{n^*-1}}\|\tilde{x}_e(0)\|e^{-\frac{\lambda_\alpha}{\varepsilon}t} + k_2\varepsilon\|X_e(0)\|e^{-\lambda_e t} + e^{-\frac{\lambda_\alpha}{\varepsilon}t} * [k_3\|\omega(t)\| + k_4], \tag{20}$$

$$\|\tilde{x}_e(t)\| \leq \frac{k_1}{\varepsilon^{n^*-1}}\|\tilde{x}_e(0)\|e^{-\frac{\lambda_\alpha}{\varepsilon}t} + k_2\varepsilon\|X_e(0)\|e^{-\lambda_e t} + \varepsilon C(t,0), \tag{21}$$

$\forall t \geq 0$, where

$$C(t,t_0) = k_5\|\omega_{t,t_0}\|_\infty + k_6. \tag{22}$$

**PROOF.** The observer output error equation (18) is rewritten as $\tilde{e} = C_M\tilde{x}_e$. Subtracting (13) from (17), the state estimation error dynamic equation results

$$\dot{\tilde{x}}_e = A_e(\varepsilon^{-1})\tilde{x}_e + B_M\left[K^{nom}\bar{U} - K\pi_e(t)\right], \tag{23}$$

where

$$A_e(\varepsilon^{-1}) = A_M - \left[\alpha(\varepsilon^{-1}) - a_M\right]C_M, \tag{24}$$

$$\bar{U} := \left[1 - (K^{\text{nom}})^{-1}K\right]U + (K^{\text{nom}})^{-1}K \times \left[(\theta^* - \theta^{\text{nom}})^T\omega - W_d(s)*d(t)\right]. \tag{25}$$

Then, applying the linear scaling transformation

$$\bar{x}_e = T(\varepsilon)\tilde{x}_e,\; T(\varepsilon) = \text{diag}\left\{1, \varepsilon, \ldots, \varepsilon^{n^*-1}\right\}, \tag{26}$$

the error equation (23) is rewritten as

$$\varepsilon\dot{\bar{x}}_e = \bar{A}_e\bar{x}_e + \varepsilon\bar{B}_M(\varepsilon)\left[K^{nom}\bar{U} - K\pi_e(t)\right], \tag{27}$$

where the companion matrix $\bar{A}_e$ has characteristic polynomial $N_\alpha(s)$, $\bar{B}_M(\varepsilon) = [0, \ldots, 0, \varepsilon^{n^*-1}]^T K_M$ and, consequently, $\|\bar{B}_M(\varepsilon)\| = \varepsilon^{n^*-1}K_M$. Therefore, the norm of the state $\bar{x}_e$ is bounded by

$$\|\bar{x}_e(t)\| \leq k_1\|\bar{x}_e(0)\|e^{-\frac{\lambda_\alpha}{\varepsilon}t} + k_2\varepsilon^{n^*}\|X_e(0)\|e^{-\lambda_e t} + k_7\frac{\varepsilon^{n^*}}{\varepsilon s + \lambda_\alpha} * \|\bar{U}(t)\|, \quad \forall t \geq 0. \tag{28}$$

Since assumption (A7) is satisfied and the signals $r(t)$ and $d(t)$ are uniformly bounded, it can be concluded from (25) that $\exists k_{\bar{U}\omega}, k_{\bar{U}} \geq 0$ such that $\|\bar{U}_{t,0}\|_\infty \leq k_{\bar{U}\omega}\|\omega_{t,0}\|_\infty + k_{\bar{U}}$, $\forall t \geq 0$. On the other hand, from the definition of the transformation (26), one has $\|T(\varepsilon)\| = 1$ and $\|T^{-1}(\varepsilon)\| = \varepsilon^{1-n^*}$, since $\varepsilon \in (0,1]$. Considering these facts, the upper bounds (20) and (21) for $\|\tilde{x}_e(t)\|$ are obtained from (28). □

### *6.2 Peaking phenomena*

The peaking phenomena is evident in the term

$$p(t,\varepsilon^{-1}) := \frac{k_1}{\varepsilon^{n^*-1}}\|\tilde{x}_e(0)\| e^{-\frac{\lambda_\alpha}{\varepsilon}t}, \qquad (29)$$

present in the upper bounds for the state estimation errors (20)–(21). The parameter $\varepsilon$ should be chosen sufficiently small in order to reduce the residual estimation error (c.f. Section 7) and to speed up the response of the observer. However, this leads to large peak amplitudes of the order of $1/\varepsilon^{n^*-1}$ in the estimation error during the initial transient.

The *peak extinction time*, an important concept regarding high gain observers, is introduced below. This concept is based on the dynamics of the state estimation error ($\tilde{x}_e(t)$) of an HGO with $\bar{U}(t) \equiv 0$ and $\pi_e(t) \equiv 0$ given by

$$\dot{\tilde{x}}_e(t) = A_e(\varepsilon^{-1})\tilde{x}_e(t), \qquad t \geq 0, \qquad (30)$$

where the Hurwitz matrix $A_e(\varepsilon^{-1})$ is defined in (24).

*Definition 1.* The *peak extinction time* ($t_e$) of the HGO is the smallest time value such that inequality

$$\|\tilde{x}_e(t)\| \leq \|\tilde{x}_e(0)\|, \quad \forall t \geq t_e \geq 0, \quad \forall \tilde{x}_e(0), \qquad (31)$$

holds for a fixed value of the parameter $\varepsilon \in (0,1]$.

The precise computation of the *peak extinction time* of an HGO may be difficult. However, a convenient upper bound $\bar{t}_e$ can be obtained from inequalities (20) and (31) ($\bar{U}(t) \equiv 0$ and $\pi_e(t) \equiv 0$), which gives

$$\frac{k_1}{\varepsilon^{n^*-1}} e^{-\frac{\lambda_\alpha}{\varepsilon}t} \leq 1, \qquad \forall t \geq \bar{t}_e \geq 0, \qquad (32)$$

where $\bar{t}_e \geq t_e$, which leads to

$$\bar{t}_e = \frac{n^*-1}{\lambda_\alpha}\varepsilon\left[(n^*-1)^{-1}\ln(k_1) - \ln(\varepsilon)\right]. \qquad (33)$$

It can be concluded that the *upper bound for the peaking extinction time* is uniformly bounded with respect to the parameter $\varepsilon \in (0,1]$ and tends to zero as $\varepsilon \to +0$, since the values of the parameters $k_1 \geq 1$, $\lambda_\alpha > 0$ and $n^* \geq 2$ are fixed.

## 7. CONTROLLER FOR PEAKING FREE SIGNALS IN THE PLANT

A first proposed controller is based on the variable structure control law (14) and the HGO (17)–(18). To avoid *peaking* in the control signal $u$ and in the plant signals, the state filter signals (9) are used to generate the modulation function $\rho$. However, the *peaking* is still present in the estimated state, since this phenomena is intrinsic in the HGO (17)–(18).

For closed-loop stability analysis purpose, the state vector is defined as $z := \left[X_e^T, \tilde{x}_e^T\right]^T$. Theorem 2 below states the stability properties of the system with all the error signals equations given by (10)–(11) and (23). In what follows, all the $k$'s and $\lambda$'s denote generic positive constants, $\mathscr{K}$'s are class $\mathscr{K}$ functions and the operator norm ($\|\cdot\|$) is induced by the norm $\mathscr{L}_{\infty e}$.

*Theorem 2.* For $n^* > 1$, consider the plant (1), the control law (14)–(15), the state filters (9) and the observer (17)–(18). If the assumptions (A1)–(A6) are satisfied and the modulation function $\rho$ satisfies the inequality (16) and assumption (A7) then, for $\varepsilon > 0$ sufficiently small, the system composed by the error equations (10)–(11) and (23) with state $z$ will be globally exponentially stable with respect to a small residual set of order $\varepsilon$, i.e., there exist constants $k_z, \lambda_z > 0$ and a class $\mathscr{K}$ function $\mathscr{K}_X(\varepsilon)$ such that, $\forall z(0), \forall t \geq 0$,

$$\|z(t)\| \leq \left[\frac{k_z}{\varepsilon^{n^*-1}}\|z(0)\| + \mathscr{K}_X(\varepsilon)\right] e^{-\lambda_z t} + O(\varepsilon), \quad (34)$$

$$|e(t)| \text{ and } \|X_e(t)\| \leq [k_z\|z(0)\| + \mathscr{K}_X(\varepsilon)]\, e^{-\lambda_z t} + O(\varepsilon). \quad (35)$$

**PROOF.** See Appendix A. □

The trajectory tracking controller proposed by (Oh and Khalil, 1997) is also based on VSC and an HGO. This controller prevents the peaking phenomena in the plant through the global saturation of the control signal. However, the results obtained *loc. cit.* guarantee only semi-global stability and the finite time convergence of the state of the tracking error equation to a residual set of order $\sqrt{\varepsilon}$. In contrast, the results in Theorem 2 are stronger, i.e., global stability and exponential convergence of the error states to a residual set of order $\varepsilon$. On the other hand, the approach of (Oh and Khalil, 1997) also considers nonlinear plants.

## 8. PEAKING FREE HGO

A second proposed controller is based on a scaled HGO for the error state (13) given by

$$\zeta = T(\varepsilon)x_e, \quad T(\varepsilon) = \text{diag}\left\{1, \varepsilon, \ldots, \varepsilon^{n^*-1}\right\}. \qquad (36)$$

The peaking free HGO is obtained by using the transformation $\hat{\zeta} = T(\varepsilon)\hat{x}_e$. When applied in the HGO (17)–(18), it gives the following observer

$$\varepsilon\dot{\hat{\zeta}} = \bar{A}_M(\varepsilon)\hat{\zeta} + \varepsilon^{n^*}B_M K^{\text{nom}} U - [\bar{\alpha} - \bar{a}_M(\varepsilon)]\,\tilde{e},$$
$$\tilde{e} = C_M\hat{\zeta} - e, \qquad (37)$$

where $\bar{A}_M(\varepsilon) = \varepsilon T(\varepsilon) A_M T^{-1}(\varepsilon)$, $\bar{\alpha} = \varepsilon T(\varepsilon)\alpha(\varepsilon^{-1})$ and $\bar{a}_M(\varepsilon) = \varepsilon T(\varepsilon) a_M$. This observer is peaking free since the state estimation error $\bar{x}_e(t) := \hat{\zeta}(t) - \zeta(t)$ is the solution of (27), which is free of peaking, as can be concluded from the upper bound (28) for the norm of the state error. In this approach, the sliding surface is $\sigma(\hat{x}_e) = 0$, where $\sigma(\hat{x}_e) := S\hat{x}_e = ST^{-1}(\varepsilon)\hat{\zeta}$. The signal $\sigma(T^{-1}(\varepsilon)\hat{\zeta})$ can exhibit the peaking phenomena since its generation applies the transform matrix $T^{-1}(\varepsilon)$ which has norm $\varepsilon^{1-n^*}$. To avoid peaking, this signal is scaled adequately as $\bar{\sigma}(\hat{\zeta}) := \varepsilon^{n^*-1} ST^{-1}(\varepsilon)\hat{\zeta}$. Therefore, the control law is given by

$$u = u^{\text{nom}} + U\,, \qquad U = -\rho\,\text{sgn}\left(\bar{S}(\varepsilon)\hat{\zeta}\right), \tag{38}$$

with $\bar{S}(\varepsilon) := \varepsilon^{n^*-1} ST^{-1}(\varepsilon)$, and the nominal control (15). For the purpose of closed-loop stability analysis, the state vector is defined as $\bar{z} := \left[X_e^T, \bar{x}_e^T\right]^T$. Theorem 3 states the stability properties of the system with all the error signals equations (10)–(11) and (27).

*Theorem 3.* For $n^* > 1$, consider the plant (1), the control law (38) and (15), the state filters (9) and the observer (37). If the assumptions (A1)–(A6) are satisfied and the modulation function $\rho$ satisfies the inequality (16) and the assumption (A7) then, for $\varepsilon > 0$ sufficiently small, the system composed by the error equations (10)–(11) and (27) with state $\bar{z}$ is globally exponentially stable with respect to a residual set of order $\varepsilon$, i.e., there exist constants $k_z, \lambda_z > 0$ and a class $\mathscr{K}$ function $\mathscr{K}_X(\varepsilon)$ such that, $\forall \bar{z}(0)$, $\forall t \geq 0$,

$$|e(t)| \text{ and } \|\bar{z}(t)\| \leq [k_z\|\bar{z}(0)\| + \mathscr{K}_X(\varepsilon)]\, e^{-\lambda_z t} + O(\varepsilon). \tag{39}$$

**PROOF.** Similar to the proof of Theorem 2, with the state estimation error given by $\bar{x}_e(t) = T(\varepsilon)\tilde{x}_e(t)$. □

## 9. CONCLUSION

The model-reference VSC for uncertain linear SISO systems developed in this paper uses simultaneously a HGO and state variable filters. The estimated state is used only in the computation of the switching law while the modulation of the control signal is generated using the state filters. The proposed scheme is *globally exponentially stable* with respect to a small residual set and *free of peaking.* This is remarkable, since the application of the HGO usually results in *peaking* (e.g., (Esfandiari and Khalil, 1992; Emelyanov *et al.*, 1992)), as it was shown in (Oh and Khalil, 1995). Experiments carried out with a positioning servomechanism have shown the robustness of the proposed scheme with respect to input disturbances, unmodeled dynamics and measurement noise (Cunha, 2004).

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## Appendix A. PROOF OF THEOREM 2

Assumption (A7) assures that all the signals can grow at most exponentially. From the uniform boundedness of $r(t)$ it follows that $\|\omega\| \leq k_M + k_\Omega \|X_e\|$. Therefore,

$$\|x_e(t)\| \text{ and } \|X_e(t)\| \leq k_{e1} e^{\lambda_{e1} t} \|X_e(0)\| + k_{e2}\left[e^{\lambda_{e1} t} - 1\right], \tag{A.1}$$

holds $\forall t \geq 0$. There exists some class $\mathscr{K}$ function of $\varepsilon$ which bounds superiorly the peaking extinction time

$t_e \leq \bar{t_e}$ given by (33). Then, an upper bound for the hight hand side of (A.1) valid $\forall t \in [0, t_e]$ is given by

$$\|x_e(t)\| \text{ and } \|X_e(t)\| \leq (k_{e3}+k_{e4}\varepsilon)\|X_e(0)\|+\mathscr{K}_{e6}(\varepsilon)\,, \quad \text{(A.2)}$$

which combined to the upper bound (21) for the state estimation error lead to the inequality ($\forall t \in [0, t_e]$)

$$\|\tilde{x}_e(t)\| \leq \frac{k_1}{\varepsilon^{n^*-1}}\|\tilde{x}_e(0)\|e^{-\frac{\lambda\alpha}{\varepsilon}t}+\varepsilon\left[k_{e9}\|X_e(0)\|+k_{e10}\right]. \quad \text{(A.3)}$$

Now, an upper bound for the state $z$ valid for $t > t_1 := t_e$ is developed. The instant $t_1$ is taken as a new initial time after the extinction of the peaking in the HGO. Thereafter,

$$\|\tilde{x}_e(t)\| \leq \left[\|\tilde{x}_e(0)\|+\varepsilon\left(k_{e13}\|X_e(0)\|+k_{e12}\right)\right]e^{-\lambda_e(t-t_1)} + \varepsilon C(t,t_1)\,, \quad \forall t \geq t_1\,. \quad \text{(A.4)}$$

The error equation (13) is rewritten as

$$\dot{x}_e = A_M x_e + B_M K\left[U + d_U + \pi_e(t)\right]\,, \quad \text{(A.5)}$$

where $d_U := \left(\theta^{\text{nom}} - \theta^*\right)^T \omega + W_d(s) * d(t)$.

Since the estimated state ($\hat{x}_e(t) = x_e(t) + \tilde{x}_e(t)$) is applied, the control law (14) can be rewritten as

$$U = -\rho\,\text{sgn}\left(\sigma(\hat{x}_e)\right)\,, \quad \text{(A.6)}$$

$$\sigma(\hat{x}_e) = S x_e + S\tilde{x}_e\,. \quad \text{(A.7)}$$

Remembering that $\{A_M, B_M, S\}$ is a controllable non-minimal realization of $W_M(s)L(s) = \frac{K_M}{(s+\gamma)}$ and $K = K_M^{-1}K_p$ then, from the dynamic equation (A.5) and the algebraic equation (A.7), the switching signal $\sigma(\hat{x}_e)$ can be represented as

$$\sigma(\hat{x}_e) = \frac{K_p}{(s+\gamma)}\left[U + d_U(t)\right] + \pi_1(t) + \beta(t)\,, \quad \text{(A.8)}$$

where $\pi_1(t) + \beta(t) = \frac{K_p}{(s+\gamma)} * \pi_e(t) + S\tilde{x}_e(t)$. Through the application of the upper bound (A.4) for $\|\tilde{x}_e\|$, one has that, $\forall t \geq t_1$,

$$|\pi_1(t)| \leq k_{e14}\left[\|\tilde{x}_e(0)\|+\|X_e(0)\|+\varepsilon k_{e12}\right]e^{-\lambda_e(t-t_1)},$$
$$\|\beta_{t,t_1}\|_\infty \leq \varepsilon k_\beta C(t,t_1)\,. \quad \text{(A.9)}$$

Since $K_p$ is assumed positive and the modulation function satisfies $\rho(t) \geq |d_U(t)|$ ($\forall t \geq 0$), the Lemma 2 of (Hsu *et al.*, 1997) can be applied to the system composed of (A.8) and the control law (A.6), which results in the upper bound, $\forall t \geq t_1$,

$$|\sigma(\hat{x}_e)| \text{ and } |\hat{\sigma}(\hat{x}_e)| \leq k_{e15}\left[\|z(0)\|+\varepsilon k_{e12}\right]e^{-\lambda_e(t-t_1)} + 2\varepsilon k_\beta C(t,t_1)\,, \quad \text{(A.10)}$$

where $\hat{\sigma} := \sigma - \pi_1(t) - \beta(t)$. Remembering that $u = u^{\text{nom}} + U$, one can note that $\frac{K_p}{(s+\gamma)}$ in (A.8) operates in the same signal $U + d_U$ in (10). From (A.8) it can be concluded that $U + d_U = K_p^{-1}[\dot{\hat{\sigma}} + \gamma\hat{\sigma}]$. Then, the tracking error can be rewritten from (10) as

$$\dot{X}_e = A_c X_e + B_c K\left[\dot{\hat{\sigma}} + \gamma\hat{\sigma}\right]\,. \quad \text{(A.11)}$$

To eliminate the derivative term $\dot{\hat{\sigma}}$, a variable transformation $\bar{X}_e := X_e - B_c K\hat{\sigma}$ is performed yielding

$$\dot{\bar{X}}_e = A_c\bar{X}_e + (A_c + \gamma I)B_c K\hat{\sigma}\,. \quad \text{(A.12)}$$

Since $A_c$ is Hurwitz and the signal $\hat{\sigma}$ satisfies the upper bound (A.10), it can be verified that

$$\|\bar{X}_e(t)\| \leq k_{e16}\left[\|z(0)\|+\varepsilon k_{e12}\right]\exp\left[-\lambda_e(t-t_1)\right] + \varepsilon\bar{k}C(t,t_1)\,, \quad \forall t \geq t_1\,. \quad \text{(A.13)}$$

Moreover, as described in what follows, $\forall t \geq t_1$,

$$\|X_e(t)\| \text{ and } \|e(t)\| \leq k_{e17}\left[\|z(0)\|+\varepsilon k_{e12}\right]e^{-\lambda_e(t-t_1)} + \varepsilon k_{e18}C(t,t_1)\,, \quad \text{(A.14)}$$

$$\|\omega_{t,t_1}\|_\infty \leq \varepsilon k_{e19}C(t,t_1) + k_{e20}\|z(0)\| + k_m\,, \quad \text{(A.15)}$$

$$C(t,t_1) \leq \frac{k'_{\text{red}} + k_{e21}\|z(0)\|}{1-\varepsilon k_{e22}}\,. \quad \text{(A.16)}$$

The inequalities in (A.14) are developed from $X_e = \bar{X}_e + B_c K\hat{\sigma}$. Inequality (A.15) is obtained from (A.14), since $\|\omega\| \leq k_M + k_\Omega\|X_e\|$. Now, from (22) and (A.15), it can be concluded that $C(t,t_1) \leq \varepsilon k_{e22}C(t,t_1) + k_{e21}\|z(0)\| + k'_{\text{red}}$, from which the upper bound (A.16) (valid for $\varepsilon < k_{e22}^{-1}$) can be obtained. The upper bound for the complete state, $\forall t \geq t_1$,

$$\|z(t)\| \leq \left[k_{e23}\|z(0)\|+\varepsilon k_{e24}\right]e^{-\lambda_e(t-t_1)} + \varepsilon k_{e25}C(t,t_1),$$

is obtained through the combination of the upper bounds (A.4) for $\|\tilde{x}_e\|$ and (A.14) for $\|X_e\|$. Then, the application of the upper bound (A.16) results in

$$\|z(t)\| \leq \left[k_{e23}\|z(0)\|+\varepsilon k_{e24}\right]e^{-\lambda_e(t-t_1)} + \varepsilon\frac{k_{e26}+k_{e27}\|z(0)\|}{1-\varepsilon k_{e22}}\,, \quad \text{(A.17)}$$

$\forall t \geq t_1$, which can be rewritten as

$$\|z(t)\| \leq \left[k_{e23}\|z(0)\|+\varepsilon k_{e24}\right]e^{-\lambda_e(t-t_1)} + \varepsilon\left[k_{e28} + k_{e29}\|z(0)\|\right]\,, \quad \forall t \geq t_1\,, \quad \text{(A.18)}$$

valid for $0 < \varepsilon \leq k_\varepsilon < \min(1, k_{e22}^{-1})$. Therefore,

$$\|z(t)\| \leq \left[k_{e23}e^{-\lambda_e(t-t_1)} + \varepsilon k_{e29}\right]\|z(0)\| + O(\varepsilon)\,,$$

holds $\forall t \geq t_1$, where the residual term $O(\varepsilon)$ is independent from the initial conditions. Noting that the initial time is irrelevant in the development of the above expressions, the inequality

$$\|z(t)\| \leq \left[k_{e30}e^{-\lambda_e(t-t_i)} + \varepsilon k_{e31}\right]\|z(t_i)\| + O(\varepsilon),$$

holds for any $t \geq t_i \geq t_1$ ($i = 1, 2, 3, \ldots$). This leads to the recursive linear inequality

$$\|z(t_{i+1})\| \leq \lambda\|z(t_i)\| + O(\varepsilon)\,, \quad \text{(A.19)}$$

with $\lambda = k_{e30}\exp(-\lambda_e T_1) + \varepsilon k_{e31}$ and some period $T_1 = t_{i+1} - t_i > 0$. For $0 < \varepsilon \leq \varepsilon^* < k_{e31}^{-1}$ and choosing $T_1 > 0$ large enough, $\lambda < 1$ is obtained. Thus, for $\varepsilon > 0$ sufficiently small, the recursion (A.19) converges exponentially to a residual set of order $\varepsilon$.

The upper bounds (34) and (35) for the norms of the error signals, valid $\forall t \geq 0$, are finally obtained, since after $t_1$ the state $z(t)$ converges exponentially to a residual set of order $\varepsilon$ and, the upper bounds (A.2) for $\|X_e\|$ and (A.3) for $\|\tilde{x}_e\|$ hold for $0 \leq t \leq t_1$.