GENERAL LQ PROBLEM FOR INFINITE JUMP LINEAR SYSTEMS AND THE MINIMAL SOLUTION OF ALGEBRAIC RICCATI EQUATIONS [1]

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Abstract: The paper addresses the LQ control problem for systems with countable Markov jump parameters, and the associated coupled algebraic Riccati equations. The problem is considered in a general optimization setting in which the solution is not required to be stabilizing in any sense. We show that a necessary and sufficient condition for a solution to the control problem to exist is that the Riccati equations have a nonempty set of solutions, which generalizes previous known results requiring stabilizability as a sufficient condition. We clarify the connection between the minimal solution of the Riccati equation and the control problem, showing that the minimal solution provides the synthesis of the optimal control. The derived results strengthen the relations of the theory of Markov jump systems with the one of linear deterministic systems. An illustrative example is included.

Keywords: Linear systems, Markov parameters, linear quadratic regulators, algebraic Riccati equations.

1. INTRODUCTION

In this paper we consider the linear quadratic problem for infinite Markov jump linear systems (IMJLS) and the associated infinite countable set of coupled algebraic Riccati equations (ICARE), which constitute solid grounds in the theory of dynamical systems.

The IMJLS form a special class of systems which can be described by a switching of countable many linear forms, in which the switching is driven by an underlying Markov chain with infinite state space. Apart from providing meaningful models for systems subject to abrupt changes in their structure, see e.g. (Athans *et al.*, 1977) and (Sworder and Rogers, 1983), IMJLS present numerous results that parallel the linear deterministic theory (Costa and do Val, 2002*b*). In the simpler context of finite Markov state space, we mention that the linear quadratic problem was studied in (Ji and Chizeck, 1990), or in (Costa *et al.*, 1997) and (Costa and Fragoso, 1995) in a convex programming perspective; methods of solutions for the associated Riccati equations were considered in (do Val and Costa, 2002), (do Val *et al.*, 1999) and (Rami and Ghaoui, 1996) and some basic concepts such as detectability were taken into account in (Costa and do Val, 2002*a*). In the infinite context considered here, (Costa and Fragoso, 1995) and (Fragoso and Baczynski, 2001) provide seminal results, including

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the derivation of the optimal control for the finite horizon linear quadratic problem and the stationary optimal control (among the stabilizing controls) for the infinite horizon problem; the concept of detectability and the role that it plays as a condition for stabilizability of solutions was devised in (Costa *et al.*, n.d.*b*) and (Costa *et al.*, n.d.*a*) in a general framework.

A common requirement in the field is that the control should belong to a class of stabilizing controls, and almost all the research effort in the area was devoted to this setup, including (Costa and Fragoso, 1995) and (Fragoso and Baczynski, 2001); this observation also holds true for previous works dedicated to IMJLS. On the other hand, in the purely deterministic setting, the LQ problem without stability constraints has long been dealt with, e.g. (Kucera, 1972), (Martensson, 1971) or (Molinari, 1977, Section 6).

The present paper deals with the general jump linear quadratic problem (GJLQ) that does not require stabilizability of controllers in any sense. Apart from the theoretical and practical interests of the results, we remark that they strengthen the relations of the IMJLS theory with that one of linear deterministic systems. The main results of the paper are summarized as follows. We show that the optimal cost reads as a quadratic form and that the optimal control is a stationary state feedback control, see Theorem 1. We also show that the set of solution of the ICARE captures the solution for the GJLQ, see Corollary 1, and the fact that the ICARE have a nonempty set of solutions if and only if C1 holds, see Theorem 2. The connection between the minimal solution to the ICARE and the GJLQ problem is then addressed and we show that the limit M is identified with the minimal solution to the ICARE, allowing us show in Theorem 3 that the solution to the GJLQ problem is synthesized by M.

2. NOTATION, BASIC CONCEPTS AND PROBLEM FORMULATION

Let $\mathbb{R}^n$ represent the usual linear space of all n-dimensional vectors. Assume that $\mathcal{R}^{r,n}$ (respectively, $\mathcal{R}^n$) represents the normed linear space formed by all $r \times n$ real matrices (respectively, $n \times n$). For $U \in \mathcal{R}^{n,r}$, U' denotes the transpose of U. For $U, V \in \mathcal{R}^n$, $U > V$ ($U \geq V$) indicates that $U - V$ is positive definite (semidefinite). $\mathcal{R}^{n0}$ ($\mathcal{R}^{n+}$) represents the closed convex cone $\{U \in \mathcal{R}^n : U = U' \geq 0\}$ (the open cone $\{U \in \mathcal{R}^n : U = U' > 0\}$).

Consider the set $\mathcal{S} = \{1, 2, \ldots\}$. Let $\mathcal{H}_\infty^{r,n}$ denote the linear space formed by sequences of matrices $H = \{H_i \in \mathcal{R}^{r,n}; i \in \mathcal{S}\}$ such that $\sup_{i \in \mathcal{S}} ||H_i|| < \infty$; also, let $\mathcal{H}_1^{r,n} = \{H \in \mathcal{H}_\infty^{r,n} : \sum_{i \in \mathcal{S}} ||H_i|| < \infty\}$. We denote $\mathcal{H}_\infty^n \equiv \mathcal{H}_\infty^{n,n}$, and $\mathcal{H}_\infty^{n0}$ (${\mathcal{H}_\infty}^{n+}$) represents the closed cone $\{H \in \mathcal{H}_\infty^n : H_i \in \mathcal{R}^{n0}, i \in \mathcal{S}\}$ (the open cone $\{H \in \mathcal{H}_\infty^n : H_i \in \mathcal{R}^{n+}, i \in \mathcal{S}\}$), and similarly for $\mathcal{H}_1$, $\mathcal{H}_1^{n0}$ and $\mathcal{H}_1^{n+}$.

For $H, W \in \mathcal{H}_\infty^n$, $H \geq W$ indicates that $H_i \geq W_i$ for each $i \in \mathcal{S}$. The notation is similar for basic mathematical relations involving elements of $\mathcal{H}_\infty^{r,n}$; e.g., $H = W$ means that $H_i = W_i$ for each $i \in \mathcal{S}$. In the sequel, capital letters denote elements of $\mathcal{H}_\infty^{r,n}$ and capital letters with an index denote elements of $\mathcal{R}^{r,n}$.

The IMJLS that we deal with in this paper are defined by the following stochastic differential equation, in a probabilistic space $(\Omega, \mathfrak{F}, \mathbb{P})$,

$$\Phi: \dot{x}(t) = A_{\theta(t)} x(t) + B_{\theta(t)} u(t), \quad t \geq 0,\ x(0) = x_0,\ \theta(0) = \theta_0 \tag{1}$$

where $x(\cdot) \in \mathbb{R}^n$, $u(\cdot) \in \mathbb{R}^r$, A_i, $i \in \mathcal{S}$, belong to a given sequence of matrices $A = \{A_i, i \in \mathcal{S}\} \in \mathcal{H}_\infty^n$ and similarly for $B \in \mathcal{H}_\infty^{n,r}$. The jump variable $\theta(t)$ is the state of an underlying continuous-time homogeneous Markov chain $\Theta = \{\theta(t); t \geq 0\}$ having the countable state space $\mathcal{S}$ and a stationary transition rate matrix $\Lambda = [\lambda_{ij}]$, $i, j \in \mathcal{S}$.

The state of system Φ is the compound variable (x, θ). The control u is assumed to belong to the class of admissible controls $\mathcal{U}^T$, $T \in [0, \infty)$, which is defined as in (Fragoso and Baczynski, 2001): $\mathcal{U}^T$ is the class of all Borel measurable functions $u : \{\mathbb{R}^n, \mathcal{S}, [0, T]\} \to \mathbb{R}^r$ such that, for each $x, z \in \mathbb{R}^n$, $i \in \mathcal{S}$ and $t \in [0, T]$,

(i) $\|u(x,i,t) - u(z,i,t)\| \leq \ell \|x - z\|$ for some $\ell \in \mathbb{R}$ (Lipschitz);
(ii) there exists $c \in \mathbb{R}$ such that $\|u(x,i,t)\| \leq c(1 + \|x\|)$ (growth condition).

The next basic concepts of stochastic stability and stabilizability are useful for the discussion that follows.

Definition 1. (S-stability). We say that (A, Λ) is stochastically (S) stable when $\int_0^\infty E\{||x(t)||^2 dt\} < \infty$, for each $x_0 \in \mathbb{R}^n$ and $\theta_0 \in \mathcal{S}$, wtih $x(t)$ given by (1) with $u \equiv 0$.

Definition 2. (S-stabilizability). We say that (A, B, Λ) is S-stabilizable when there exist $G \in \mathcal{H}_\infty^{r,n}$ such that $(A + BG, \Lambda)$ is S-stable.

We shall deal with the cost functionals

$$W_{u,S}^T(x,i) := E\Big\{ \int_0^T x(\tau)' Q_{\theta(\tau)} x(\tau) + u(\tau)' R_{\theta(\tau)} u(\tau)\, d\tau + x(T)' S_{\theta(T)} x(T) \Big\} \tag{2}$$

where $T \in [0, \infty)$ is the horizon length, matrices Q_i, S_i and R_i belong to the sequences of matrices $Q, S \in \mathcal{H}_\infty^{n,0}$ and $R \in \mathcal{H}_\infty^{n,+}$ respectively, and the expected value is with respect to the initial condition $x_0 = x$ and $\theta_0 = i$. For ease of notation, we define

$$W_u(x,i) := \lim_{T \to \infty} W_{u,S \equiv 0}^T(x,i), \tag{3}$$

$$W_S^T(x,i) := \inf_{u \in \mathcal{U}^T} W_{u,S}^T(x,i). \tag{4}$$

Problem Formulation. The general infinite jump linear quadratic (GJLQ) problem is the infinite-horizon control problem that consists of minimizing over $\mathcal{U}^\infty$

the cost functional $W_u(x,i)$, i.e., for each initial condition $x_0 \in \mathbb{R}^n$ and $\theta_0 \in \mathcal{S}$, to find

$$W(x_0,\theta_0) := \inf_{u \in \mathcal{U}^\infty} W_u(x_0,\theta_0)$$

and the associate optimal control $u_{\text{opt}} \in \mathcal{U}^\infty$. We say that a solution exists when the optimal cost is bounded.

Remark 1. In this paper, the GJLQ problem is a strict optimization problem in the sense that we do not require that the solution stabilizes the system in any sense. Accordingly, the class of controls $\mathcal{U}$ is not necessarily stabilizing.

The following condition on the boundedness of the optimal cost is connected to the idea of existence of solutions, a key issue in the paper.

C1. **(Existence of Solution to the GJLQ Problem)**. There exists $\gamma \in \mathcal{R}$ such that, for each $x \in \mathbb{R}^n$ and $i \in \mathcal{S}$, $W(x,i) \leq \gamma \|x\|^2$.

Remark 2. It can be shown following the arguments of (Fragoso and Baczynski, 2001, Proof of Proposition 6.9) that S-stabilizability implies C1. This is surprisingly in the sense that C1 requires an uniform bound for W whereas stabilizability requires only finiteness of W_u with $u(t) = G_{\theta(t)}x(t)$ and G given in the definition. The converse assertion does not hold true, even in the purely deterministic case (e.g. when $A = I$, $B = 0$ and $Q = 0$, leading to $W = 0$).

In connection to the GJLQ problem, in this paper we address the following ICARE. Consider $H = (H_1, H_2, \ldots) \in \mathcal{H}_1^{n0}$ and we introduce the nonlinear operator $\mathcal{L} : \mathcal{H}_1^{n0} \to \mathcal{H}_1^n$, $\mathcal{L}(H) = (\mathcal{L}_1(H), \mathcal{L}_2(H), \ldots)$,

$$\mathcal{L}_i(H) = A_i' H_i + H_i A_i - H_i B_i R_i^{-1} B_i' H_i + Q_i + \sum_{j \in \mathcal{S}} \lambda_{ij} H_j. \tag{5}$$

The ICARE are a set of countable interconnected Riccati equation in the variable $P = (P_1, P_2, \ldots) \in \mathcal{H}_1^{n0}$, which reads as follows

$$\mathcal{L}(P) = 0. \tag{6}$$

Definition 3. (Minimal solution to the ICARE). We say that a solution $M \in \mathcal{H}_1^{n0}$ of the ICARE in (6) is minimal if $M \leq P$ for any solution $P \in \mathcal{H}_1^{n0}$ of the ICARE.

We finish the section by introducing, for a fixed $T \in [0,\infty)$ and for $L = (L_1, L_2, \ldots) \in \mathcal{H}_1^{n0}$, the Riccati differential equation

$$\dot{P}^T(t) + \mathcal{L}(P^T(t)) = 0, \quad P^T(T) = L,\ 0 \leq t \leq T. \tag{7}$$

The following result is presented for ease of reference, see (Fragoso and Baczynski, 2001, Proposition 4.9).

Proposition 1. There exists a unique solution $P^T(t) \in \mathcal{H}_1^{n0}$, $t \in [0,T]$, $T \in [0,\infty)$, for (7).

3. PRELIMINARY RESULTS

Next we present some useful inequalities concerning the finite and infinite horizon optimal costs. The proof is ommited.

Lemma 1. The following assertions hold:

(i) $W_V^T(\cdot) \geq W_S^T(\cdot)$ whenever $V \geq S$;

(ii) $W_{u,V}^T(\cdot) \geq W_{u,S}^T(\cdot)$ whenever $V \geq S$;

(iii) $W(\cdot) \geq W_{S\equiv 0}^T(\cdot)$.

The next result concerning the finite-horizon jump linear quadratic control problem follows from (Fragoso and Baczynski, 2001, Proposition 5.6 and 5.8).

Proposition 2. Consider $P^T(t)$ the solution for the Riccati differential equation (7) with terminal condition $L = S$. Then, the optimal control for the finite-horizon control problem is given by

$$u_{\text{opt}}(t) = R_{\theta(t)}^{-1} B_{\theta(t)}' P_{\theta(t)}^T(t) x(t), \quad t \in [0,T],$$

and the optimal cost reads as follows

$$W_L^T(x,i) = min_{u \in \mathcal{U}^T} W_{u,L}^T(x,i) = x' P_i^T(0) x.$$

Next we present a sufficient condition for convergence of $P^T(0)$ as $T \to \infty$ to a solution of the ICARE. The result is adapted from the proof of Proposition 6.9 of (Fragoso and Baczynski, 2001), which is of particular interest here; the proof is omitted.

Proposition 3. Consider the Riccati differential equation (7). Assume that $L \equiv 0$ and there exists $\gamma \geq 0$ such that $P_i^T(0) \leq \gamma I$, $\forall i, T$. Then,

$$P_i^T(0) \to M \quad \text{as } T \to \infty,$$

where $M \in \mathcal{H}_1^{n0}$ is a solution for the ICARE.

4. MAIN RESULTS

The first goal in this section is to show how the solution to the GJLQ problem is connected to the solution of the Riccati differential equation (7) and the fact that the optimal solution is in the stationary state feedback form. The next result will be needed.

Lemma 2. Consider the Riccati differential equation (7). Assume that $L \equiv 0$ and that C1 holds. Then,

$$P^T(0) \to M \quad \text{as } T \to \infty, \tag{8}$$

where $M \in \mathcal{H}_1^{n0}$ is a solution for the ICARE.

Proof. Employing Proposition 2 and item (iii) of Lemma 1, respectively, we have for each $x \in \mathbb{R}^n$ and $i \in \mathcal{S}$ that

$$x' P_i^T(0) x = W_{L\equiv 0}^T(x,i) \leq W(x,i)$$

Now, from assumption C1 we have that $W(x,i) \leq \gamma\|x\|^2 = x'(\gamma I)x$, thus leading to

$$x'P_i^T(0)x \leq x'(\gamma I)x$$

for each $x \in \mathbb{R}^n$ and $i \in \mathcal{S}$, thus attending the condition in Proposition 3, which provides the result. ∎

The optimal control is derived in the next theorem. The control is synthesized via M the limit of $P^T(0)$ as $T \to \infty$, as in Lemma 2.

Theorem 1. Assume that condition C1 holds and let M be the limit in (8). Then, the optimal control to the GJLQ problem is the stationary feedback policy

$$u_{\text{opt}}(t) = G_{\theta(t)}x(t), \quad t \geq 0$$

where $G = -R^{-1}B'M$, and the optimal cost reads as

$$W(x,i) = x'M_i x.$$

Proof. From Proposition 2 we have that $W^T_{L\equiv 0}(x,i) = x'P_i^T(0)x$ and taking limits and employing Lemma 2 we obtain:

$$W(x,i) = \lim_{T\to\infty} W^T_{L\equiv 0}(x,i) = \lim_{T\to\infty} x'P_i^T(0)x = x'M_i x. \tag{9}$$

Notice that, in principle, the optimal control obtained above, via Proposition 2, is not stationary. Next we define the stationary control

$$\bar{u}(t) = -R^{-1}_{\theta(t)}B'_{\theta(t)}M_{\theta(t)}x(t)$$

and we show that it is optimal. Let $L = S = M$, recalling that S is the terminal data in (2) and L is the terminal condition in (7). Let $t \to X^T(t)$ represent the solution to the Riccati differential equations (7) with $X^T(T) = L = M$. Since Lemma 2 provides that $\mathcal{L}(M) = 0$, one can easily check from (7) that

$$X^T(t) = M, \quad \forall T \geq 0,\ t \in [0,T],$$

In this context, Proposition 2 ensures that $\bar{u}$ is the optimal control for the finite horizon control problem with $S = M$ and

$$W^T_{\bar{u},S\equiv M}(x,i) = x'X_i^T(0)x = x'M_i x, \quad \forall T \geq 0. \tag{10}$$

Now, from Lemma 1 (ii) we have that $W^T_{\bar{u},S\equiv 0}(x,i) \leq W^T_{\bar{u},S\equiv M}(x,i)$, $\forall T \geq 0$. This inequality, (9) and (10) lead to

$$\begin{aligned} W_{\bar{u}}(x,i) &= \lim_{T\to\infty} W^T_{\bar{u},S\equiv 0}(x,i) \\ &\leq \lim_{T\to\infty} W^T_{\bar{u},S\equiv M}(x,i) = x'M_i x = W(x,i). \end{aligned} \tag{11}$$

The opposite relation $W_{\bar{u}}(x,i) \geq W(x,i)$, comes from definition of the problem **P**, completing the proof. ∎

Now we turn our attention to the relationship between the GJLQ problem and the ICARE. We start showing that the ICARE serve as a necessary condition for optimality in the GJLQ problem, as follows.

Corollary 1. Assume that $M \in \mathcal{H}_1^{n0}$ is such that $W(x,i) = x'M_i x$, for each $x \in \mathbb{R}^n$ and $i \in \mathcal{S}$. Then, M is a solution to the ICARE, i.e., $\mathcal{L}(M) = 0$.

Proof. We omit the details. The fact that $W(x,i) = x'M_i x$ and $M \in \mathcal{H}_1^{n0}$ implies that C1 holds with $\gamma := \sup_{i\in\mathcal{S}} \|M_i\|$. Theorem 1 provides that the optimal cost reads as $W(x,i) = x'X_i x$ where $X = \lim_{T\to\infty} P^T(0) \in \mathcal{H}_1^{n0}$, thus leading to $M = X$. Lemma 2 provides that $\mathcal{L}(M) = 0$. ∎

The fact that a solution to the GJLQ problem exists (i.e., C1 holds) if and only if a solution to the ICARE exists is presented in what follows. We shall need the next result.

Lemma 3. Let P be a solution for the ICARE, i.e., $P \in \mathcal{H}_1^{n0}$ is such that $\mathcal{L}(P) = 0$. Then,

$$W(x,i) \leq x'P_i x.$$

Proof. The arguments are similar to the ones in the proof of Theorem 1 and we omit the details. Let $L = S = P$ and let $X^T(t)$ represent the solution to the Riccati differential equations (7) with $L = P$. Employing Lemma 2 one can check that

$$X^T(t) = P, \quad \forall T \geq 0,\ t \in [0,T],$$

and employing Lemma 1 (ii) and Proposition 2 respectively we get that

$$W^T_{S\equiv 0}(x,i) \leq W^T_{S\equiv P}(x,i) = x'X_i^T(0)x = x'P_i x, \quad \forall T \geq 0.$$

Taking limits, we finally obtain $W(x,i) \leq x'P_i x$. ∎

Theorem 2. A solution to the GJLQ problem exists if and only if there exists a solution to the ICARE, i.e., there exists $P \in \mathcal{H}_1^{n0}$ such that $\mathcal{L}(P) = 0$.

Proof. Sufficiency. Let

$$\gamma := \sup_{i\in\mathcal{S}} \|P_i\|.$$

Notice that γ is well defined since $P \in \mathcal{H}_1^{n0} \subset \mathcal{H}_\infty^{n0}$. Then, from Lemma 3 we evaluate

$$W(x,i) \leq x'P_i x \leq \gamma\|x\|^2$$

and C1 holds.

Necessity. Assuming that C1 holds, Lemma 2 ensures that $M = \lim_{T\to\infty} P^T(0) \in \mathcal{H}_1^{n0}$ is a solution to the ICARE. ∎

Remark 3. In the general context of IMJLS, both S-stabilizability and solvability of the ICARE are complex to check. The conditions are testable in the finite state space case with $\mathcal{S} = \{1,\ldots,N\}$, see (Costa and Marques, 2000) and (Costa and do Val, n.d.).

In the sequel we clarify the connection between the minimal solution of the ICARE and the solution to the GJLQ problem. We need the next interesting result.

Lemma 4. Assume that C1 holds and let $M \in \mathcal{H}_1^{n0}$. M is the limit in (8) if and only if M is the minimal solution to the ICARE.

Proof. Sufficiency. Since M is a solution to the ICARE, Theorem 2 provides that C1 holds and Lemma 2 ensures that the limit in (8) exists. Now we assume that the limit is $\bar{M}$ and we show that $\bar{M} = M$. Theorem 1 and Lemma 3 allow us to write

$$x'\bar{M}_i x = W(x,i) \le x' M_i x, \quad \forall x \in \mathbb{R}^n, i \in \mathcal{S},$$

which leads to

$$\bar{M} \le M.$$

On the other hand, Lemma 2 provides that $\bar{M}$ is a solution for the ICARE, and from Definition 3,

$$\bar{M} \ge M.$$

Necessity. Since M is the limit in (8), from Lemma 2 we get that M is a solution for the ICARE. Let us deny the assertion in the theorem and assume that M is not the minimal solution, i.e., that there exists a solution P to the ICARE and some $x \in \mathbb{R}^n$ and $i \in \mathcal{S}$ for which $x'M_i x > x'P_i x$. This and Lemma 3 lead to

$$x'M_i x > x'P_i x \ge W(x,i). \qquad (12)$$

However, Theorem 1 provides that

$$W(x,i) = x'M_i x$$

thus contradicting (12). ∎

The fact that the minimal solution to the ICARE provides the strictly optimal control for the GJLQ problem is now easy to show.

Theorem 3. Let M be the minimal solution to the ICARE. Then, the optimal control for the GJLQ problem is the stationary feedback control

$$u_{\rm opt}(t) = G_{\theta(t)} x(t)$$

where $G = -R^{-1}B'M$, and the optimal cost reads as

$$W(x,i) = x'M_i x.$$

Proof. The proof follows immediately from Theorems 1 and 2, and Lemma 4. ∎

Remark 4. It is a well known fact that the solution to the LQ problem restricted to stabilizing controllers is given by a stationary feedback law, in parallel with Theorem 3. However, it is associated with the stabilizing solution of the ICARE, see (Fragoso and Baczynski, 2001, Proposition 6.11), which in general is not equal to M (even in the linear deterministic case, see (Molinari, 1977, Theorem 8)).

5. EXAMPLE

In this section we present an illustrative example employing a IMJLS with finite Markov state space. This class of systems is simpler to deal with and still exhibits the desired properties in the examples.

Example 1. (The minimal solution of the ICARE provides the synthesis of the optimal control) Consider the system Φ with

$$A_1 = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix};\ A_2 = \begin{bmatrix} 1 & 1/4 \\ 0 & 2 \end{bmatrix};\ \Lambda = \begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix};$$
$$B_1 = \begin{bmatrix} 1/2 \\ 2 \end{bmatrix};\ B_2 = \begin{bmatrix} 3/2 \\ 2 \end{bmatrix};$$
$$Q_1 = Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1/4 \end{bmatrix};\ R_1 = R_2 = 1. \qquad (13)$$

Next, in the quest for the optimal control, we consider the Riccati differential equations in (7); one can check that their (unique) solution is of the form

$$P_i^T(t) = \begin{bmatrix} 0 & 0 \\ 0 & p_i^T(t) \end{bmatrix}, \quad i = 1,2,$$

and (7) leads to the set of differential equations

$$\dot{p}_1^T(t) = -p_1^T(t) - p_2^T(t) + 4(p_1^T(t))^2 - \frac{1}{4},$$
$$\dot{p}_2^T(t) = -5p_1^T(t) + p_2^T(t) + 4(p_2^T(t))^2 - \frac{1}{4},$$
$$p_1^T(T) = 0, p_2^T(T) = 0.$$

Now one can solve the above equations for $T \to \infty$ or equivalently, check that they have a unique positive stationary solution $p_1 = 0.6580 \ge 0$, $p_2 = 0.8240 \ge 0$, thus obtaining $\lim_{T\to\infty} P^T(0) = M$ where

$$M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0.6580 \end{bmatrix};\ M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0.8240 \end{bmatrix}. \qquad (14)$$

Theorem 1 provides that

$$u_{\rm opt}(t) = \begin{cases} [0 \ \ -1.3161]x(t), & \theta(t) = 1, \\ [0 \ \ -1.6481]x(t), & \theta(t) = 2; \end{cases}$$
$$W(x,i) = x'M_i x.$$

On the other hand, it is presented in (Costa and do Val, n.d.) a method for solving the ICARE, which reaches the minimal solution if and only if it exists. Employing the method, one can check that M is indeed the minimal solution to the ICARE, thus verifying the result of Lemma 4 and Theorem 3.

It is interesting to mention that the ICARE have multiple solutions in this example; e.g., employing the method presented in (Costa *et al.*, 1999) we find that

$$X_1 = \begin{bmatrix} 53.35 & -8.977 \\ -8.977 & 2.685 \end{bmatrix};\ X_2 = \begin{bmatrix} 22.73 & -10.00 \\ -10.00 & 6.820 \end{bmatrix}$$

is a (stabilizing) solution to the ICARE. Notice that $X > M$, in accordance with Theorem 3.

6. CONCLUSION

In this paper we examine the GJLQ problem, a linear quadratic problem involving IMJLS that does not require stabilizability of solutions in any sense, and its relationship with the associated Riccati equations.

In connection to the GJLQ problem, we assume that a solution exists when the optimal cost is bounded (i.e., C1 holds), a condition that can be relaxed to finiteness of the cost in the scenario of IMJLS with finite Markov state space. An important feature is that condition C1 is weaker than the stabilizability condition that usually serves as a sufficient condition for existence of solutions to the ICARE, see Remark 2.

Assuming that C1 holds, the paper shows that the solution to the related Riccati differential equations (7) converges to a certain $M \in \mathcal{H}_1^{n0}$ satisfying the ICARE, as in (8). The paper also shows that M synthesizes the optimal control, see Theorem 1.

As regards to the ICARE, in Theorem 3 we clarify that if X is the minimal solution to the ICARE, then X provides the optimal, stationary, linear state feedback control, and the optimal cost, as follows:

$$u_{\text{opt}}(t) = -R_{\theta(t)}^{-1} B_{\theta(t)}' X_{\theta(t)} x(t)$$
$$W(x,i) = x' X_i x,$$

and we show in Lemma 4 that X is identical to the limit M. We also show that the ICARE have a nonempty set of solutions if and only if C1 holds, see Theorem 2, in a generalization of previous results.

The results obtained in the paper parallel existing results for linear deterministic systems, see e.g. (Molinari, 1977) and (Kucera, 1972), thus strengthening the relations among the theory of IMJLS and that of linear deterministic systems.

REFERENCES

Athans, M., D. Castanon, K. P. Dunn, C. S. Greene, W. H. Lee, N. R. Sandell and A. S. Willsky (1977). The stochastic control of the F-8C aircraft using a multiple model adaptive control (MMAC) method - Part i: Equilibrium flight. *IEEE Transactions on Automatic Control* **22**, 768–780.

Costa, E. F. and J. B. R. do Val (2002*a*). On the detectability and observability of continuous-time Markov jump linear systems. *SIAM Journal on Control and Optimization* **41**(4), 1295–1314.

Costa, E. F. and J. B. R. do Val (2002*b*). Weak detectability and the linear quadratic control problem of discrete-time Markov jump linear systems. *International Journal of Control. Special issue on Switched and Polytopic Linear Systems* **75**, 1282–1292.

Costa, E. F. and J. B. R. do Val (n.d.). An algorithm for solving a perturbed algebraic Riccati equation. *To appear in the European Journal of Control*.

Costa, E. F., J. B. R. do Val and M. D. Fragoso (n.d.*a*). A new approach to detectability of discrete-time infinite Markov jump linear systems. *To appear in the SIAM Journal on Control and Optimization*.

Costa, E. F., J. B. R. do Val and M. D. Fragoso (n.d.*b*). On a detectability concept of discrete-time infinite Markov jump linear systems. *To appear in the Stochastic Analysis and Applications*.

Costa, O. L. V. and M.D. Fragoso (1995). Discrete-time LQ-optimal control problems for infinite Markov jump parameter systems. *IEEE Transactions on Automatic Control* **AC-40**, 2076–2088.

Costa, O. L. V. and R.P. Marques (2000). Robust H_2-control for discrete-time Markovian jump linear systems. *International Journal of Control* **73**(1), 11–21.

Costa, O. L. V., J. B. R. do Val and J. C. Geromel (1997). A convex programming approach to H_2 control of discrete-time Markovian jump linear systems. *Int. J. Control* **66**(4), 557–579.

Costa, O. L. V., J. B. R. do Val and J. C. Geromel (1999). Continuous-time state-feedback H_2-control of Markovian jump linear systems via convex analysis. *Automatica* **35**, 259–268.

do Val, J.B.R. and E. F. Costa (2002). Numerical solution for the linear-quadratic control problem of Markov jump linear systems and a weak detectability concept. *Journal of Optimization Theory and Applications* **114**(1), 69–96.

do Val, J.B.R., J.C. Geromel and O.L.V. Costa (1999). Solutions for the linear quadratic control problem of Markov jump linear systems. *Journal of Optimization Theory and Applications* **103**, 283–311.

Fragoso, M. and J. Baczynski (2001). Optimal control for continuous time LQ problems with infinite Markov jump parameters. *SIAM Journal on Control and Optimization* **40**(1), 270– 297.

Ji, Y. and H. J. Chizeck (1990). Controlability, stabilizability and continuous time Markovian jump linear quadratic control. *IEEE Transactions on Automatic Control* **35**(7), 777–788.

Kucera, V. (1972). On nonnegative definite solutions to matrix quadratic equations. *Automatica* **8**, 413–423.

Martensson, K. (1971). On the matrix Riccati equation. *Inform. Sci.* **3**, 17–49.

Molinari, B. P. (1977). The time-invariant linear-quadratic optimal control problem. *Automatica* **13**, 347–357.

Rami, M. A. and L. E. Ghaoui (1996). LMI optimization for nonstandard Riccati equations arising in stochastic control. *IEEE Transactions on Automatic Control* **41**(11), 1666–1671.

Sworder, D. D. and R. O. Rogers (1983). An LQ-solution to a control problem associated with a solar thermal central receiver. *IEEE Transactions on Automatic Control* **28**(10), 971–978.