$\mathcal{H}_\infty$ FILTERING FOR DISCRETE-TIME PIECEWISE LINEAR SYSTEMS

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Abstract: This paper deals with the design of piecewise linear $\mathcal{H}_\infty$ filters for discrete-time piecewise linear systems. Both the system and filter have structures that change according to their switching rules. The system switching rule is characterized in terms of a given partition of the system state-space. The filter switching rule is also characterized in terms of a suitable filter state-space partition which is determined, jointly with the filter matrices, in order to ensure a prescribed $\mathcal{H}_\infty$ performance for the estimation error system. The filter design is based on a piecewise quadratic Lyapunov function and is given in terms of linear matrix inequalities (LMIs).

Keywords: Filtering theory, $\mathcal{H}_\infty$ filtering, piecewise linear systems.

1. INTRODUCTION

The discrete-time $\mathcal{H}_\infty$ filtering problem has been largely studied over the past decade and a considerable number of important results are now available; see, for instance, Yaesh and Shaked (1992) for $\mathcal{H}_\infty$ filtering for linear systems, Barbosa *et al.* (2002), Geromel *et al.* (2000) and Xie *et al.* (1991) for robust $\mathcal{H}_\infty$ filtering for uncertain linear systems, de Souza (2003) and de Souza and Fragoso (2003) for $\mathcal{H}_\infty$ filtering for Markov jump linear systems, and Shaked and Berman (1995) and Xie *et al.* (1996) for $\mathcal{H}_\infty$ filtering for nonlinear systems.

The main purpose of this paper is to study the problem of $\mathcal{H}_\infty$ filtering for discrete-time piecewise linear systems. This class of systems, which has been attracting an increasing attention in the literature (see, e.g. Bemporad *et al.* (2000), Feng (2003) and the references therein) is very appropriate to model plants whose structure is subject to changes depending on whether the state belongs to a specific subregion, for instance, when piecewise linear components are encountered, such as dead-zone, saturation, relays and hysteresis. The design of switching observers for piecewise linear systems has been addressed in Alessandri and Coletta (2001), Juloski *et al.* (2002) and Juloski *et al.* (2003) where both continuous-time and discrete-time systems have been considered. On the other hand, the design of piecewise linear $\mathcal{H}_\infty$ and $\mathcal{H}_2$ filters in the continuous-time case has

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been recently proposed in Xu *et al.* (2003). To the best of the authors' knowledge, to date the problem of $\mathcal{H}_\infty$ filtering for discrete-time piecewise linear systems has not yet been fully addressed.

In this paper we propose an LMI technique to solve the $\mathcal{H}_\infty$ filtering problem for a class of signals described by a discrete-time piecewise linear state-space model. The filter to be designed is also piecewise linear and the corresponding switchings are based on the filter states. In order to solve the filtering problem in terms of LMIs, we propose a new filter parameterization and a congruent transformation to render the final design conditions affine in the new decision variables. The proposed design also uses the so-called *S-procedure* and scaling matrix variables, which incorporate information on the partitions of the system and filter state-spaces, in order to reduce the design conservatism.

The paper is organized as follows. Section 2 presents the system and the filter models, whereas the filter design is addressed in Section 3. Finally, concluding remarks are presented in Section 4.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, I_n is the $n \times n$ identity matrix, 0_n and $0_{m\times n}$ are the $n \times n$ and $m \times n$ zero matrices, respectively, $col_i(I_n)$ denotes the i-th column of I_n, and diag$\{\cdots\}$ stands for a block-diagonal matrix. The notation $S_k > 0$, for a real matrix sequence S_k, means that S_k is symmetric and positive definite for all integers $k \geq 0$, and $M \succeq 0$ ($M \succ 0$), for a real matrix M, means that the elements of M are non-negative (positive). ℓ_2 denotes the space of square summable vector sequences over the non-negative integers with norm $\|x_k\|_2 := (\sum_{k=0}^{\infty} x_k' x_k)^{\frac{1}{2}}$ and $\tilde{x}_k$ stands for the one-step-ahead operator defined by $\tilde{x}_k = x_{k+1}$. For a symmetric block matrix, the symbol $\star$ denotes the transpose of the blocks outside the main diagonal block and $\mathcal{I}_m$ stands for the set of integers $\{1, \ldots, m\}$. For a polytope $\mathcal{X}$, the notation $\mathcal{X}^{int}$ denotes its interior.

2. SYSTEM AND FILTER DEFINITIONS

Consider a discrete-time piecewise linear system with the following characteristics:

$$\mathcal{X} := \bigcup_{i=1}^{m} \mathcal{X}_i \tag{1}$$

is the region of interest of the state-space in which the system dynamics will be represented and the cells $\mathcal{X}_i$ are given polytopes.

The system dynamics in $\mathcal{X}_i$ is represented by the state-space model:

$$\mathcal{S}_i \ : \ \begin{aligned} x_{k+1} &= A_i\, x_k + B_i d_k \\ y_k &= C_i\, x_k + D_i d_k \\ z_k &= L_i\, x_k + M_i d_k \end{aligned} \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $d_k \in \mathbb{R}^{n_d}$ is the noise signal (including process and measurement noise vectors), which is an arbitrary signal in ℓ_2, $y_k \in \mathbb{R}^{n_y}$ is the measurement, $z_k \in \mathbb{R}^{n_z}$ is the signal to be estimated, and A_i, B_i, C_i, D_i, L_i and M_i are given constant matrices of appropriate dimensions.

Assumptions 1. It is assumed that when the system state transits from one cell to itself, or to another cell, say from $\mathcal{X}_i$ to $\mathcal{X}_j$, it always moves to the interior of $\mathcal{X}_j$. Also, let the set $\mathcal{T}_i \subseteq \mathcal{I}_m$, $\forall\, i \in \mathcal{I}_m$ represent all admissible one-step transitions from $\mathcal{X}_i$ to itself, or to the interior of another cell of the system state-space, namely

$$\mathcal{T}_i = \big\{ j : \ x_k \in \mathcal{X}_i \text{ and } x_{k+1} \in \mathcal{X}_j^{int} \big\},\ \forall\, i \in \mathcal{I}_m \tag{3}$$

The set $\{\mathcal{T}_1, \ldots, \mathcal{T}_m\}$ will be referred to as the system transition paths. Moreover, it is assumed that any cells intersection has empty interior and is free of sliding modes (Wang *et al.*, 1994). □

The model $\mathcal{S}_i$ of (2) represents the system dynamics whenever the following condition holds, hereafter referred to as system switching rule:

$$\big\{ x_k \in \mathcal{X}_i^{int}, \ \text{ if } \ \Lambda_i\, x_k + \lambda_i \succ 0 \,\big\}, \ \ \forall\, i \in \mathcal{I}_m \tag{4}$$

where $\Lambda_i \in \mathbb{R}^{m_i \times n_x}$ and $\lambda_i \in \mathbb{R}^{m_i}$ are given matrices and vectors, and m_i are given integers. Note that, in view of (4), the polytope $\mathcal{X}_i$ must be defined as $\mathcal{X}_i = \{x : \Lambda_i x + \lambda_i \succeq 0\}$.

Remark 1. From the definition of $\mathcal{T}_i$ and the system equations (2), it follows that a given index j belongs to $\mathcal{T}_i$ if there exist vectors x_k and d_k satisfying:

$$\Lambda_i x_k + \lambda_i \succeq 0, \quad \Lambda_j (A_i x_k + B_i d_k) + \lambda_j \succ 0.$$

Note that this is an LMI problem in x_k and d_k that can be easily solved. Thus, the transition paths of the system can be easily determined. See Bemporad *et al.* (2000) for a related result. □

This paper is concerned with designing a discrete-time piecewise linear filter as well as its switching rule. To this end, let

$$\hat{\mathcal{X}} := \bigcup_{i=1}^{m} \hat{\mathcal{X}}_i \tag{5}$$

be the region of interest of the filter state-space in which the filter dynamics will be represented and the cells $\hat{\mathcal{X}}_i$ are given polytopes.

The filter dynamics in $\hat{\mathcal{X}}_i$ is described by the state-space model:

$$\mathcal{F}_i \ : \ \begin{aligned} \hat{x}_{k+1} &= \mathbf{A}_i \hat{x}_k + \mathbf{B}_i y_k \\ \hat{z}_k &= \mathbf{L}_i \hat{x}_k + \mathbf{M}_i y_k \end{aligned} \tag{6}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the filter state, $\hat{z}_k \in \mathbb{R}^{n_z}$ is the estimate of z_k, and $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{L}_i$ and $\mathbf{M}_i$ are matrices to be determined.

Similarly as for the system state trajectory, the following assumptions are imposed to the filter state trajectory:

Assumptions 2. It is assumed that when the filter state transits from one cell to itself, or to another cell, say from $\hat{\mathcal{X}}_i$ to $\hat{\mathcal{X}}_j$, it always moves to the interior of $\hat{\mathcal{X}}_j$. Also, let the set $\mathcal{U}_i \subseteq \mathcal{I}_m,\ \forall i \in \mathcal{I}_m$ represent all admissible one-step transitions from $\hat{\mathcal{X}}_i$ to itself, or to the interior of another cell of the filter state-space, namely

$$\mathcal{U}_i = \left\{ j : \hat{x}_k \in \hat{\mathcal{X}}_i \text{ and } \hat{x}_{k+1} \in \hat{\mathcal{X}}_j^{int} \right\},\ \forall i \in \mathcal{I}_m \quad (7)$$

The set $\{\mathcal{U}_1, \ldots, \mathcal{U}_m\}$ will be referred to as the filter transition paths. Moreover, it is assumed that any cells intersection has empty interior and does not contain unstable sliding modes. □

The model $\mathcal{F}_i$ of (6) represents the filter dynamics whenever the following condition holds, hereafter referred to as filter switching rules

$$\left\{ \hat{x}_k \in \hat{\mathcal{X}}_i^{int}, \text{ if } \hat{\Lambda}_i \hat{x}_k + \hat{\lambda}_i \succ 0 \right\},\ \forall i \in \mathcal{I}_m \quad (8)$$

where $\hat{\Lambda}_i \in \mathbb{R}^{m_i \times n_x}$ and $\hat{\lambda}_i \in \mathbb{R}^{m_i}$ are matrices and vectors to be determined jointly with the filter matrices. In the light of (8), the polytope $\hat{\mathcal{X}}_i$ must be defined as $\hat{\mathcal{X}}_i = \{\hat{x}_k : \hat{\Lambda}_i \hat{x}_k + \hat{\lambda}_i \succeq 0\}$.

Observe that as the system matrices and switching rules are given, it is possible to determine in advance the system transition paths as pointed out in Remark 1. However, this is not the case of the filter transition paths because the filter matrices and switching rules are to be determined. For this reason it is assumed that the filter transition paths are given and their choice can be regarded as degrees-of-freedom in the filter design. A natural choice is to consider for the filter the same transition paths as for the system.

It should be remarked that once the filter design problem is solved, the filter switching rules can be computed on-line because the signal $\hat{x}_k$ will be available on-line. However, due to the transition paths constraints (3) and (7) and the presence of noise in the system, it is not possible to know in advance the precise cell sequences corresponding to the system and filter trajectories. As a result, all possible cell to cell transitions specified in the system and filter transition paths sets must be considered in the filter design.

In order to represent the dynamics of the estimation error,

$$e_k := z_k - \hat{z}_k,$$

in a compact form, in the sequel we introduce some notations. Consider the binary variables δ_{i_k} and α_{i_k}, $i = 1, \ldots, m$, referred to as switching signals and defined according to (4) and (8) by

$$\delta_{i_k} := \begin{cases} 1, & \text{if } x_k \in \mathcal{X}_i^{int} \\ 0, & \text{otherwise} \end{cases},\quad \alpha_{i_k} := \begin{cases} 1, & \text{if } \hat{x}_k \in \hat{\mathcal{X}}_i^{int} \\ 0, & \text{otherwise} \end{cases}$$

and let the m-dimensional vectors

$$\delta_k := \left[\delta_{1_k} \cdots\ \delta_{m_k} \right]',\ \alpha_k := \left[\alpha_{1_k} \cdots\ \alpha_{m_k} \right]'. \quad (9)$$

With the above notation, the dynamics of the estimation error can be described by the following state-space model:

$$\mathcal{S}_e : \begin{aligned} \rho_{k+1} &= \hat{A}(\delta_k, \alpha_k)\rho_k + \hat{B}(\delta_k, \alpha_k) d_k \\ e_k &= \hat{C}(\delta_k, \alpha_k)\rho_k + \hat{D}(\delta_k, \alpha_k) d_k \end{aligned} \quad (10)$$

where $\rho_k = \left[\, x_k' \ \hat{x}_k' \,\right]'$ and

$$\hat{A}(\delta_k, \alpha_k) = \begin{bmatrix} A(\delta_k) & 0 \\ \mathbf{B}(\alpha_k)C(\delta_k) & \mathbf{A}(\alpha_k) \end{bmatrix} \quad (11)$$

$$\hat{B}(\delta_k, \alpha_k) = \begin{bmatrix} B(\delta_k) \\ \mathbf{B}(\alpha_k)D(\delta_k) \end{bmatrix} \quad (12)$$

$$\hat{C}(\delta_k, \alpha_k) = \left[\, L(\delta_k) - \mathbf{M}(\alpha_k)C(\delta_k) \quad -\mathbf{L}(\alpha_k) \,\right] \quad (13)$$

$$\hat{D}(\delta_k, \alpha_k) = M(\delta_k) - \mathbf{M}(\alpha_k)D(\delta_k). \quad (14)$$

In the above, $A(\delta_k) = \sum_{i=1}^m A_i \delta_{i_k}$, $\mathbf{A}(\alpha_k) = \sum_{i=1}^m \mathbf{A}_i \alpha_{i_k}$ and similarly for the remaining matrices. Observe that $A(\delta_k) = A_i$ whenever $x_k \in \mathcal{X}_i^{int}$ and $\mathbf{A}(\alpha_k) = \mathbf{A}_i$ whenever $\hat{x}_k \in \hat{\mathcal{X}}_j^{int}$, or equivalently $\delta_k = col_i(I_m)$ and $\alpha_k = col_j(I_m)$.

3. $\mathcal{H}_\infty$ FILTER DESIGN

This paper is concerned with the following filtering problem for the piecewise linear system described by (2)-(4):

Given a scalar $\gamma > 0$ and the filter transition paths (7), design a piecewise linear filter (6) as well as its switching rules (8) such that the estimation error system (10) is exponentially stable and its $\mathcal{H}_\infty$ norm is less than γ, namely

$$\|\mathcal{S}_e\|_\infty := \sup_{d \in \ell_2} \left\{ \frac{\|e\|_2}{\|d\|_2};\ d \not\equiv 0,\ \rho_0 = 0 \right\} < \gamma. \quad (15)$$

Consider the piecewise quadratic Lyapunov function candidate

$$V(\rho_k, \delta_k, \alpha_k) := \rho_k' P(\delta_k, \alpha_k)\rho_k \quad (16)$$

$$P(\delta_k, \alpha_k) := \sum_{i=1}^m \sum_{j=1}^m P_{ij} \delta_{i_k} \alpha_{j_k},\quad P_{ij} > 0 \quad (17)$$

and its time-variation

$$\begin{aligned} \Delta V(\rho_k, \delta_k, \alpha_k) &= \tilde{V}(\rho_k, \delta_k, \alpha_k) - V(\rho_k, \delta_k, \alpha_k) \\ \tilde{V}(\rho_k, \delta_k, \alpha_k) &= \tilde{\rho}_k' \tilde{P} \tilde{\rho}_k,\quad \tilde{P} = P(\tilde{\delta}_k, \tilde{\alpha}_k). \end{aligned}$$

It is well known that the $\mathcal{H}_\infty$ performance requirement (15) is fulfilled if there exists a Lyapunov function $V(\rho_k)$ such that

$$\Delta V(\rho_k, \delta_k, \alpha_k) + \frac{e_k' e_k}{\gamma} - \gamma\, d_k' d_k < 0 \quad (18)$$

Let the scaling matrix

$$G(\delta_k, \alpha_k) := \sum_{i=1}^m \sum_{j=1}^m G_{ij} \delta_{i_k} \alpha_{j_k}$$

where $G_{ij} \in \mathbb{R}^{2n_x \times 2n_x}$, $\forall i, j \in \mathcal{I}_m$ are nonsingular matrices to be determined. In addition, let $\tilde{G}$

denote $G(\tilde{\delta}_k, \tilde{\alpha}_k)$. For notation simplicity, in the sequel the dependence on k, δ_k and α_k of the variables and matrices will be omitted.

Next, define

$$\xi = \left[\, (\tilde{G}^{-1}\tilde{\rho})' \;\; (G^{-1}\rho)' \;\; d' \;\; \gamma^{-1}e' \,\right]' \tag{19}$$

$$\Psi = \mathrm{diag}\left\{\tilde{W},\ -W,\ -\gamma I_{n_d},\ \gamma I_{n_z}\right\} \tag{20}$$

$$W = G'PG, \quad \tilde{W} = \tilde{G}'\tilde{P}\tilde{G}. \tag{21}$$

Note that ξ is such that

$$\mathcal{H}\xi = 0, \quad \mathcal{H} = \begin{bmatrix} -I_{2n_x} & \tilde{G}^{-1}\hat{A}G & \tilde{G}^{-1}\hat{B} & 0 \\ 0 & \hat{C}G & \hat{D} & -\gamma I_{n_z} \end{bmatrix}.$$

With the above notation, (18) can be rewritten as

$$\xi'\Psi\xi < 0, \quad \forall\, \xi \neq 0 : \mathcal{H}\xi = 0. \tag{22}$$

By Finsler's lemma, (22) holds if and only if there exists a matrix F of appropriate dimensions such that

$$\xi'\bar{\Psi}\xi < 0, \quad \bar{\Psi} = \Psi + F\mathcal{H} + \mathcal{H}'F'. \tag{23}$$

Next, let us choose F as

$$F' = \begin{bmatrix} \tilde{G}' & 0_{2n_x} & 0_{2n_x \times n_d} & 0 \\ 0 & 0 & 0 & I_{n_z} \end{bmatrix} \tag{24}$$

which implies that

$$\bar{\Psi} = \begin{bmatrix} \tilde{W} - \tilde{G}' - \tilde{G} & \star & \star & \star \\ G'\hat{A}' & -W & \star & \star \\ \hat{B}' & 0 & -\gamma I & \star \\ 0 & \hat{C}G & \hat{D} & -\gamma I \end{bmatrix}. \tag{25}$$

Note that the above choice of F is non-conservative in the standard H_∞ filtering problem for linear systems without switchings ($m = 1$ and $\mathcal{X}_1 \equiv \mathbb{R}^{n_x}$). In this situation $\xi'\bar{\Psi}\xi < 0$ becomes equivalent to the matrix inequality $\bar{\Psi} < 0$, which turns out to be identical to the well known bounded real lemma inequality with the change of variable $W_{br} = -W + G + G'$ and $\tilde{W}_{br} = -\tilde{W} + \tilde{G} + \tilde{G}'$.

Observe that $\bar{\Psi}$ is not affine in the decision variable G and the filter matrices, but it may be rendered affine via appropriate variables parameterizations and a congruent transformation. This issue will be discussed in the sequel.

First, the matrices P_{ij}, G_{ij} and the filter matrices $\mathbf{A}_i, \mathbf{B}_i$ and $\mathbf{L}_i$ are parameterized in terms of new decision variables, namely

$$\mathbf{Z}(\alpha_k) := \sum_{i=1}^{m} \mathbf{Z}_i \alpha_{i_k}\ , \quad \mathbf{X}(\alpha_k) := \sum_{i=1}^{m} \mathbf{X}_i \alpha_{i_k} \tag{26}$$

$$\mathbf{Y}(\alpha_k) := \sum_{i=1}^{m} \mathbf{Y}_i \alpha_{i_k}\ , \quad Z(\alpha_k) := \sum_{i=1}^{m} Z_i \alpha_{i_k} \tag{27}$$

$$Q(\delta_k, \alpha_k) = \sum_{i=1}^{m}\sum_{j=1}^{m} Q_{ij}\delta_{i_k}\alpha_{j_k} \tag{28}$$

$$S(\delta_k, \alpha_k) := \sum_{i=1}^{m}\sum_{j=1}^{m} S_{ij}\delta_{i_k}\alpha_{j_k} \tag{29}$$

$$T(\delta_k, \alpha_k) := \sum_{i=1}^{m}\sum_{j=1}^{m} T_{ij}\delta_{i_k}\alpha_{j_k} \tag{30}$$

Specifically, for $\delta_k = col_i(I_m)$, $\alpha_k = col_j(I_m)$ let

$$G_{ij} = \begin{bmatrix} T_{ij}^{-1} & T_{ij}^{-1}S_{ij}U^{-1} \\ R_j T_{ij}^{-1} & (R_j + R_j T_{ij}^{-1}S_{ij})U^{-1} \end{bmatrix} \tag{31}$$

$$P_{ij} = \begin{bmatrix} I_{n_x} & R_j' \\ 0 & R_j' \end{bmatrix}^{-1} Q_{ij} \begin{bmatrix} I_{n_x} & 0 \\ R_j & R_j \end{bmatrix}^{-1} \tag{32}$$

$$\mathbf{A}_j = (U')^{-1}\mathbf{X}_j R_j^{-1}, \quad \mathbf{B}_j = (U')^{-1}\mathbf{Z}_j, \quad \mathbf{L}_j = \mathbf{Y}_j R_j^{-1} \tag{33}$$

where U and R_j are any $n \times n$ nonsingular matrices satisfying $U'R_j = Z_j$. Next, condition (23) is rewritten as

$$\xi_a'\Psi_a\xi_a < 0 \tag{34}$$

where

$$\xi_a = \mathcal{N}^{-1}\xi, \quad \mathcal{N} = \mathrm{diag}\left\{\tilde{N},\ N,\ I_{n_x},\ I_{n_z}\right\} \tag{35}$$

$$N(\delta_k, \alpha_k) = \begin{bmatrix} T(\delta_k, \alpha_k) & -S(\delta_k, \alpha_k) \\ 0_{n_x} & U \end{bmatrix} \tag{36}$$

$$\tilde{N} = N(\tilde{\delta}_k, \tilde{\alpha}_k) \tag{37}$$

$$\begin{aligned} \Psi_a &= \mathcal{N}'\bar{\Psi}\mathcal{N} \\ &= \begin{bmatrix} \tilde{N}'(\tilde{W} - \tilde{G}' - \tilde{G})\tilde{N} & \star & \star & \star \\ N'G'\hat{A}'\tilde{N} & -N'WN & \star & \star \\ \hat{B}'\tilde{N} & 0 & -\gamma I & \star \\ 0 & \hat{C}GN & \hat{D} & -\gamma I \end{bmatrix} \end{aligned} \tag{38}$$

Observe that for $\delta_k = col_i(I_m)$, $\alpha_k = col_j(I_m)$

$$N(\delta_k, \alpha_k) = \sum_{i=1}^{m}\sum_{j=1}^{m} N_{ij}\delta_{i_k}\alpha_{j_k}, \quad N_{ij} = \begin{bmatrix} T_{ij} & -S_{ij} \\ 0_{n_x} & U \end{bmatrix} \tag{39}$$

$$GN = \sum_{i=1}^{m}\sum_{j=1}^{m} G_{ij}N_{ij}\delta_{i_k}\alpha_{j_k}, \quad G_{ij}N_{ij} = \begin{bmatrix} I_{n_x} & 0 \\ R_j & R_j \end{bmatrix} \tag{40}$$

The interest of the above parameterization is that (38) becomes affine in the decision variables of (26)-(30), as it will be shown in the sequel. Moreover, the congruent transformation $\mathcal{N}'\bar{\Psi}\mathcal{N}$ has the feature that x_k and x_{k+1} are elements of ξ_a, which will be fundamental in the filter design.

Let us suppose, without loss of generality, that

$$\begin{cases} x_k \in \mathcal{X}_i^{int} \text{ and } \hat{x}_k \in \hat{\mathcal{X}}_j^{int}, \\ \text{for some } i \in \mathcal{I}_m \text{ and } j \in \mathcal{I}_m \\ x_{k+1} \in \mathcal{X}_s^{int} \text{ and } \hat{x}_{k+1} \in \hat{\mathcal{X}}_r^{int}, \\ \text{for some } s \in \mathcal{T}_i \text{ and } r \in \mathcal{U}_j \end{cases} \tag{41}$$

In this situation, the matrix Ψ_a, referred to as $\Psi_{a_{ij}}^{sr}$, takes the form

$$\Psi^{sr}_{a_{ij}} = \begin{bmatrix} Q_{sr} - \Xi_{1_{sr}} & \star & \star & \star \\ \Xi'_{2^{sr}_{ij}} & -Q_{ij} & \star & \star \\ \Xi'_{3^{sr}_{ij}} & 0 & -\gamma I & \star \\ 0 & \Xi_{4_{ij}} & M_i - \mathbf{M}_j D_i & -\gamma I \end{bmatrix} \quad (42)$$

where

$$\Xi_{1_{sr}} = \begin{bmatrix} T_{sr} + T'_{sr} & Z'_r - S_{sr} \\ Z_r - S'_{sr} & Z_r + Z'_r \end{bmatrix} \quad (43)$$

$$\Xi_{2^{sr}_{ij}} = \begin{bmatrix} T'_{sr} A_i & 0 \\ -S'_{sr} A_i + \mathbf{Z}_j C_i + \mathbf{X}_j & \mathbf{X}_j \end{bmatrix} \quad (44)$$

$$\Xi_{3^{sr}_{ij}} = \begin{bmatrix} T'_{sr} B_i \\ -S'_{sr} B_i + \mathbf{Z}_j D_i \end{bmatrix} \quad (45)$$

$$\Xi_{4_{ij}} = \begin{bmatrix} L_i - \mathbf{Y}_j - \mathbf{M}_j C_i & -\mathbf{Y}_j \end{bmatrix} \quad (46)$$

On the other hand, notice from (19) and (35) that

$$\xi_a = \begin{bmatrix} (\tilde{N}^{-1}\tilde{G}^{-1}\tilde{\rho})' & (N^{-1}G^{-1}\rho)' & d' & \gamma^{-1}e' \end{bmatrix}'. \quad (47)$$

Further, by considering (40), we have

$$N^{-1}G^{-1}\rho = \begin{bmatrix} x_k \\ -x_k + R^{-1}\hat{x}_k \end{bmatrix} \quad (48)$$

$$\tilde{N}^{-1}\tilde{G}^{-1}\tilde{\rho} = \begin{bmatrix} x_{k+1} \\ -x_{k+1} + \tilde{R}^{-1}\hat{x}_{k+1} \end{bmatrix}. \quad (49)$$

Thus, the transformation performed to render Ψ_a affine in the decision variable is such that the elements x_k and x_{k+1} of ρ_k and ρ_{k+1}, respectively, are also elements of ξ_a, which is a desired feature.

For each value of (i,j,s,r) from (41), let L^{sr}_{ij}, E^{sr}_{ij}, $\hat{L}^{sr}_{ij}$ and $\hat{E}^{sr}_{ij}$ be symmetric matrices with non-negative entries. Observe from (4) that for any $x_k \in \mathcal{X}^{int}_i$ and $x_{k+1} \in \mathcal{X}^{int}_s$, the scalar inequalities $(\Lambda_i x_k + \lambda_i)' L^{sr}_{ij} (\Lambda_i x_k + \lambda_i) > 0$ and $(\Lambda_s x_{k+1} + \lambda_s)' E^{sr}_{ij} (\Lambda_s x_{k+1} + \lambda_s) > 0$ hold. Similarly, from (8) it follows that for any $\hat{x}_k \in \hat{\mathcal{X}}^{int}_j$ and $\hat{x}_{k+1} \in \hat{\mathcal{X}}^{int}_r$, then $(\hat{\Lambda}_j \hat{x}_k + \hat{\lambda}_j)' \hat{L}^{sr}_{ij} (\hat{\Lambda}_j \hat{x}_k + \hat{\lambda}_j) > 0$ and $(\hat{\Lambda}_r \hat{x}_{k+1} + \hat{\lambda}_r)' \hat{E}^{sr}_{ij} (\hat{\Lambda}_r \hat{x}_{k+1} + \hat{\lambda}_r) > 0$.

Using the *S-Procedure*, we have $\xi'_a \Psi_a \xi_a < 0$ satisfied for the situation (41) if

$$\begin{aligned} &\xi'_a \Psi^{sr}_{a_{ij}} \xi_a + (\Lambda_i x_k + \lambda_i)' L^{sr}_{ij} (\Lambda_i x_k + \lambda_i) \\ &+ (\Lambda_s x_{k+1} + \lambda_s)' E^{sr}_{ij} (\Lambda_s x_{k+1} + \lambda_s) \\ &+ (\hat{\Lambda}_j \hat{x}_k + \hat{\lambda}_j)' \hat{L}^{sr}_{ij} (\hat{\Lambda}_j \hat{x}_k + \hat{\lambda}_j) \\ &+ (\hat{\Lambda}_r \hat{x}_{k+1} + \hat{\lambda}_r)' \hat{E}^{sr}_{ij} (\hat{\Lambda}_r \hat{x}_{k+1} + \hat{\lambda}_r) < 0 \end{aligned} \quad (50)$$

holds for some symmetric matrices with non-negative entries L^{sr}_{ij}, E^{sr}_{ij}, $\hat{L}^{sr}_{ij}$ and $\hat{E}^{sr}_{ij}$.

Motivated by the structure of the vector ξ_a in (47), we shall assume the following particular structure for the matrix $\hat{\Lambda}_i$ and vector $\hat{\lambda}_i$ of the filter switching rules (8):

$$\hat{\Lambda}_i = \Lambda_i R_i^{-1}, \quad \hat{\lambda}_i = \lambda_i \quad (51)$$

where R_i is the same matrix used to parameterize the matrix G_{ij} as in (31), i.e. the filter switching rule of (8) is of the form

$$\left\{ \hat{x}_k \in \hat{\mathcal{X}}^{int}_i, \text{ if } \Lambda_i R_i^{-1} \hat{x}_k + \lambda_i \succ 0 \right\}, \quad i \in \mathcal{I}_m \quad (52)$$

Considering (41), from (48) and (49) we get

$$\begin{aligned} \hat{\Lambda}_r \hat{x}_{k+1} &= \Lambda_r \Lambda_0 N^{-1}_{sr} G^{-1}_{sr} \rho_{k+1} = \Lambda_r \Lambda_0 H_1 \xi_a \\ \hat{\Lambda}_j \hat{x}_k &= \Lambda_j \Lambda_0 N^{-1}_{ij} G^{-1}_{ij} \rho_k = \Lambda_j \Lambda_0 H_2 \xi_a \end{aligned}$$

where $\Lambda_0 = \begin{bmatrix} I_{n_x} & I_{n_x} \end{bmatrix}$ and

$$H_1 = \begin{bmatrix} I_{2n_x} & 0_{2n_x} & 0_{2n_x \times (n_d + n_z)} \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0_{2n_x} & I_{2n_x} & 0_{2n_x \times (n_d + n_z)} \end{bmatrix}.$$

In order to transform (50) into an LMI, consider the matrix $\Psi^{sr}_{a_{ij}}$ of (42) and define

$$\xi_b = \begin{bmatrix} \xi'_a & 1 \end{bmatrix}', \quad \mathcal{J} = \begin{bmatrix} I_n & 0_n \end{bmatrix} \quad (53)$$

$$\begin{aligned} \Omega^{sr}_{ij} &= H'_1 \Lambda'_0 \Lambda'_r \hat{E}^{sr}_{ij} \Lambda_r \Lambda_0 H_1 + H'_2 \Lambda'_0 \Lambda'_j \hat{L}^{sr}_{ij} \Lambda_j \Lambda_0 H_2 \\ &\quad + H'_3 \Lambda'_s E^{sr}_{ij} \Lambda_s H_3 + H'_4 \Lambda'_i L^{sr}_{ij} \Lambda_i \mathcal{J} \end{aligned} \quad (54)$$

$$\begin{aligned} \Phi^{sr}_{ij} &= H'_1 \Lambda'_0 \Lambda'_r \hat{E}^{sr}_{ij} \lambda_r + H'_2 \Lambda'_0 \Lambda'_j \hat{L}^{sr}_{ij} \lambda_j \\ &\quad + H'_3 \Lambda'_s E^{sr}_{ij} \lambda_s + H'_4 \Lambda'_i L^{sr}_{ij} \lambda_i \end{aligned} \quad (55)$$

$$\varpi^{sr}_{ij} = \lambda'_i L^{sr}_{ij} \lambda_i + \lambda'_s E^{sr}_{ij} \lambda_s + \lambda'_j \hat{L}^{sr}_{ij} \lambda_j + \lambda'_r \hat{E}^{sr}_{ij} \lambda_r \quad (56)$$

$$H_3 = \begin{bmatrix} \mathcal{J} & 0_{n_x \times 2n_x} & 0_{n_x \times (n_d + n_z)} \end{bmatrix} \quad (57)$$

$$H_4 = \begin{bmatrix} 0_{n_x \times 2n_x} & \mathcal{J} & 0_{n_x \times (n_d + n_z)} \end{bmatrix}. \quad (58)$$

Hence, it can be readily shown that (50) can be rewritten as $\xi'_b \Gamma^{sr}_{ij} \xi_b < 0$, where

$$\Gamma^{sr}_{ij} = \begin{bmatrix} \Psi^{sr}_{a_{ij}} + \Omega^{sr}_{ij} & \Phi^{sr}_{ij} \\ \star & \varpi^{sr}_{ij} \end{bmatrix}. \quad (59)$$

Thus, (50) holds if the following LMIs are feasible:

$$\bar{\Gamma}^{sr}_{ij} < 0, \ \forall \ i, \ j, \ s, \ r \ \text{ as in (41)}. \quad (60)$$

The set of decision variables in the above LMIs are the matrices in (26)-(30), the filter matrix $\mathbf{M}_i$ and the scaling matrices L^{sr}_{ij}, E^{sr}_{ij}, $\hat{L}^{sr}_{ij}$ and $\hat{E}^{sr}_{ij}$.

The above development leads to the following theorem.

Theorem 1. Consider the piecewise linear system (2) with given transition paths (3) and switching rules (4). Given a scalar $\gamma > 0$, there exists a piecewise linear filter (6) with given transition paths (7) and switching rules (52), where the matrices R_i are to be found, and such that $\|\mathcal{S}_e\|_\infty < \gamma$, if there exist matrices Q_{ij}, S_{ij}, T_{ij}, Z_j, $\mathbf{X}_j$, $\mathbf{Y}_j$, $\mathbf{Z}_j$, $\mathbf{M}_j$, $L^{sr}_{ij} \succeq 0$, $E^{sr}_{ij} \succeq 0$, $\hat{L}^{sr}_{ij} \succeq 0$ and $\hat{E}^{sr}_{ij} \succeq 0$ satisfying the LMIs of (60). Moreover, the filter matrices $\mathbf{A}_i$, $\mathbf{B}_i$ and $\mathbf{L}_i$ are given by (33), where U and R_i are any nonsingular matrices satisfying $U' R_i = Z_i$.

Proof: First, note that by performing appropriate congruent transformations on (60), it can be easily shown that the matrices R_i and T_{ij} are nonsingular and thus the nonsingular matrices U and R_i such that $U' R_i = Z_i$ are well defined. Hence, the matrices N_{ij} and $G_{ij} N_{ij}$ are nonsingular and thus G_{ij} is a nonsingular matrix as well.

The proof then follows directly from the fact that (60) implies the desired $\mathcal{H}_\infty$ condition (15), even in the presence of switchings from cell to cell in the system and filter models. Observe from the definition of the system and filter transition paths (3) and (7), that $i \in \mathcal{T}_i$ and $j \in \mathcal{U}_j$. The condition $\bar{\Gamma}_{ij}^{ij} < 0$, $\forall i, j \in \mathcal{I}_m$ ensure that the error dynamics satisfy (15) whenever the system and filter trajectories remain inside a cell. On the other hand, if any admissible cell to cell transition takes place in one step of time, the condition $\bar{\Gamma}_{ij}^{sr} < 0$, $\forall i, j \in \mathcal{I}_m$ and $s \in \mathcal{T}_i$ and $r \in \mathcal{U}_j$ ensures that (15) is also satisfied. To complete the proof observe that, by assumption, no unstable sliding surfaces exist at any cells intersection. Note that, this is indeed a necessary condition to have (60) satisfied for all $x_k \in \mathcal{X}$ and $\hat{x}_k \in \hat{\mathcal{X}}$. $\nabla\nabla\nabla$

Remark 2. Note that since the LMIs of (60) do not dependent on the matrix U and as this matrix can be arbitrarily chosen such that $U'R_i = Z_i$, without loss of generality, we can set $U = I_n$. Under this condition, $R_i = Z_i$ and the filter matrices of (33) become:

$$\mathbf{A}_i = \mathbf{X}_i Z_i^{-1}, \quad \mathbf{B}_i = \mathbf{Z}_i, \quad \mathbf{L}_i = \mathbf{Y}_i Z_i^{-1} \qquad (61)$$

Remark 3. Under the assumption that any intersection among the filter cells has empty interior, the filter model is uniquely characterized through the filter switching signal in (9). If the filter state cells obtained from the Theorem 1 do not satisfy this condition, the filter model corresponding to a point at any non-empty intersection of cells can be arbitrarily chosen from the set of cells belonging to that intersection. For instance, if i, j, s and r are as in (41) and $\hat{x}_{k+1} \in \hat{\mathcal{X}}_r^{int} \cap \hat{\mathcal{X}}_q^{int}$ for some $q \in \mathcal{I}_m$ $(q \neq r)$, the filter model at instant $k+1$ can be arbitrarily chosen from the models associated with either the cell $\hat{\mathcal{X}}_r^{int}$, or $\hat{\mathcal{X}}_q^{int}$. This is possible because the conditions leading to the desired $\mathcal{H}_\infty$ condition (15) are satisfied for all cells of the filter state space given by Theorem 1. A natural filter choice is to use the filter associated with the cell for which the trace of $\bar{\Gamma}_{ij}^{sr}$ is the smallest one.

Remark 4. If the LMIs of (60) are feasible, then (16)-(18) are satisfied pointwise $\forall x_k \in \mathcal{X}$ and $\forall \hat{x}_k \in \hat{\mathcal{X}}$. Once the filter is given by Theorem 1, the problem of estimating a region of attraction of the origin of the estimation error system may be addressed as in Coutinho and Trofino (2003).

4. CONCLUSIONS

This paper has addressed the problem of $\mathcal{H}_\infty$ filtering for piecewise linear discrete-time state-space models. Attention was focused on deriving LMI conditions for the design of a piecewise linear filter whose matrices and switching rules are determined in order to minimize an upper bound on the $\mathcal{H}_\infty$ norm of the estimation error system. An interesting feature of the method is that the proposed filter parameterization leads, after a congruent transformation, to a performance condition based on a quadratic form where the system and filter states appear explicitly. This allows for the use of the *S-Procedure* to take the structure of the cells of the system and filter state-space partitions into account leading to a less conservative result.

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