

CONTROLLER DESIGN FOR A CLASS OF NONLINEAR STOCHASTIC SYSTEMS WITH TIME-DELAYS

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Abstract: In this paper, the stabilization problem is considered for a class of nonlinear continuous stochastic systems with state delays. The purpose of this problem is to design a state feedback controller such that the closed-loop system is exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We first investigate the sufficient conditions for the nonlinear stochastic time-delay systems to be stable, and then derive the explicit expression of the desired controller gains. A numerical simulation example is provided to show the usefulness of the proposed design method. *Copyright © 2005 IFAC*

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1. INTRODUCTION

Nonlinear stochastic control has long been an important research field that has attracted many researchers, and enormous results have been published in the literature. In particular, the fundamental nonlinear stochastic stabilization issue has received considerable research interests, and has found successful applications in control and communication problems, such as attitude control of satellites and missile control, macroeconomic

system control, chemical process control, etc., see McEneaney et al. (1999) for a survey.

Recently, there have appeared many methods to tackle different kinds of nonlinear stochastic systems. For example, in Charalambous (2003), a minimax dynamic game approach has been developed for the controller design problem of the nonlinear stochastic systems that employ risk-sensitive performance criteria. The stabilization problem has been investigated in Deng et al. (2001) for nonlinear stochastic systems, and a stochastic counterpart of the input-to-state stabilization results has been provided. In Liu et al. (2003), under an infinite-horizon risk-sensitive cost criterion, the problem of output feedback control design has been studied for a class of strict feedback stochastic nonlinear systems. In Xie and Xie (2000), the decentralized global stabilization problem has been dealt with by using a Lyapunov-based recursive design method. On the

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other hand, the dual nonlinear stochastic filtering problem has also been an active area for three decades.

It is now a recognized fact that the time delay is frequently a source of instability and encountered in various engineering systems such as chemical processes, long transmission lines in pneumatic systems, and so on. Recently, increasing attention has been focused on robust and/or H_∞ control problems for linear systems with certain types of time-delays, see Boukas and Liu (2002) for a survey. As for nonlinear stochastic time-delay systems, the related results have been scattered, and most of the results have been concerned with the stability analysis issue, see e.g. Mao (1997). So far, the stabilization problem for general nonlinear time-delay systems has not been fully investigated and remains important.

In this paper, we will consider the stabilization problem for a class of nonlinear continuous stochastic systems with state delays. Such a class of systems have been intensively investigated in Wang and Ho (2003), Wang et al. (2003), Yaz and Azemi (1993) for the nonlinear filtering problems. An effective algebraic matrix inequality approach is proposed to design the state feedback controllers, such that the closed-loop system is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square, for all admissible nonlinearities and time-delays. We first investigate the sufficient conditions for the nonlinear stochastic systems to be exponentially stable (or exponentially ultimately bounded), and then derive the explicit expression of the desired controller gains. A numerical simulation example is provided to show the usefulness and effectiveness of the proposed design method.

2. PROBLEM FORMULATION AND ASSUMPTIONS

Consider the following nonlinear continuous-time state delayed stochastic system in a fixed complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$:

$$dx(t) = [f(x(t), u(t)) + g(x(t - \tau))]dt + Dx(t)dw(t), \quad (1)$$

$$x(t) = \varphi(t), \quad t \in [-\tau, 0], \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the deterministic input, and $f(\cdot, \cdot) \in \mathbb{R}^n$ and $g(\cdot) \in \mathbb{R}^n$ are nonlinear vector functions. $\tau > 0$ denotes the state delay and $\varphi(t)$ is a continuous vector valued initial function. Here, $w(t) := [w_1(t)w_2(t) \cdots w_m(t)]^T \in \mathbb{R}^m$ is an m -dimensional Brownian motion. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$, and is uncorrelated with $w(t)$. D is a known constant matrices with appropriate dimensions.

Assumption 1. The nonlinear vector functions $f(\cdot, \cdot)$ and $g(\cdot)$ satisfy $f(0, 0) = 0$, $g(0) = 0$ and

$$\left| f(x(t), u(t)) - \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right| \leq a_{11} \left| \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right| + a_{12}, \quad (3)$$

$$\begin{aligned} |g(x(t - \tau)) - A_d x(t - \tau)| \\ \leq a_{21} |x(t - \tau)| + a_{22}, \end{aligned} \quad (4)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $A_d \in \mathbb{R}^{n \times n}$ are known constant matrices, and $a_{11} > 0$, $a_{12} \geq 0$, $a_{21} > 0$ and $a_{22} \geq 0$ are known scalars.

Remark 1. The system (1)-(2) can be used to represent many important physical nonlinear systems subject to inherent state delays and stochastic exogenous noises with known statistics. Similar to Wang and Ho (2003), Wang et al. (2003), Yaz and Azemi (1993), the nonlinear descriptions (3)-(4) quantify the maximum possible derivations from a linear model with (A, B, A_d) as its system parameter matrices.

When a state feedback control law $u(t) = Kx(t)$ is applied to the system (1)-(2), the closed-loop system is governed by

$$\begin{aligned} dx(t) = [f(x(t), Kx(t)) + g(x(t - \tau))]dt \\ + Dx(t)dw(t) \end{aligned} \quad (5)$$

For notation convenience, we define:

$$A_c = A + BK \quad (6)$$

$$p(t) = f(x(t), Kx(t)) - A_c x(t), \quad (7)$$

$$q(t) = g(x(t - \tau)) - A_d x(t - \tau), \quad (8)$$

and then obtain from (5) that

$$\begin{aligned} dx(t) = [A_c x(t) + A_d x(t - \tau) + p(t) + q(t)]dt \\ + Dx(t)dw(t). \end{aligned} \quad (9)$$

Now, let $x(t; \xi)$ denote the state trajectory from the initial data $x(\theta) = \xi(\theta)$ on $-\tau \leq \theta \leq 0$ in $L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$. It is clear from Assumption 1 that the system (9) admits a trivial solution $x(t; 0) \equiv 0$ corresponding to the initial data $\xi = 0$.

Definition 1. Consider the system (9) and let \mathbb{E} denote the mathematical expectation operator. For every $\xi \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

- the trivial solution is exponentially stable in the mean square if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\mathbb{E}|x(t; \xi)|^2 \leq \alpha x^{-\beta t} \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2; \quad (10)$$

- the trivial solution is exponentially ultimately bounded in the mean square if there exist constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$ such that

$$\mathbb{E}|x(t; \xi)|^2 \leq \alpha x^{-\beta t} \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 + \gamma. \quad (11)$$

The objective of this paper is to design a controller for the nonlinear time-delay system (1)-(2), such that the closed-loop systems is exponentially stable (or exponentially ultimately bounded) in the mean square. More specifically, we are interested in designing a controller parameter K such that:

- in the case of $a_{12} = 0$ and $a_{22} = 0$ (i.e., there are no bounded nonlinearities and uncertain disturbances), the solution of the system (9) is guaranteed to be exponentially stable;
- in the case of $a_{12} \neq 0$ or $a_{22} \neq 0$ (i.e., there are bounded nonlinearities or uncertain disturbances), the solution of the system (9) is guaranteed to be exponentially ultimately bounded in the mean square.

3. MAIN RESULTS AND PROOFS

The following theorem will play an essential role in the design of the expected controllers. It reveals that the exponential stability (or exponential ultimate boundedness) of the controlled nonlinear time-delay stochastic system (9) can be guaranteed if a positive definite solution to a modified algebraic Riccati-like matrix inequality (quadratic matrix inequality) is known to exist.

Theorem 1. Let the controller parameter K be given. If there exist positive scalars $\varepsilon_1, \varepsilon_2$ such that the following matrix inequality

$$A_c^T P + P A_c + D^T P D + (\varepsilon_1 + \varepsilon_2) P^2 + 4\varepsilon_2^{-1} a_{11}^2 (I + K^T K) + Q < 0 \quad (12)$$

where

$$Q := \varepsilon_1^{-1} A_d^T A_d + 4\varepsilon_2^{-1} a_{21}^2 I \quad (13)$$

has a solution $P > 0$, then in the mean square, the system (9) is

- exponentially stable in the case of $a_{12} = 0$ and $a_{22} = 0$;
- exponentially ultimately bounded in the case of $a_{12} \neq 0$ or $a_{22} \neq 0$.

Proof. Fix $\xi \in L_{\mathcal{F}_0}^2([-\tau, 0]; \mathbb{R}^n)$ arbitrarily and write $x(t; \xi) = x(t)$. For $(x(t), t) \in \mathbb{R}^n \times \mathbb{R}_+$, we define the Lyapunov function candidate

$$V(x(t), t) = x^T(t) P x(t) + \int_{t-\tau}^t x^T(s) Q x(s) ds \quad (14)$$

where P is the positive definite solution to the matrix inequality (12) and $Q > 0$ is defined in (13).

By Itô's formula (see, e.g. Mao, 1997), the stochastic derivative of V along a given trajectory is obtained as

$$\begin{aligned} dV(x(t), t) &= \{x^T(t) P [A_c x(t) + A_d x(t-\tau) + p(t) + q(t)] \\ &\quad + [A_c x(t) + A_d x(t-\tau) + p(t) + q(t)]^T P x(t) \\ &\quad + x^T(t) Q x(t) - x^T(t-\tau) Q x(t-\tau) \\ &\quad + x^T(t) D^T P D x(t)\} dt + 2x^T(t) P D x(t) dw(t) \\ &= \{x^T(t) [A_c^T P + P A_c + D^T P D + Q] x(t) \\ &\quad + x^T(t) P A_d x(t-\tau) + x^T(t-\tau) A_d^T P x(t) \\ &\quad + x^T(t) P [p(t) + q(t)] + [p(t) + q(t)]^T P x(t) \\ &\quad - x^T(t-\tau) Q x(t-\tau)\} dt \\ &\quad + 2x^T(t) P D x(t) dw(t). \end{aligned} \quad (15)$$

Let $\varepsilon_1 > 0, \varepsilon_2 > 0$. It is easy to have

$$\begin{aligned} &x^T(t) P A_d x(t-\tau) + x^T(t-\tau) A_d^T P x(t) \\ &\leq \varepsilon_1 x^T(t) P^2 x(t) \\ &\quad + \varepsilon_1^{-1} x^T(t-\tau) A_d^T A_d x(t-\tau). \end{aligned} \quad (16)$$

In the sequel, we will use several times the following simple inequality

$$(u + v)^T (u + v) \leq 2u^T u + 2v^T v,$$

where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

Noticing the Assumption 1 and (6)-(8), we have

$$\begin{aligned} p^T(t) p(t) &= |f(x(t), Kx(t)) - A_c x(t)|^2 \\ &\leq \left\{ a_{11} \left| \begin{bmatrix} x(t) \\ Kx(t) \end{bmatrix} \right| + a_{12} \right\}^2 \\ &\leq 2a_{11}^2 \left| \begin{bmatrix} x(t) \\ Kx(t) \end{bmatrix} \right|^2 + 2a_{12}^2 \\ &\leq 2a_{11}^2 x^T(t) (I + K^T K) x(t) + 2a_{12}^2 \quad (17) \\ q^T(t) q(t) &= |g(x(t-\tau)) - A_d x(t-\tau)|^2 \\ &\leq \{a_{21} |x(t-\tau)| + a_{22}\}^2 \\ &\leq 2a_{21}^2 x^T(t-\tau) x(t-\tau) + 2a_{22}^2. \end{aligned} \quad (18)$$

Then, it follows from (17), (18) and

$$\Psi_1 := \varepsilon_2^{1/2} x^T(t) P - \varepsilon_2^{-1/2} [p(t) + q(t)]^T, \quad \Psi_1 \Psi_1^T \geq 0$$

that

$$\begin{aligned} &x^T(t) P [p(t) + q(t)] + [p(t) + q(t)]^T P x(t) \\ &\leq \varepsilon_2 x^T(t) P^2 x(t) + \varepsilon_2^{-1} [p(t) + q(t)]^T [p(t) + q(t)] \\ &\leq \varepsilon_2 x^T(t) P^2 x(t) + 2\varepsilon_2^{-1} [p^T(t) p(t) + q^T(t) q(t)] \\ &= x^T(t) [\varepsilon_2 P^2 + 4\varepsilon_2^{-1} a_{11}^2 (I + K^T K)] x(t) \\ &\quad + 4\varepsilon_2^{-1} a_{21}^2 x^T(t-\tau) x(t-\tau) \\ &\quad + 4\varepsilon_2^{-1} (a_{12}^2 + a_{22}^2). \end{aligned} \quad (19)$$

For simplicity, we denote

$$\begin{aligned} \Pi := & A_c^T P + P A_c + D^T P D + (\varepsilon_1 + \varepsilon_2) P^2 \\ & + 4\varepsilon_2^{-1} a_{11}^2 (I + K^T K) + \varepsilon_1^{-1} A_d^T A_d + 4\varepsilon_2^{-1} a_{21}^2 I, \end{aligned} \quad (20)$$

and then (12) and (13) indicate that $\Pi < 0$.

Substituting (13), (16) and (19) into (15) gives

$$\begin{aligned} dV(x(t), t) \leq & [x^T(t) \Pi x(t) + 4\varepsilon_2^{-1} (a_{12}^2 + a_{22}^2)] dt \\ & + 2x^T(t) P D x(t) dw(t). \end{aligned} \quad (21)$$

Let $\beta > 0$ be the unique root of the equation

$$\lambda_{\min}(-\Pi) - \beta \lambda_{\max}(P) - \beta \tau \lambda_{\max}(Q) x^{\beta\tau} = 0 \quad (22)$$

where Π and Q are defined, respectively, in (20) and (13), P is the positive definite solution to (12), and τ is the time-delay.

We can obtain from (21) that

$$\begin{aligned} d[x^{\beta t} V(x(t), t)] &= x^{\beta t} [\beta V(x(t), t) dt + dV(x(t), t)] \\ \leq x^{\beta t} & \left(- [\lambda_{\min}(-\Pi) - \beta \lambda_{\max}(P)] |x(t)|^2 \right. \\ & + \beta \lambda_{\max}(Q) \int_{t-\tau}^t |x(s)|^2 ds dt \\ & \left. + 4\varepsilon_2^{-1} (a_{12}^2 + a_{22}^2) x^{\beta t} dt + 2x^{\beta t} x^T(t) P D x(t) w(t) dt. \right) \end{aligned}$$

Then, integrating both sides from 0 to $T > 0$ and taking the expectation result in

$$\begin{aligned} & x^{\beta T} \mathbb{E}V(x(T), T) \\ \leq & [\lambda_{\max}(P) + \tau \lambda_{\max}(Q)] \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \\ & - [\lambda_{\min}(-\Pi) - \beta \lambda_{\max}(P)] \mathbb{E} \int_0^T x^{\beta t} |x(t)|^2 dt \\ & + \beta \lambda_{\max}(Q) \mathbb{E} \int_0^T x^{\beta t} \int_{t-\tau}^t |x(s)|^2 ds dt \\ & + 4\varepsilon_2^{-1} (a_{12}^2 + a_{22}^2) \beta^{-1} (x^{\beta T} - 1). \end{aligned}$$

Note that

$$\begin{aligned} & \int_0^T x^{\beta t} \int_{t-\tau}^t |x(s)|^2 ds dt \\ \leq & \int_{-\tau}^T \left(\int_{\max(s, 0)}^{\min(s+\tau, T)} x^{\beta t} dt \right) |x(s)|^2 ds \\ \leq & \int_{-\tau}^T \tau x^{\beta(s+\tau)} |x(s)|^2 ds \\ \leq & \tau x^{\beta\tau} \int_0^T x^{\beta t} |x(t)|^2 dt + \tau x^{\beta\tau} \int_{-\tau}^0 |\xi(\theta)|^2 d\theta. \end{aligned}$$

Considering the definition of β in (22), we have

$$\begin{aligned} & x^{\beta T} \mathbb{E}V(x(T), T) \\ \leq & [\lambda_{\max}(P) + \tau \lambda_{\max}(Q)] \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \\ & + \beta \lambda_{\max}(Q) \tau^2 x^{\beta\tau} \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \\ & + 4\varepsilon_2^{-1} (a_{12}^2 + a_{22}^2) \beta^{-1} (x^{\beta T} - 1), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}|x(T)|^2 \leq & \lambda_{\min}^{-1}(P) \left([\lambda_{\max}(P) + \tau \lambda_{\max}(Q)] \right. \\ & \cdot \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 + \beta \lambda_{\max}(Q) \tau^2 x^{\beta\tau} \\ & \cdot \sup_{-\tau \leq \theta \leq 0} \mathbb{E}|\xi(\theta)|^2 \Big) x^{-\beta T} \\ & + 4\varepsilon_2^{-1} (a_{12}^2 + a_{22}^2) \beta^{-1} \lambda_{\min}^{-1}(P) \\ & \cdot (x^{\beta T} - 1) x^{-\beta T}. \end{aligned}$$

Notice that $(x^{\beta T} - 1)x^{-\beta T} < 1$ and let

$$\begin{aligned} \alpha := & \lambda_{\min}^{-1}(P) [\lambda_{\max}(P) + \tau \lambda_{\max}(Q) (1 + \beta \tau x^{\beta\tau})], \\ \gamma := & 4\varepsilon_2^{-1} (a_{12}^2 + a_{22}^2) \beta^{-1} \lambda_{\min}^{-1}(P). \end{aligned}$$

Since $T > 0$ is arbitrary, the definition of exponential ultimate boundedness in (11) is then satisfied if $a_{12} \neq 0$ or $a_{22} \neq 0$. If $a_{12} = a_{22} = 0$, it is obvious that the definition of exponential stability in (10) is met. This completes the proof of Theorem 1.

Next, let us focus on deriving the *explicit* expression of expected controller gains by using an algebraic matrix inequality approach. It is worth mentioning that, in most literature concerning nonlinear stochastic stabilization problems, the solution has not been given as an explicit representation.

Based on Theorem 1, we can see that the controller design problem can be transformed into the following two-step problem: (i) find a necessary and sufficient condition for the existence of the positive definite matrix P such that there exists a controller gain K satisfying (12); and (ii) if the controller gain K exists, give the characterization of the set of expected controller gains in terms of the positive definite matrix P and some other free parameters.

For presentation convenience, we define

$$\begin{aligned} \Gamma(\varepsilon_1, \varepsilon_2, P) := & A^T P + P A + D^T P D \\ & + (\varepsilon_1 + \varepsilon_2) P^2 + 4\varepsilon_2^{-1} a_{11}^2 I + Q, \quad (23) \\ \Xi(\varepsilon_1, \varepsilon_2, P) := & A^T P + P A + D^T P D \\ & + P[(\varepsilon_1 + \varepsilon_2) I - 0.25\varepsilon_2 a_{11}^{-2} B B^T] P \\ & + 4\varepsilon_2^{-1} (a_{11}^2 + a_{21}^2) I \\ & + \varepsilon_1^{-1} A_d^T A_d, \quad (24) \end{aligned}$$

where Q is defined in (13).

The aforementioned two-step problem is solved in the following theorem.

Theorem 2. There exist positive scalars $\varepsilon_1, \varepsilon_2$ and a positive definite matrix P such that the matrix inequality (12) has a solution K if and only if the following quadratic matrix inequality

$$\Xi(\varepsilon_1, \varepsilon_2, P) < 0 \quad (25)$$

holds, where $\Xi(\varepsilon_1, \varepsilon_2, P)$ is defined in (24). Furthermore, if (25) is true, all gain matrices K satisfying the matrix inequality (12) can be parameterized by

$$K = (0.5a_{11}^{-1}\varepsilon_2^{1/2}\Lambda U - 0.25a_{11}^{-2}\varepsilon_2 PB)^T \quad (26)$$

where $\Lambda \in \mathbb{R}^{n \times m}$ is any matrix satisfying

$$\Lambda \Lambda^T < -\Xi(\varepsilon_1, \varepsilon_2, P) \quad (27)$$

and $U \in \mathbb{R}^{m \times m}$ is arbitrary orthogonal matrix (i.e., $UU^T = I$).

Proof. Rewrite the matrix inequality (12) as

$$K^T B^T P + PBK + 4\varepsilon_2^{-1} a_{11}^2 K^T K + \Gamma(\varepsilon_1, \varepsilon_2, P) < 0, \quad (28)$$

where $\Gamma(\varepsilon_1, \varepsilon_2, P)$ is defined in (23).

In terms of the definition of $\Xi(\varepsilon_1, \varepsilon_2, P)$ in (24), we can rearrange (28) as

$$\begin{aligned} & (2\varepsilon_2^{-1/2} a_{11} K^T + 0.5\varepsilon_2^{1/2} a_{11}^{-1} PB) \\ & \cdot (2\varepsilon_2^{-1/2} a_{11} K^T + 0.5\varepsilon_2^{1/2} a_{11}^{-1} PB)^T \\ & < -\Xi(\varepsilon_1, \varepsilon_2, P). \end{aligned} \quad (29)$$

Obviously, there exists a controller gain matrix K such that the inequality (12) (or equivalently (29)) holds for some positive scalars $\varepsilon_1, \varepsilon_2$ and positive definite matrix P if and only if the right-hand side of (29) is positive definite, i.e., $-\Xi(\varepsilon_1, \varepsilon_2, P) > 0$ or (25) holds. The first part of this theorem is proved.

Assume now that (25) is true. Note that the dimension of the controller gain K is $m \times n$. From (29) and the definition of $\Lambda \in \mathbb{R}^{n \times m}$ in (27), we could relate a Λ such that

$$\begin{aligned} & (2\varepsilon_2^{-1/2} a_{11} K^T + 0.5\varepsilon_2^{1/2} a_{11}^{-1} PB) \\ & \cdot (2\varepsilon_2^{-1/2} a_{11} K^T + 0.5\varepsilon_2^{1/2} a_{11}^{-1} PB)^T = \Lambda \Lambda^T. \end{aligned}$$

It then follows that (3) holds if and only if

$$2\varepsilon_2^{-1/2} a_{11} K^T + 0.5\varepsilon_2^{1/2} a_{11}^{-1} PB = \Lambda U, \quad (30)$$

where $U \in \mathbb{R}^{m \times m}$ is an arbitrary orthogonal matrix. Therefore, the expression (26) follows immediately. This completes the proof of the theorem.

Finally, our main results can be summarized in the following corollary.

Corollary 1. Consider the nonlinear continuous-time state delayed stochastic system (1)-(2) with the state feedback controller $u(t) = Kx(t)$. If

there exist positive scalars $\varepsilon_1, \varepsilon_2$, and a positive definite matrix P such that the matrix inequality (25) holds, then the state feedback controller with its gain given in (26) will be such that the system (9) is exponentially stable in the case of $a_{12} = 0$ and $a_{22} = 0$; or exponentially ultimately bounded in the case of $a_{12} \neq 0$ or $a_{22} \neq 0$, both in the mean square.

Remark 2. Corollary 1 solves the addressed stabilization problem for the class of nonlinear time-delay stochastic systems in this paper. In implementation, we could first solve the quadratic matrix inequality (25), and then obtain the expected control parameters from (26) easily. Firstly, based on the algorithms provided in Xie and Soh (1994) and references therein, we may select appropriate positive scalar parameters ε_1 and ε_2 so as to reduce the conservatism that may have resulted from the inequalities (16) and (19). Then, (25) will be a standard quadratic matrix inequality (QMI) for P . For details concerning the *general* QMIs and relevant algorithms, we refer the reader to Saberi et al. (1995). It can also be noticed that, there exists a lot of design freedom in our proposed procedure, such as the choices of matrices Λ and U , which could be used to achieve other expected performance specifications, e.g., reliability constraints.

4. NUMERICAL SIMULATION

In this section, for the purpose of illustrating the usefulness and flexibility of the theory developed in this paper, we present a simulation example.

Assume that the nonlinear continuous-time stochastic state delayed system (1)-(2) is given by

$$\begin{aligned} dx_1(t) &= [-2x_1(t) - 0.1x_2(t) + 0.2 \cos(x_1(t) + x_2(t)) \\ & \quad + 0.1x_1(t - 0.1) + 0.16 \sin x_2(t) \\ & \quad + 2.9u_1(t) + 0.2u_2(t)]dt + 0.2x_1 dw(t) \\ dx_2(t) &= [-0.1x_1(t) + x_2(t) + 0.15 \sin x_2(t) \\ & \quad + 0.1x_2(t - 0.1) + 0.15 \cos x_1(t) \\ & \quad + 0.1u_1(t) - 2.1u_2(t)]dt + 0.2x_2 dw(t). \end{aligned}$$

Considering the system (1)-(2) with the constraints (3)-(4), we can obtain that

$$\begin{aligned} A &= \begin{bmatrix} -2 & -0.1 \\ -0.1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2.9 & 0.2 \\ 0.1 & -2.1 \end{bmatrix}, \\ A_d &= 0.1I_2, \quad D = 0.2I_2 \\ \tau &= 0.1, \quad a_{11} = 0.25; \quad a_{12} = 0.12; \\ & \quad a_{21} = 0; \quad a_{22} = 0. \end{aligned}$$

We choose $\varepsilon_1 = 4.8$, $\varepsilon_2 = 8.2$, and solve (25) to obtain

$$P = \begin{bmatrix} 0.1287 & 0.0013 \\ 0.0013 & 0.2003 \end{bmatrix}.$$

Then, setting $\Lambda = 2I_2$ which meets (27) and considering two cases of $U = I_2$ and $U = -I_2$, we have two desired gain matrices as follows:

$$\text{Case 1: } K_1 = \begin{bmatrix} -0.7938 & -0.7764 \\ -0.7580 & 25.2439 \end{bmatrix},$$

$$\text{Case 2: } K_2 = \begin{bmatrix} -23.7023 & -0.7764 \\ -0.7580 & 2.3354 \end{bmatrix}.$$

The responses of closed-loop system dynamics to initial conditions are shown in Fig. 1 and Fig. 2. The simulation results imply that the desired goal is well achieved, i.e., the closed-loop system is exponentially stable in the mean square.

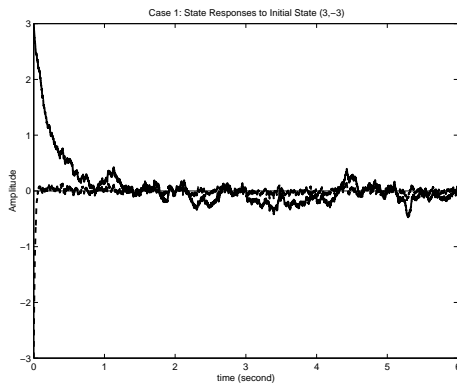


Fig. 1. x_1 (solid), x_2 (dashed).

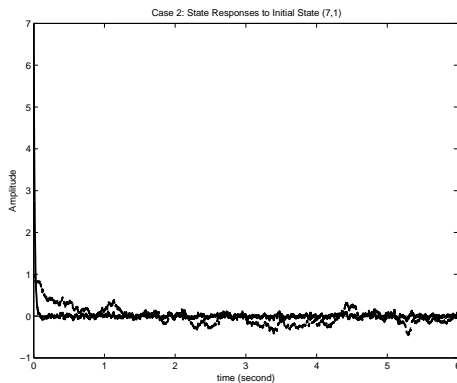


Fig. 2. x_1 (solid), x_2 (dashed).

5. CONCLUSIONS

In this paper, we have studied the stabilization problem for a class of nonlinear stochastic time-delay systems. The nonlinearities are assumed to have the similar form as those in Wang and Ho (2003), Wang et al. (2003), Yaz and Azemi (1993). We have developed an effective algebraic matrix inequality approach to designing the state feedback controllers, such that the closed-loop system is stochastically exponentially stable (or exponentially ultimately bounded) in the mean square, for

all admissible nonlinearities and time-delays. We have investigated the sufficient conditions for the nonlinear stochastic systems to be exponentially stable (or exponentially ultimately bounded), and have derived the explicit expression of the desired controller gains. A numerical simulation example has been provided to show the usefulness and effectiveness of the proposed design method.

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