NONLINEAR CONTROL ANALYSIS ON KINEMATICALLY ASYMMETRICALLY AFFINE CONTROL SYSTEMS WITH NONHOLONOMIC AFFINE CONSTRAINTS

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Abstract: This paper is concerned with kinematically asymmetrically affine control systems with nonholonomic affine constraints (KAACS). We first show the nonintegrability condition for affine constraints using their geometric representation. Next, the KAACS is uniquely derived from affine constraints and control inputs. We then analyze the KAACS based on nonlinear control theory and provide some conditions of accessibility, controllability and stabilizability of the system. Finally, two physical examples are illustrated to verify our results.

Keywords: Nonlinear control systems, Kinematics, Geometric approach, Controllability, Stabilizability

1. INTRODUCTION

Many researches on nonholonomic systems have been done in the fields of control engineering and robotics. Nonholonomic systems are defined in analytical mechanics as systems that there exist nonintegrable constraints and the behavior is subject to the constraints. In researches of nonholonomic systems, almost all constraints are linear in velocities, called *linear constraints*. For example, mobile robots, trailers, space robots and so on, many systems in engineering have such linear constraints. In general, integrability and nonintegrability of linear constraints can be checked using concepts of Lie brackets, involutivity of distributions and Frobenius's theorem. It is well known that linear constraints are transformed into symmetrically affine control systems and such systems are not locally asymptotically stabilizable by any nonlinear smooth state feedback from Brockett's theorem (Brockett, 1983). Therefore, many control laws such as nonsmooth feedback law, time-varying feedback and switching control to avoid the property have been proposed. On the other hand, there is a larger class of constraints which are affine in velocities, called *affine constraints*. Space robots with initial angular momentum, a rolling ball on a rotating table and a pneumatic tire (Neimark, *et al.* 1972) are subject to such affine constraints. However, few researches on affine constraints have been done so far. Moreover, there is no study on asymmetrically affine control systems which are derived from affine constraints.

The purpose of this paper is to analyze kinematically asymmetrically affine control systems with nonholonomic affine constraints based on nonlinear control theory. In Section 2, we first give a classification method, geometric representation of affine constraints and then show their complete nonholonomicity condition. We next derive the kinematically asymmetrically affine control system (KAACS) in Section 3. Section 4 contains the main results of this paper and it devotes nonlinear control analysis of the KAACS. Accessibility, controllability and stabilizability of the KAACS are investigated here. Finally, we illustrate two physical examples to check our theory in Section 5. Through this paper, manifolds, vector fields, functions and distributions are all assumed to be smooth.

2. NONHOLONOMIC AFFINE CONSTRAINTS

2.1 Preliminaries

In this subsection, we give some preliminaries on affine constraints. We first define affine constraints which we treat through this paper. Let Q be an n-dimensional configuration manifold and a column vector $q = [\ q_1 \ \ \cdots \ \ q_n\ ]^T \in \boldsymbol{R}^n$ be a generalized configuration coordinate of Q. In this paper, we consider $n-m$ $(n > m)$ affine constraints:

$$A(q) + B(q)\dot{q} = 0, \tag{1}$$

where $A(q)$ is an $(n-m)$-dimensional column vector and $B(q)$ is an $(n-m) \times n$ matrix. Now we assume independence of affine constrains as follows.

Assumption 1. *$B(q)$ in affine constraints (1) has row full-rank, that is,* $\mathrm{rank}\, B(q) = n - m$, $\forall q \in Q$. □

We next introduce new concepts to classify affine constraints. A point $q \in Q$ such that $A(q) = 0$ is called *an affine equilibrium point* and the set of affine equilibrium points is defined by

$$U^e = \{q \in Q \mid A(q) = 0\}. \tag{2}$$

We then define *the affine index at* $q^e \in U^e$:

$$r(q^e) := \mathrm{rank}\frac{\partial A}{\partial q}(q^e) \tag{3}$$

and *the affine index of the affine constraints*:

$$r := \max_{q^e \in U^e} r(q^e). \tag{4}$$

Using these, we classify affine constraints as follows.

Definition 1. *Affine constraints (1) are categorized into three types by affine indices as follows.*

(A) $r = 0$ *: completely linear constraints*

(B) $1 \le r \le n-m-1$ *: r-th order partially affine constraints*

(C) $r = n - m$ *: completely affine constraints* □

In Definition 1, "completely linear constraints" correspond with "linear constraints" which have been studied so far. "Partially affine constraints" are constraints that linear and affine constraints are blended such as a coin on a rotating table in Section 5. "Completely affine constraints" are constraints which consist of only affine constraints such as a ball on a rotating table in Section 5.

Finally, we explain geometric representation of affine constraints. Geometric representation plays important roles through this paper. Since $n - m$ row vectors of $B(q)$ are independent each other from Assumption 1, we consider m vector fields $Y_1, \cdots, Y_m$ which are independent each other and annihilate $n - m$ row vectors of $B(q)$. Let us denote a space spanned by $Y_1, \cdots, Y_m$, that is, a distribution on Q as

$$D = \mathrm{span}\{Y_1, \cdots, Y_m\}. \tag{5}$$

A curve $q : I \to Q$ is said to satisfy affine constraints for a time interval I if and only if there exists a vector field X and the curve satisfies

$$\dot{q}(t) - X(q(t)) \in D(q(t)) \qquad \forall\, t \in I. \tag{6}$$

X is called *an affine vector field* and satisfies

$$A(q) + B(q)X(q) = 0, \ \ \forall q \in Q. \tag{7}$$

Therefore, affine constraints (1) are geometrically represented by a pair (D, X) (Lewis *et al.*, 1995; Bloch, 2003). The next holds for geometric representation.

Proposition 1. *For the geometric representation of affine constraints (D, X), $X(q) \in D(q)$ holds at a point $q \in Q$ if and only if the point is an affine equilibrium point. Conversely, $X(q) \notin D(q)$ holds at a point $q \in Q$ if and only if the point is an affine regular point.* □

2.2 Completely Nonholonomicity

In this subsection, we discuss completely nonholonomicity of affine constraints. If all the $n - m$ affine constraints (1) are nonintegrable, they are said to be *completely nonholonomic* or *completely nonintegrable*. Now we define a smallest and involutive distribution C_0 which contains $Y_1, \cdots, Y_m$ and satisfies $[X, W] \in C_0, \forall W \in C_0$. The necessary and sufficient condition of complete nonholonomicity for affine constraints is given as follows.

Theorem 1. *Affine constraints (1) are completely nonholonomic if and only if*

$$\dim C_0 = n \tag{8}$$

holds.

(Proof) Consider the product space $\bar{Q} := \boldsymbol{R} \times Q$ with $(n + 1)$-dimension, where $\boldsymbol{R}$ is the space of the time variable. In $\bar{Q}$, affine constraints (1) are represented by Pfaffian equations of $n - m$ differential forms:

$$A(q)dt + B(q)dq = 0. \tag{9}$$

Since an affine vector field X of geometric representation satisfies (7), $m + 1$ vector fields on $\bar{Q}$ which annihilate (9) are given by

$$\bar{X} := \frac{\partial}{\partial t} \oplus X, \ \ \bar{Y}_i := 0 \oplus Y_i \ \ (i = 1, \cdots, m). \tag{10}$$

Now we define an involutive distribution $\bar{C}$ defined on $\bar{Q}$, which contains $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$ and iterated Lie brackets that consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$. Therefore, the necessary and sufficient condition for complete nonintegrability is given by

$$\dim \bar{C} = n + 1 \tag{11}$$

(cf. Frobenius's theorem (Nijmeijer *et al.*, 1990; Isidori 1995)). Calculate iterated Lie brackets which consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$, we have

$$\begin{aligned}
&[\bar{X}, \bar{Y}_i] = 0 \oplus [X, Y_i]\,, \\
&[\bar{X}, [\bar{X}, \bar{Y}_i]] = 0 \oplus [X, [X, Y_i]], \ \cdots \\
&[\bar{Y}_j, \bar{Y}_i] = 0 \oplus [Y_j, Y_i]\,, \\
&[\bar{Y}_k, [\bar{Y}_j, \bar{Y}_i]] = 0 \oplus [Y_k, [Y_j, Y_i]], \ \cdots
\end{aligned} \tag{12}$$

We can find that $\bar{X}$ is independent of $\bar{Y}_i, \cdots, \bar{Y}_m$ and iterated Lie brackets (12) each other. Therefore, the necessary and sufficient condition (11) is changed into the condition that $\bar{Y}_1, \cdots, \bar{Y}_m$ and iterated Lie brackets which consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$ span an n-dimensional space. From (10) and (12), we only consider $Y_1, \cdots, Y_m$ on Q instead of $\bar{Y}_1, \cdots, \bar{Y}_m$ on $\bar{Q}$, and iterated Lie brackets which consist of $X, Y_1, \cdots, Y_m$ on Q instead of those which consist of $\bar{X}, \bar{Y}_1, \cdots, \bar{Y}_m$ on $\bar{Q}$. Consequently, the necessary and sufficient condition for the complete nonintegrability is that $Y_1, \cdots, Y_m$ and iterated Lie brackets which consist of $X, Y_1, \cdots, Y_m$ span an n-dimensional space, that is, (8) holds. □

We then assume the following, that is, we deal with nonholonomic affine constraints.

Assumption 2. *Affine constraints (1) are completely nonholonomic, that is, (8) holds.* □

3. KINEMATICALLY ASYMMETRICALLY AFFINE CONTROL SYSTEMS

In this section, we consider *kinematically asymmetrically affine control systems with nonholonomic affine constraint (KAACS)*. Let an m-dimensional column vector $u = [\, u_1 \ \cdots \ u_m \,]^T \in \Omega \subset \boldsymbol{R}^m$ be control inputs, where u is admissible control inputs and Ω denotes the set of admissible control inputs, which is a bounded closed subset of $\boldsymbol{R}^m$. We can think of velocities of the system $\dot{q}$ as control inputs, since we deal with kinematic models. Then we set control inputs u as

$$u := E(q)\dot{q}, \tag{13}$$

where $E(q)$ is an $m \times n$ matrix. Here we claim the next assumption on control inputs.

Assumption 3. *m control inputs (13) are assumed to be independent each other. Therefore, $E(q)$ in (13) has row full-rank, that is,* $\operatorname{rank} E(q) = m \ \forall q \in Q$. *Moreover, all the control inputs do not lie on the constrained space* $D^{\perp} := \operatorname{span}\{B_1, \cdots, B_{n-m}\}$, *where $B_1, \cdots, B_{n-m}$ are $n-m$ row vectors of $B(q)$, that is,*

$$\operatorname{rank} \begin{bmatrix} B(q) \\ E(q) \end{bmatrix} = n, \ \ \forall q \in Q \tag{14}$$

holds. □

We next derive a KAACS from affine constraints (1) and the control inputs (13) as follows.

Proposition 2. *From affine constraints (1) and control inputs (13) under Assumption 3, an affine vector field $\hat{X}$ and m base vector fields $\hat{Y}_1, \cdots, \hat{Y}_m$ of a distribution D are uniquely given by*

$$\begin{aligned} \hat{X}(q) &:= -\begin{bmatrix} B(q) \\ E(q) \end{bmatrix}^{-1} \begin{bmatrix} A(q) \\ 0 \end{bmatrix}, \\ \hat{Y}_i(q) &:= -\begin{bmatrix} B(q) \\ E(q) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ e_i \end{bmatrix} \ \ (i = 1, \cdots, m), \end{aligned} \tag{15}$$

where e_i is an m-dimensinal unit column vector. Moreover, the kinematically asymmetrically affine control systems with nonholonomic affine constraints:

$$\dot{q} = \hat{X}(q) + \sum_{i=1}^{m} \hat{Y}_i(q) u_i \tag{16}$$

can be uniquely derived using $\hat{X}, \hat{Y}_1, \cdots, \hat{Y}_m$. □

Since the drift term $\hat{X}$ does not lie in the distribution D at any affine regular point from Proposition 1, the KAACS is essentially asymmetric in control inputs. From (15), we can find that affine equilibrium points of affine constraints (1) correspond with equilibria of the KAACS, that is, $\hat{X}(q) = 0$ if and only if $A(q) = 0$.

Finally, we consider the linear approximation of the KAACS, which plays an important role in nonlinear control analysis in Section 4. Now we define

$$\left[\hat{Z}(q) \ \hat{Y}(q) \right] := \begin{bmatrix} B(q) \\ E(q) \end{bmatrix}^{-1},$$

where $\hat{Z}(q)$ is an $n \times (n-m)$ matrix and $\hat{Y}(q) = [\, \hat{Y}_1(q) \ \cdots \ \hat{Y}_m(q) \,]$. Then the linear approximation of the KAACS (16) at an equilibrium $q^e \in U^e$ is given by

$$\begin{aligned} \dot{q} &= \frac{\partial \hat{X}}{\partial q}(q^e)(q - q^e) + \hat{Y}(q^e) u \\ &= \underbrace{-\hat{Z}(q^e)\frac{\partial A}{\partial q}(q^e)}_{\mathcal{A}}(q - q^e) + \underbrace{\hat{Y}(q^e)}_{\mathcal{B}} u \\ &=: \mathcal{A}(q - q^e) + \mathcal{B}u, \end{aligned} \tag{17}$$

where $\mathcal{A}$ is an $n \times n$ matrix and $\mathcal{B}$ is an $n \times m$ matrix.

4. NONLINEAR CONTROL ANALYSIS

4.1 Accessibility

In this subsection, we first consider accessibility of the KAACS. Now we sum up some concepts of accessibility of nonlinear control systems. Given $q_0 \in Q$, let $\Lambda^V(t, q_0)$ be the set of points $q \in Q$ that there exists a neighborhood V of q_0 and an admissible control u such that there is a trajectory $q(\tau)$ of (16) which satisfies $q(\tau) \in V \ (0 \le \tau \le t)$ and $q(0) = q_0, q(t) = q$. This set is called *the accessible set from q_0 at time t*. Let $\Lambda^V_t(q_0)$ be the other set defined by a sum of $\Lambda^V(\tau, q_0)$ from time 0 to t. This set is called *the accessible set from q_0 in up to time t*. If $\Lambda^V_t(q_0)$ contains a non-empty open set of the configuration manifold Q for all neighborhoods V of q_0, then the system is called *locally accessible from q_0*. Moreover if for any neighborhood V of q_0, $\Lambda^V(t, q_0)$ contains a non-empty open set of the configuration manifold Q for any $t > 0$ sufficiently small, then the system is called *strongly locally accessible from q_0*. Now we can prove the following theorem.

Theorem 2. *The KAACS (16) is strongly locally accessible from any point $q \in Q$.*
(Proof) It is known that a system is strongly locally accessible at $q \in Q$ if and only if the dimension of the strong accessibility distribution at the point is equal to that of the configuration manifold (Nijmeijer *et al.*, 1990). We can find that the distribution C_0 defined in Section 2 is equivalent to the strong accessibility distribution. Since affine constraints (1) are assumed

to be completely nonholonomic, that is, (8) holds, then the KAACS is strongly locally accessible from any point $q \in Q$. □

In case of completely linear constraints, it is well known that the condition for complete nonintegrability is equivalent to Chow's theorem (Nijmeijer *et al.*, 1990).

4.2 Controllability

We next investigate controllability of the KAACS. If a system is locally accessible and $\Lambda_t^V(q_0)$ contains the point q_0, then the system is called *locally controllable at* q_0. There are two approaches to check local controllability for nonlinear control systems, the one is based on linear approximation and the other is Sussmann's theorem (Sussmann, 1987). We here use the former approach. It is known that if the linear approximation of a nonlinear control system at an equilibrium is controllable, then the nonlinear system is locally controllable at the equilibrium. Now we show the following theorem for controllability of the linear approximation of the KAACS.

Theorem 3. *The linear approximation of the KAACS at an equilibrium $q^e \in U^e$ (17) is controllable if and only if the matrix defined by*

$$\mathcal{V} := \left[\frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) \ \cdots \ \frac{\partial A}{\partial q}(q^e)\left\{\hat{Z}(q^e)\frac{\partial A}{\partial q}(q^e)\right\}^{n-2}\hat{Y}(q^e)\right] \quad (18)$$

has row full-rank, that is, $\mathrm{rank}\,\mathcal{V} = n - m$ *holds.*
(Proof) The necessary and sufficient condition of controllability of the linear approximation (17) is that rank of the controllability matrix

$$\begin{aligned}\mathcal{W} :&= \left[\mathcal{B} \ \mathcal{A}B \ \cdots \ \mathcal{A}^{n-1}\mathcal{B}\right] \\ &= \left[\hat{Y}(q^e) \ -\hat{Z}(q^e)\frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) \ \cdots \ (-1)^{n-1}\left\{\hat{Z}(q^e)\frac{\partial A}{\partial q}(q^e)\right\}^{n-1}\hat{Y}(q^e)\right]\end{aligned} \quad (19)$$

is equal to n. Multiplying the $n \times n$ square non-singular matrix $\begin{bmatrix} B(q^e) \\ E(q^e) \end{bmatrix}$ by (19) from the left-hand side, then we have

$$\begin{bmatrix} B(q^e) \\ E(q^e) \end{bmatrix}\mathcal{W} = \begin{bmatrix} O_{n-m,m} & \frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) & \cdots & (-1)^{n-1}\frac{\partial A}{\partial q}(q^e)\left\{\hat{Z}(q^e)\frac{\partial A}{\partial q}(q^e)\right\}^{n-2}\hat{Y}(q^e) \\ I_m & O_m & \cdots & O_m \end{bmatrix}.$$

Note that rank of $\mathcal{W}$ does not change by multiplying above. Consequently, $\mathrm{rank}\,\mathcal{W} = n$ holds if and only if $\mathrm{rank}\,\mathcal{V} = n - m$ holds. This completes the proof. □

From Theorem 3, the following can be derived in case of completely linear and partially affine constraints.

Corollary 1. *In case of completely linear and partially affine constraints, the linear approximation of the KAACS at any equilibrium $q^e \in U^e$ (17) is uncontrollable.*
(Proof) Multiplying a row elementary operation matrix F by (18) from the left-hand side, then we have

$$F\mathcal{V} = \begin{bmatrix} * & \cdots & * \\ O_{n-m-r,m} & \cdots & O_{n-m-r,m} \end{bmatrix}, \quad (20)$$

where we use the following property:

$$F\frac{\partial A}{\partial q}(q^e) = \begin{bmatrix} I_r & * \\ O_{n-m-r,r} & O_{n-m-r,n-r} \end{bmatrix}.$$

In this case, rank of $\mathcal{V}$ is smaller than $n - m$ because of $0 \le r \le n - m - 1$, then the linear approximation (17) is uncontrollable from Theorem 3. □

In case of completely linear constraints, the linear approximation is obviously uncontrollable due to $\mathcal{A} \equiv 0$. In case of partially affine constraints, the linear approximation is also uncontrollable, and then we have to adopt Sussmann's theorem to check local controllability. From Theorem 3, the following can be derived in case of completely affine constraints.

Corollary 2. *In case of completely affine constraints with $n \le 2m$, if the affine index at an equilibrium $q^e \in U^e$ is $n - m$ and*

$$\mathrm{rank}\,\frac{\partial A}{\partial q}(q^e)\hat{Y}(q^e) = n - m \quad (21)$$

holds at q^e, then the linear approximation of the KAACS at q^e (17) is controllable. Therefore, the KAACS (16) is locally controllable at q^e.
(Proof) If the affine index at q^e is $n - m$ and (21) holds, then $\mathrm{rank}\,\mathcal{V} = n - m$ holds and the linear approximation (17) is controllable from Theorem 3. Therefore, the KAACS (16) is also locally controllable at q^e. □

If $n > 2m$ holds or (21) does not hold in case of completely affine constraints, we cannot check local controllability of the KAACS by the linear approximation approach, and then we have to rely on Sussmann's theorem (Sussmann, 1987).

4.3 Stabilizability

In the previous subsection, we have studied local controllability of the KAACS. In general, there exists a gap between controllability and stabilizability in nonlinear control systems. In this subsection, we investigate stabilizability to equilibria for the KAACS. We first consider stabilizability of the KAACS by linear state feedback. It is known that if the linear approximation of a nonlinear system at an equilibrium is controllable or all its uncontrollable modes are stable, then the nonlinear system is locally asymptotically stabilizable to the equilibrium by a linear state feedback. In case of completely linear constraints, uncontrollable modes of the linear approximation system are not stable due to $\mathcal{A} \equiv 0$. In case of partially affine constraints, if uncontrollable modes of the linear approximation are stable, then the KAACS is locally asymptotically stabilizable by a linear state feedback. In case of completely affine constraints, we can derive the following from Corollary 2.

Corollary 3. *In case of completely affine constraints with $n \le 2m$, if the affine index at an equilibrium $q^e \in U^e$ is $n-m$ and (21) holds at q^e, then the KAACS (16) is locally asymptotically stabilizable to any equilibrium $q^e \in U^e$ by a linear state feedback.* □

We next consider stabilizability of the KAACS by nonlinear smooth state feedback, which is a larger class than linear state feedback. The necessary condition of locally asymptotic stabilizability by nonlinear smooth state feedback can be derived as follows.

Theorem 4. *If the KAACS (16) is locally asymptotically stabilizable to an equilibrium $q^e \in U^e$ by a nonlinear smooth state feedback, then the affine index at q^e is $n-m$, that is, $r(q^e) = n-m$.*
(Proof) Consider $A(q)$ as a map $A : U \to \boldsymbol{R}^{n-m}$, where U is an open set of Q. By the implicit function theorem, if the affine index at q^e is $n-m$, then there exists a diffeomorphism $\sigma : V \to W$ such that

$$\begin{aligned} A \circ \sigma^{-1}&(q_1, \cdots, q_m, q_{m+1}, \cdots, q_n) \\ &= (q_{m+1}, \cdots, q_n) + A(q^e), \quad q \in W \end{aligned}$$

and $\sigma(q^e) = 0$, where $V\ (\subset U)$ is an open neighborhood of q^e in Q and W is an open neighborhood of 0 in $\boldsymbol{R}^n$. Now $A(q^e) = 0$, we have

$$\begin{aligned} \sigma \circ A^{-1}&(q_{m+1}, \cdots, q_n) \\ &= (q_1, \cdots, q_m, q_{m+1}, \cdots, q_n). \end{aligned}$$

Therefore, the subset of Q defined by

$$\begin{aligned} M :&= \sigma \circ A^{-1}(A(q^e)) = \sigma \circ A^{-1}(0) \\ &= (q_1, \cdots, q_m, 0, \cdots, 0) \end{aligned}$$

can be parameterized by m variables, and then M is m-dimensional submanifold of Q. On the other hand, Ishikawa and Sampei (1998) have shown that if a nonlinear control system is locally asymptotically stabilizable, then the dimension of equilibria set is equal to the number of control inputs. Consequently, both the dimension of M and the number of inputs are m, this proves the theorem. □

From Theorem 4, the following can be derived in case of completely linear and partially affine constraints.

Corollary 4. *In case of completely linear and partially affine constraints, the KAACS (16) is not locally asymptotically stabilizable to any equilibrium $q^e \in U^e$ by any nonlinear smooth state feedback.*
(Proof) In this case, the affine index at any equilibrium q^e is smaller than $n-m$. Hence from Theorem 4, the proof is completed. □

From Corollary 4, the KAACS is not locally asymptotically stabilizable by any smooth nonlinear state feedback in not only completely linear constraints case but also partially affine constraints case. On the other hand in case of completely affine constraints, the KAACS has a possibility of locally asymptotic stabilizability by a nonlinear smooth state feedback even though Corollary 3 does not hold.

5. PHYSICAL EXAMPLES

In this section, we apply our results to two physical examples. We first consider *a coin on a rotating table* as shown in Fig. 1. Set the xy-coordinate whose origin corresponds to the center of rotation of the table. Let R be the radius of the coin and $\varOmega$ be the angular rate of the table. (x, y) denotes the point that the coin contacts with the table and θ and ϕ denote the heading angle and self-rotation angle of the coin, respectively. Then the generalized configuration coordinate of the system is denoted by $q = [\,x\ \ y\ \ \theta\ \ \phi\,]^T \in SE(2) \times \boldsymbol{S}$ with $n = 4$. Considering equilibrium of velocities in the heading and side directions of the coin, we obtain affine constraints of this system:

$$\underbrace{\begin{bmatrix} 0 \\ \varOmega(y\cos\theta - x\sin\theta) \end{bmatrix}}_{A(q)} + \underbrace{\begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & R \end{bmatrix}}_{B(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = 0, \tag{22}$$

where $m = 2$. Therefore, the equilibria set is given by $U^e = \{q \in Q \mid y\cos\theta - x\sin\theta = 0\}$. We can find that affine constraints (22) are completely nonholonomic by calculating C_0. Since the affine index at any equilibrium $q^e \in U^e$ is $r(q^e) = 1 < n-m = 2$, then affine constraints (22) are first order partially affine constraints. Assuming that $\dot{\theta}$ and $\dot{\phi}$ can be controlled, that is, we set control inputs $u = [\,u_1\ \ u_2\,]^T$ as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{E(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}. \tag{23}$$

Therefore, from affine constraints (22) and control inputs (23), the KAACS is given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} &= \underbrace{\begin{bmatrix} \varOmega\cos\theta(x\sin\theta - y\cos\theta) \\ \varOmega\sin\theta(x\sin\theta - y\cos\theta) \\ 0 \\ 0 \end{bmatrix}}_{\hat{X}} \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\hat{Y}_1} u_1 + \underbrace{\begin{bmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{bmatrix}}_{\hat{Y}_2} u_2. \end{aligned} \tag{24}$$

Since affine constraints (22) are completely nonholonomic, then the KAACS (24) is strongly locally accessible at any point $\forall q \in Q$ from Theorem 2. Due to partially affine constraints, the linear approximation of (24) at any equilibrium $q^e \in U^e$ is uncontrollable from Corollary 1. However we can find that the KAACS (24) is locally controllable at any equilibrium q^e by Sussmann's Theorem. Finally, it is shown from Corollary 4 that the KAACS (24) is not locally asymptotically stabilizable to any equilibrium q^e by any nonlinear smooth state feedback.

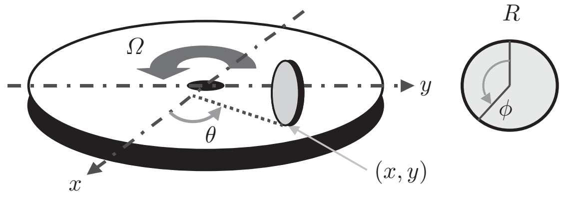

Fig. 1. A coin on a rotating table.

The second example is *a ball on a rotating table* as depicted in Fig. 2. Set the xy-coordinate whose origin corresponds to the center of rotation of the table. Let R be the radius of the ball and Ω be the angular rate of the table. (x, y) denotes the point that the ball contacts with the table and (θ, ϕ, ψ) denotes the Eulerian angles of the ball. Then the generalized configuration coordinate of the system is denoted by $q = [\, x \;\; y \;\; \theta \;\; \phi \;\; \psi \,]^T \in \boldsymbol{R}^2 \times SO(3)$ with $n = 5$. Considering equilibration of velocities in the x and y directions of the ball, we obtain affine constraints of the system as follows.

$$\underbrace{\begin{bmatrix} \Omega\, y \\ -\Omega\, x \end{bmatrix}}_{A(q)} + \underbrace{\begin{bmatrix} 1 & 0 & -R\sin\psi & R\sin\theta\cos\psi & 0 \\ 0 & 1 & R\cos\psi & R\sin\theta\sin\psi & 0 \end{bmatrix}}_{B(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = 0, \tag{25}$$

where $m = 2$. Therefore, the equilibria set is given by $U^e = \{q \in Q \mid x = y = 0\}$. We can find that affine constraints (25) are completely nonholonomic by calculating C_0. Since the affine index at any equilibrium $q^e \in U^e$ is $r(q^e) = 2 = n - m$, then affine constraints (25) are completely affine constraints. Assuming that $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$ can be controlled, that is, we set control inputs $u = [\, u_1 \;\; u_2 \;\; u_3 \,]^T$ as

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{E(q)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix}. \tag{26}$$

Therefore, from affine constraints (25) and control inputs (26), the KAACS is given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} &= \underbrace{\begin{bmatrix} -\Omega y \\ \Omega x \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\hat{X}} + \underbrace{\begin{bmatrix} R\sin\psi \\ -R\cos\psi \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\hat{Y}_1} u_1 \\ &\quad + \underbrace{\begin{bmatrix} -R\sin\theta\cos\psi \\ -R\sin\theta\sin\psi \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\hat{Y}_2} u_2 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\hat{Y}_3} u_3. \end{aligned} \tag{27}$$

Since affine constraints (25) are completely nonholonomic, then the KAACS (27) is strongly locally accessible at any point $\forall q \in Q$ from Theorem 2. Since $n < 2m$ and (21) hold, we can find that the KAACS (27) is locally controllable at any equilibrium $q^e \in U^e$ from Corollary 2. Finally, it is shown from Corollary 3 that the KAACS (27) is locally asymptotically stabilizable to any equilibrium q^e by a linear state feedback.

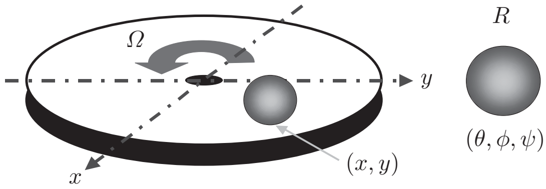

Fig. 2. A ball on a rotating table

6. CONCLUSION

We have analyzed the kinematically asymmetrically affine control system with nonholonomic affine constraints based on nonlinear control theory. As a result, we have shown that there exists a class of systems whose linear approximation systems are controllable and which are locally asymptotically stabilizable by nonlinear smooth state feedback. These properties go against the facts that are known for nonholonomic systems so far. Moreover we have found that there exist systems which are locally asymptotically stabilizable by linear state feedback.

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