OPTIMAL CONTROL FOR SWITCHED POINT DELAY SYSTEMS WITH REFRACTORY PERIOD

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Abstract: The problem of optimal switching of a multi-mode time delay system is considered. The class of multi-mode systems consists of systems where the control variables are the switching times in a sequence of fixed vector fields. We assume that the systems considered all have a refractory period, in the sense that once an action is taken, it takes a non-infinitesimal amount of time before a subsequent action can be taken. Necessary conditions for a stationary solution are derived for systems with a single or commensurate delays, and shown to extend those of the delay free case in (Egerstedt *et al.* 2003).

Keywords: Time-delay, Optimal Control, Switched Systems, Hybrid Systems

1. INTRODUCTION

The switching control problem for a finite dimensional multi-mode system involves control actions at discrete instants. We assume that at the switching times, the state is carried over from one mode to the other in a continuous fashion. The control variables are the switching times between the fixed systems $\dot{x} = f_i(x,t)$, where $i \in \{1,\ldots,m\} = \Xi$, assuming continuity of the state at the switching times, and the particular sequence of modes. Switched systems belong to a general class of *hybrid systems*, discussed in (Branicky *et al.* 1998, Sussmann 1999). Such a control is parameterized by the number of switches, $N-1$, a "word" of length N with alphabet, Ξ, and a sequence of switching times $\{T_1,\ldots,T_{N-1}\}$. We show first that without restrictions on the duration of modes, the *optimal sequencing* problem with $N-1$ switches follows from the solution for optimal timing given a *fixed* sequence. However, we shall assume that the systems considered all have a refractory period, in the sense that once an action is taken, it takes a non-infinitesimal amount of time before a subsequent action can be taken. Consequently, the optimal sequencing no longer follows from the solution of the fixed sequence problem, except by comparison of the N^m possible time optimized strategies. Refractory periods are ubiquitous in physiological systems, and many technological systems, (e.g., time required to recharge a capacitor). A refractory time provides a safeguard towards unwanted high frequency switching. Other instances of such systems appear in chemical process technology, automotive systems, electromechanical and manufacturing systems. The paper extends the results of (Egerstedt *et al.* 2003) to systems with delays, but derives the optimality conditions via a classical variational approach (Bryson and Ho 1975). We believe its derivation to be somewhat more straightforward. The presence of delays adds a nontrivial twist to the original problem posed in (Xu *et al.* 2002). We point out that the optimal switching problem bears some relation to the optimal impulsive control problem in (Verriest *et al.* 2004). Necessary conditions for the optimal switching policy are determined in Section 2.

2. VARIATIONAL APPROACH TO OPTIMAL SWITCHING

Consider a system with a delay, τ. We follow standard notation (Hale and Verduyn Lunel 1993) and denote by x_t the data $\{\, x(t+\theta) \mid -\tau \leq \theta \leq 0 \,\}$. Let a finite set of autonomous vector fields, $\{f^{(a)}(x_t)\}$, be given. The dynamical system discussed in this paper is a switched delay system with a *fixed* sequence of vector fields: $f_i(x_t); i = 1, \ldots, N$, and the *instants of switching* are the sole control variables. We take therefore the space $C([-\tau, 0], \mathbb{R}^n)$ as the state space for this multi-mode delay system. The vector $x(t) \in \mathbb{R}^n$ is called the *partial state* at t. As in the delay free problem, we assume that the entire state x_t is carried over from one mode to the next at the switch. This is implied by continuity of x. The problem is to determine these optimal switching times in order to minimize

$$J = \int_0^T L(x, \xi)\, dt + \Phi(x(T)) \tag{1}$$

for a fixed terminal time T. Here, $\xi(t)$ is a discrete state, taking values in the finite set, Ξ, and denotes the operating mode at time t. If $\xi(t) = a$, then the dynamical system at t is the autonomous system

$$\dot{x}(t) = f^{(a)}(x_t). \tag{2}$$

Denote the nominal switching times by $T_i; i = 1, \ldots, N-1$, and define $T_0 = 0$ and $T_N = T$ (assumed fixed). For simplicity, of notation, set $L(x, \xi(t)) = L_i(x)$ and $f^{(\xi(t))} = f_i$ in the interval (T_{i-1}, T_i). The performance index (1) expands to

$$\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} L_i(x)\, dt + \Phi(x(T)), \quad \text{with } \dot{x} = f_i(x_t) \tag{3}$$

for $T_{i-1} \leq t \leq T_i$. Consider now arbitrary, independent perturbations of the nominal T_i with scale parameter, ϵ, which we will let eventually tend to zero, i.e., $T_i \to T_i + \epsilon\theta_i$. Adjoining the dynamical constraints with *different* Lagrange multipliers, defined in each appropriate subinterval, will not alter the value of J. Assume further that optimal values T_i exist, giving the optimal nominal performance index, $\overline{J}_0$, equal to

$$\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [L_i(x) + \lambda_i'(f_i(x_t) - \dot{x})]\, dt + \Phi(x(T)) \tag{4}$$

It is very important to remark that due to the requisite continuity of the state, a change in T_j say will have an effect on all the modes $i > j$. This happens because the change from T_j to $T_j + \epsilon\theta_i$ (keeping everything else the same) now changes the final (partial) state in the j-th mode from $x(T_j)$ to ($\theta_j > 0$ is assumed, the other case being similar)

$$x(T_j + \epsilon\theta_j) = x(T_j) + \dot{x}(T_j^-)\, \epsilon\theta_j.$$

Note also that the *left* derivative needs to be taken with mode j, and this is

$$\dot{x}(T_j^-) = f_j(x_{T_i}).$$

If however, no perturbation were made to T_j, then the value of the partial state $x(T_j + \epsilon\theta_j)$ would have been

$$x(T_j + \epsilon\theta_j) = x(T_j) + \dot{x}(T_j^+)\, \epsilon\theta_j.$$

This gives a difference in the state at the beginning of the $j+1$-st mode of

$$\Delta_{T_j} x = [f_j(x_{T_j}) - f_{j+1}(x_{T_j})]\, \epsilon\theta_j. \tag{5}$$

As each subsequent switch will add such a term, it is clear that the effects of all such perturbations will accumulate in subsequent modes, and keeping track of all these effects will complicate the derivation requiring the explicit computation of perturbations as done in (Egerstedt *et al.* 2003). In keeping with the philosophy of calculus of variations, we shall avoid having to keep track of these by introducing a *sequence* of induced variations, $\{\eta_j\}$, in the same way as we introduced independent Lagrange multipliers λ_j for each mode, i.e., in each subinterval.Equivalently, we may model the induced partial state variation $\eta(t)$ as a *discontinuous* function with discontinuities at the switching times. These can then be *chosen* in a very convenient way in order to avoid computation of the induced variations. Since continuity of the state of the delay systems implies the continuity of the vector function $x(t)$, a partial state, we shall adjoin the constraints $x(T_i^-) = x(T_i^+)$ at the switching times by a sequence of Lagrange multipliers μ_i.

Defining the Hamiltonian functionals,

$$H_i(x_t, \lambda) = L_i(x) + \lambda_i' f_i(x_t), \tag{6}$$

we consider thus

$$\overline{J}_0 = \Phi(x(T)) + \sum_{i=1}^{N} \mu_i'[x(T_i^+) - x(T_i^-)] + \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [H_i(x_t) - \lambda_i'\dot{x}]\, dt. \tag{7}$$

A neighboring solution (with η possibly discontinuous) yields the perturbed performance index

$$\overline{J}_\epsilon = \Phi(x(T)+\epsilon\eta(T)) \; + \sum_{i=1}^{N} \mu_i' \; (x+\epsilon\eta_i)|_{(T_i+\epsilon\theta_i)^-}^{(T_i+\epsilon\theta_i)^+}$$
$$+ \sum_{i=1}^{N} \int\limits_{T_{i-1}+\epsilon\theta_{i-1}}^{T_i+\epsilon\theta_i} [H_i(x_t+\epsilon\eta_t) - \lambda_i'(\dot{x}+\epsilon\dot{\eta})] \, dt. \tag{8}$$

Note that $\theta_0 = \theta_N = 0$, since initial and final time were considered fixed.

2.1 Separable Mode Systems

We shall leave the full generality of the problem behind and consider from now on only systems having a single crisp delay (point delay) and with *separable* modes defined by the functional differential equations

$$\dot{x}(t) = f_i(x(t)) + g_i(x(t-\tau)). \tag{9}$$

Thus, the Hamiltonians, $H_i(x_t, \lambda_i)$ and final cost adjoined with the state continuity constraints are respectively defined by

$$L_i(x(t)) + \lambda_i'(t)\,[f_i(x(t)) + g_i(x(t-\tau))] \tag{10}$$

$$\Psi(x,\mu) = \Phi(x) + \sum_{i=1}^{N} \mu_i'[x(T_i^+) - x(T_i^-)]. \tag{11}$$

Precisely definition (11) obviates the need for computing the perturbations at switching times.

2.2 Delay effect of a single switch

Definition[1] *A function y is said to be C_k at t_0, if the k-th derivative of y is continuous at t_0, but the $(k+1)$-st is not.*
Obviously, this implies that the derivatives of order i are all continuous at t_0 for $i \leq k$.

Assume that a single controlled switch occurs at time T, switching from mode i to $i{+}1$. This makes $\dot{x}$ discontinuous at T. Consequently, x and f have a 'kink' (i.e., are non-differentiable) at T. From the continuity assumption, x and f are C_0 at T. But then $x(t-\tau)$ and g are C_0 at $t = T+\tau$. In turn this implies that $\dot{x}$ is C_0 at $T+\tau$, inducing again C_1 behavior in x and f at $T+\tau$ and C_1 behavior in $x(t-\tau)$ and g at time $T+2\tau$, and so on. We summarize the chain of events:

Lemma 1: *A controlled switch at time T induces the following behavior in the delay system* (9)

	T	$T+\tau$	$T+2\tau$		$T+k\tau$
$\dot{x}(t)$	jump	C_0	C_1		C_{k-1}
$x(t)$	C_0	C_1	C_2		C_k
$f(x(t))$	C_0	C_1	C_2		C_k
$x(t-\tau)$		C_0	C_1		C_{k-1}
$g(x(t-\tau))$		C_0	C_1		C_{k-1}

2.3 Effect of variation in a single switching time

Let us first recall a simple result:
Lemma 2: *If y is C_k (see Definition) at t_0, then the* variation *of y induced by the perturbation $t_0 \to t_0+\theta$ is of order $k+1$ in θ.*
Proof: We note that the variation involves only the effect due to the shift of the switch of behavior from t_0 to $t_0+\theta$. Since this is only visible in the $(k+1)$-st derivative,

$$\Delta_{t_0} y = \left[y_+^{(k+1)} - y_-^{(k+1)}\right] \frac{\theta^{k+1}}{(k+1)!}. \qquad \square$$

In particular, if y is a continuous integrand, the first order variation of its integral (as in a performance index) will not involve 'future' additional contributions due to the propagation effect of the delay. Such additional contributions do occur however, if the integrand has jumps, as with impulsive control (Verriest *et al.* 2004).

If another switch occurs, say at T_1, then for some k, $T+k\tau < T_1 < T+(k+1)\tau$, and another ϵ perturbation may be induced. Therefore the bookkeeping of all perturbation terms will be quite complicated, in this more general case, especially in view of the fact that that all possibilities (of relative positions of switching instants) need to be taken into account.

2.4 Optimal sequencing

So far, it was assumed that the sequence of modes was fixed, and only control of the switching instants was assumed. However, if the m modes are cycled k times as: $1\to2\to\ldots\to m\to1\to2\to m\to\ldots\to m$, then we can optimize their switching times. If it is found that the optimal switching time sequence has $T_{i-1} = T_i$ for some i, it means that mode i only occurs for duration 0, and therefore should be excised. Performing all such excisions will leave the *optimal mode sequence* associated with at most $km-1$ switches. Thus by formulating first the optimal control problem for a fixed mode sequence, we really do not loose the generality of the global optimal control problem if an upper bound on the number of switches is imposed on the latter.

2.5 Refractory period

Let us now assume the existence of a refractory period, which means that another switch cannot

[1] This is a slight departure from the usual definition, which implies $C_k \subseteq C_{k-1} \subseteq \ldots \subseteq C_0$.

occur until a sufficiently long time has elapsed. This avoids the existence of switching cluster points and chattering. However, at once this also invalidates the search for the optimal sequencing of the modes as explained in the previous section, since modes persisting in intervals of length zero are no longer feasible.

In particular, we shall consider the case where this refractory time exceeds the delay time, τ. In this case, the problem greatly simplifies, and the aforementioned complexity disappears, as only two adjacent intervals need to be considered. Indeed the delayed effect of the k-th switch hits before the $k+1$-st switch. This yields a situation which is akin to the case with a single switch in the previous section.

With the Hamiltonian functionals defined in (10), we express the first variation in the performance index (4) as the limit for $\epsilon \to 0$ of

$$\delta J = \lim_{\epsilon\to 0} \frac{\overline{J}_\epsilon - \overline{J}}{\epsilon}.$$

For the separable mode form, (4) is

$$\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [L_i(x) + \lambda_i'(f_i(x(t)) + g_i(x(t-\tau)) - \dot{x})]\, dt \\ + \Phi(x(T)) + \sum_{i=1}^{N} \mu_i'[x(T_i^+) - x(T_i^-)]. \quad (12)$$

Hence, up to first order, the perturbed performance index (8) evaluates to

$$\overline{J}_\epsilon = \Phi(x(T)+\epsilon\eta_N(T)) + \sum_{i=1}^{N} \mu_i'[\eta_i(T_i^+) - \eta_i(T_i^-)] + \\ + \sum_{i=1}^{N} \mu_i'[x((T_i+\epsilon\theta_i)^+) - x((T_i+\epsilon\theta_i)^-)] + \\ + \sum_{i=1}^{N} \int_{T_{i-1}+\epsilon\theta_{i-1}}^{T_i+\epsilon\theta_i} [H_i(x+\epsilon\eta_i) - \lambda_i'\dot{x})]\, dt. \quad (13)$$

This expands to

$$\overline{J}_\epsilon = \Phi(x(T)) + \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [H_i(x) - \lambda_i'\dot{x})]\, dt + \\ + \sum_{i=1}^{N} \int_{T_i}^{T_i+\epsilon\theta_i} L_i(x)\, dt - \sum_{i=1}^{N} \int_{T_{i-1}}^{T_{i-1}+\epsilon\theta_{i-1}} L_i(x)\, dt + \\ + \sum_{i=1}^{N} \mu_i'[x((T_i+\epsilon\eta_i)^+) - x((T_i+\epsilon\eta_i)^-)] + \\ + \epsilon\sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [\mathbf{D}_\eta H - \lambda_i'\dot{\eta}_i]\, dt + \\ + \epsilon\sum_{i=1}^{N} \mu_i'[\eta_i(T_i^+) - \eta_i(T_i^-)] + \epsilon\frac{\partial\Phi}{\partial x}\bigg|_T \eta_N(T). \quad (14)$$

Note that we explicitly used the fact that $f_i(x(t)) + g_i(x(t-\tau)) - \dot{x}$ is zero in the intervals $[T_{i-1}, T_{i-1} + \epsilon\theta_{i-1}]$ and $[T_i, T_i + \epsilon\theta_i]$. Also, here

$$\mathbf{D}_\eta H = \frac{\partial H_i(x,y)}{\partial x}\eta_i(t) + \frac{\partial H_i(x,y)}{\partial y}\eta_i(t-\tau),$$

with $H_i(x,y) = H(x(t), x(t-\tau))$.

It readily follows that

$$\delta J = \sum_{i=1}^{N} [L_i(x(T_{i-1}))\,\theta_i - L_i(x(T_{i-1})\,\theta_{i-1}] + \\ + \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} [\mathbf{D}_\eta H - \lambda_i'\dot{\eta}_i]\, dt + \\ + \sum_{i=1}^{N} \mu_i'[\eta_i(T_i^+) - \eta_i(T_i^-)] + \frac{\partial\Phi}{\partial x}\bigg|_T \eta_N(T) + \\ + \sum_{i=1}^{N} \mu_i'[\,\dot{x}(T_i)^+) - \dot{x}(T_i)^-]\,\theta_i, \quad (15)$$

in which we separate the delayed terms and reorder the summation, remembering that $\theta_0 = \theta_N = 0$

$$\delta J = \sum_{i=1}^{N-1} L_i(x(T_{i-1}))\,\theta_i - \sum_{i=1}^{N-1} L_{i+1}(x(T_i))\,\theta_i + \\ + \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} \left[\left(\frac{\partial L_i(x)}{\partial x} + \lambda_i'\frac{\partial f_i(x)}{\partial x}\right)\eta_i - \lambda_i'\dot{\eta}_i\right] dt \\ + \sum_{i=1}^{N} \int_{T_{i-1}}^{T_i} \lambda_i'\frac{\partial g_i(y)}{\partial y}\bigg|_{y=x(t-\tau)} \eta_i(t-\tau)\, dt + \\ + \sum_{i=1}^{N} \mu_i'[\eta_i(T_i^+) - \eta_i(T_i^-)] + \frac{\partial\Phi}{\partial x}\bigg|_T \eta_N(T) + \\ + \sum_{i=1}^{N} \mu_i'[\,\dot{x}(T_i^+) - \dot{x}(T_i^-)]\,\theta_i, \quad (16)$$

The integrals of the delayed terms in (16) may be rearranged as follows:

$$\delta\mathcal{K} \stackrel{\text{def}}{=} \int_{T_{i-1}}^{T_i} \lambda_i'(t)\frac{\partial g_i(y)}{\partial y}\bigg|_{x(t-\tau)} \eta_i(t-\tau)\, dt \\ = \int_{T_{i-1}-\tau}^{T_i-\tau} \lambda_i'(t+\tau)\frac{\partial g_i(y)}{\partial y}\bigg|_{x(t)} \eta_i(t)\, dt$$

$$= \int_{T_{i-2}}^{T_{i-1}} \chi_{[T_{i-1}-\tau,T_{i-1}]}(t)\lambda_i'(t+\tau)\frac{\partial g_i(x)}{\partial x}\bigg|_{x(t)} \eta_i(t)\,dt$$
$$+ \int_{T_{i-1}}^{T_i} \chi_{[T_{i-1},T_i-\tau]}(t)\lambda_i'(t+\tau)\frac{\partial g_i(x)}{\partial x}\bigg|_{x(t)} \eta_i(t)dt.$$

In these expressions, $\chi_{[a,b]}$ is the indicator function of the interval $[a,b]$.

Let the advanced term $\lambda(t+\tau)$ be denoted by λ^τ, and define the *pseudo-Hamiltonians*, $\mathcal{H}_i(x,\lambda,\lambda^\tau)$,

$$\mathcal{H}_i(x,\lambda,\lambda^\tau) = L_i(x) + \lambda_i' f_i(x) + \chi_i^+(\lambda_i^\tau)' g_i(x) + + \chi_{i+1}^-(\lambda_{i+1}^\tau)' g_{i+1}(x), \tag{17}$$

for $i = 1,\ldots,N-1$, and for $i = N$,

$$\mathcal{H}_N(x,\lambda,\lambda^\tau) = L_N(x) + \lambda_N' f_N(x) + \chi_N^+(\lambda_N^\tau)' g_N(x)$$

where for simplicity of notation, we also set $\chi_{i+1}^+ = \chi_{[T_i,T_{i+1}-\tau]}$ and $\chi_{i+1}^- = \chi_{[T_i-\tau,T_i]}$. The integral terms in the expression (16) reduce further to

$$\delta\mathcal{K} = \sum_{i=1}^{N-1} \int_{T_{i-1}}^{T_i} \left[\frac{\partial \mathcal{H}_i}{\partial x}\eta_i(t) - \lambda_i'\dot{\eta}_i\right] dt + \\ + \int_{T_{N-1}}^{T_N} \left[\frac{\partial \mathcal{H}_N}{\partial x}\eta_N(t) - \lambda_N'\dot{\eta}_N\right] dt + \\ + \int_{-\tau}^{0} \chi_1^-(\lambda_1)'^\tau \frac{\partial g_1}{\partial x}\eta_1(t)\,dt \tag{18}$$

The last integral is zero, as the initial data is specified in the problem (thus making $\eta_1(t) = 0$ for $t < 0$.) As usual, integration by parts gives for the other terms in (18)

$$\delta\mathcal{K} = \sum_{i=1}^{N-1} \int_{T_{i-1}}^{T_i} \left[\frac{\partial \mathcal{H}_i}{\partial x}\eta_i(t) + \dot{\lambda}_i'\eta_i(t)\right] dt + \\ + \int_{T_{N-1}}^{T_N} \left[\frac{\partial \mathcal{H}_N}{\partial x}\eta_N(t) + \dot{\lambda}_N'\eta_N(t)\right] dt + \\ + \sum_{i=1}^{N} \left[-\lambda_i'(T_i^-)\eta_i(T_i^-) + \lambda_i'(T_{i-1}^+)\eta_i(T_{i-1}^+)\right]. \tag{19}$$

Note that for $i = 0$, $\theta_0 = 0$ and for $i = N$, $\theta_N = 0$, since initial and final time were fixed.

The $\{\theta_i\}$-induced perturbation of the non-integral term in (16), follows from (5),

$$\delta\Psi = \frac{\partial \Phi}{\partial x}\eta(T) + \sum_{i=1}^{N-1} \mu_i'\big[(\dot{x}(T_i^-) - \dot{x}(T_i^+))\theta_i + \\ + \eta_i(T_i^-) - \eta_{i+1}(T_i^+)\big]. \tag{20}$$

Substituting the $\delta\mathcal{K}$ and $\delta\Psi$ in δJ, and choosing λ_i in the intervals $[T_{i-1}, T_i]$ to solve

$$\dot{\lambda}_i = -\left(\frac{\partial \mathcal{H}_i}{\partial x}\right)', \tag{21}$$

this finally yields the expression

$$\sum_{i=1}^{N-1} \left[A_i\theta_i + B_i'\eta_{i+1}(T_i^+) + C_i'\eta_i(T_i^-)\right] + \tag{22}$$
$$+\lambda_1(0^+)\eta_1(0^+) + \left(\frac{\partial \Phi}{\partial x} - \lambda_N'(T_N^-)\right)\eta_N(T_N^-),$$

for δJ, where

$$A_i = L_i(x(T_i^-)) + \mu_i'\dot{x}(T_i^-) + \\ -L_{i+1}(x(T_i^+)) - \mu_i'\dot{x}(T_i^+) \tag{23}$$
$$B_i = -\mu_i + \lambda_{i+1}(T_i^+) \tag{24}$$
$$C_i = \mu_i - \lambda_i(T_i^-) \tag{25}$$

With the initial data given $\eta_1(0^+)$ must be zero. Computation of the requisite perturbations $\{\eta_i\}$ is avoided if one *chooses*

$$\lambda_i(T_i^-) = \mu_i = \lambda_{i+1}(T_i^+), \tag{26}$$

with final condition

$$\lambda_N(T_N^-) = \left(\frac{\partial \Phi}{\partial x}\right)', \tag{27}$$

thus specifying the the boundary conditions for the differential equations (21). This implies that we can choose the *costates* λ_i in $[T_{i-1}, T_i]$ to concatenate to a *continuous* functions in $[0,T]$. It follows that the first order variation of J reduces to

$$\delta J = \sum_{i=1}^{N-1} A_i\,\theta_i. \tag{28}$$

2.6 Main Result

Since the θ_i are independent, necessary conditions for optimality are the vanishing of the A_i in (28). In view of the choice (26) of the multipliers μ_i and boundary conditions, it gives for $i = 1$ to $N-1$

$$H_i(x(T_i^-)) = H_{i+1}(x(T_i^+)). \tag{29}$$

Simply stated, it means the continuity of a Hamiltonian, H, at the switching times.

We summarize this result as the main theorem below. We will assume that the vector fields $f_i(x)$

and $g_i(x)$ as well as the functions $L_i(x)$ are smooth, and we let $N-1$ be the total number of switches, with $T_0 = 0$ and $T_N = t_f$ being fixed.

Theorem 3:
The separable mode switched system in equation (9) minimizes the performance index J in (1), with fixed initial time ($T_0 = 0$) and terminal time ($T_N = T$) if the switching times T_i, $i = 1, \ldots, N-1$ are chosen as follows:
Euler-Lagrange Equations:

$$\dot{\lambda}_i = -\left(\frac{\partial L_i}{\partial x}\right)^T - \left(\frac{\partial f_i}{\partial x}\right)^T \lambda_i - \chi_i^+ \left(\frac{\partial g_i}{\partial x}\right)^T \lambda_i^\tau + \\ -\chi_{i+1}^- \left(\frac{\partial g_{i+1}}{\partial x}\right)^T \lambda_{i+1}^\tau, \qquad (30)$$

with $T_{i-1} < t < T_i$, $i = 1, \ldots, N-1$, and where $\chi_i^+(t) = 1$ if $t \in [T_{i-1}, T_i - \tau]$ and 0 otherwise, $\chi_{i+1}^-(t) = 1$ if $t \in [T_i - \tau, T_i]$ and 0 otherwise, and $\lambda_i^\tau = \lambda_i(t+\tau)$. Moreover,

$$\dot{\lambda}_N = -\left(\frac{\partial L_N}{\partial x}\right)^T - \left(\frac{\partial f_N}{\partial x}\right)^T \lambda_N - \chi_N^+ \left(\frac{\partial g_N}{\partial x}\right)^T \lambda_N^\tau. \qquad (31)$$

where $\chi_N^+ = 0$ is understood if $T_{N-1} > T - \tau$.
Boundary Conditions:

$$\lambda_N(T_N) = \left(\frac{\partial \Phi}{\partial x}\right)^T \qquad (32)$$

$$\lambda_i(T_i^-) = \lambda_{i+1}(T_i^+) \qquad (33)$$

Optimality Conditions:

$$H_i(T_i^-) = H_{i+1}(T_{i+1}^+) \qquad (34)$$

where H_i is the Hamiltonian

$$H_i = L_i(x) + \lambda_i(f(x) + g(x_\tau)). \qquad (35)$$

Proof: All one has to do is to recall what the function K was and realize that both indicator functions in its definition evaluate to 1 at the switching point T_i $\square$.

2.7 Commensurate delays

The problem for systems with multiple delays can be reduced to the problem solved in Theorem 3, if the delays are *commensurate*. Indeed, assume that for $x \in \mathbb{R}^n$

$$\dot{x}(t) = f(x(t)) + \sum_{k=1}^{m} g_k(x(t - k\tau)), \qquad (36)$$

then the embedding $\mathbf{x}'(t) = [x'(t), \ldots x'(t-(k-1)\tau)]$ reduces the dynamics to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{g}(\mathbf{x}(t - m\tau)), \qquad (37)$$

a system with partial state $\mathbf{x} \in \mathbb{R}^{mn}$ and a single delay equal to $m\tau$.

3. CONCLUSIONS AND BEYOND

We derived necessary conditions for stationarity of the performance index of a multi-mode delay system controlled by switchings between a prespecified mode sequence. This is a first step in the complete optimal control of a multi-mode system, where also the *optimal sequence* of the modes needs to be found. Whereas in the absence of a refractionary period, the optimal sequencing follows from the fixed sequence problem, with a refractionary period, in principle a search is required to find the global optimum. Whereas this quickly leads to a combinatorial explosion, regularization methods as for instance presented in (Verriest 2003) could be invoked to obtain a first approximation and thus narrow down the search. The result is amenable to numerical solution using gradient methods (Xu *et al.* 2002, Egerstedt *et al.* 2003).

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