MIMO PID CONTROLLERS SYNTHESIS USING ORTHOGONAL FUNCTIONS

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Abstract: This paper presents a new method for the synthesis of a PID controller structure for multi-input multi-output (MIMO) linear dynamic systems in closed loop. The parameters setting of the MIMO PID controller (composed by many simple standard PID controllers) uses interesting properties of a complete set of orthogonal functions in general and shifted Legendre polynomials in particular and specifically the operational matrix of integration. The considered technique allows the conversion of differential state equations in a set of algebraic ones depending on PID structure parameters by expanding the system inputs and outputs variables into orthogonal functions. The parameters setting of the MIMO PID controller is leaded by reference to a model system in open loop having all desired performances.
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Keywords: PID Controller, MIMO systems, orthogonal functions, operational matrix of integration, Legendre polynomials.

1. INTRODUCTION

In industrial control, proportional integral and derivative (PID) controllers still have an undisputed lead. In spite of system theory evolution, the most controllers in use are still PIDs because they can resolve almost all control problems. The PID controllers offer many advantages:

- they have a standard and simple structure with the P, I and D blocks.
- they can be found in all technologies: analogical or digital electronics, hydraulic or pneumatic
- they are presented to the user under a unique form in all technologies.
- their tuning is quite easy and can be leaded even when they are on the industrial plant.

Over the past decades, an enormous amount of effort has been expended in designing these controllers beginning by the well-known Ziegler & Nichols method for SISO systems (Ziegler and Nichols, 1942). Hundreds of research papers, a number of M.Phil./Ph.D. thesis and books have been written on this subject (Unar *et al.*, 1996). Despite these advancements and improvements, the design of PID controllers, especially for MIMO systems (Multi-Input Multi-Output), is still a challenge for engineers and researchers (Unar *et al.*, 1996). Many methods were developed and improved as:

- Generalised Ziegler-Nichols Method (Neiderlinski, 1971),
- Seraji's Method (Seraji and Tarokh, 1977),
- Biggest log Modulus (BLT) Method (Luyben, 1986),
- Characteristic Locus Design Method (Zhuang, 1992),
- Zhuang and Atherton's Method for Optimisation (Zhuang and Atherton, 1994),

and many interesting others using iterative LMI approach (Lin *et al.*, 2004), a fuzzy neural network (Lee and Teng, 2003) or multiobjective genetic algorithms (Herreros *et al.*, 2002).

In this paper, a new analytical method for MIMO PID controllers synthesis by using the orthogonal functions as a tool of approximation is presented.

In recent decades, the problem of analysis, modelling and control of linear systems has been approached via orthogonal functions. The main characteristic of this technique is that it reduces the system of differential equations to an algebraic one, thus greatly simplifying the problem. For this purpose, the operational matrix of integration that approximates the integral of the basis function is used.

This approach originated from the use of Walsh (Chen and Hsiao, 1975) and block-pulse (Shih *et al.*, 1978) functions was later extended to orthogonal polynomial series such as the Laguerre (King and Paraskevopoulos, 1979), the Chebychev (Paraskevopoulos, 1983), the Hermite (Paraskevopoulos and Kekkeris, 1983) and the Legendre polynomials (Paraskevopoulos, 1985). They were also used with non linear systems (Benhadj Braiek, 1990).

This paper is organised as follows: in section 2, the orthogonal functions are presented with interesting properties and their use for systems description. The section 3 is reserved to the description of the shifted Legendre polynomials. The proposed method for MIMO PID controllers synthesis using orthogonal functions is derived in section 4. In the last section, an example is presented to emphasise the effectiveness of this method.

2. ORTHOGONAL FUNCTIONS

The continuous orthogonal functions have been adopted by many researchers as a convenient and sharp tool to approximate the solution of physical systems. The key idea of this technique is that all analytical function $f(t)$ absolutely integrable can be developed as follows:

$$f(t) = \sum_{i=0}^{\infty} f_i \phi_i(t) \tag{1}$$

where the elements $\phi_0(t), \phi_1(t), \ldots, \phi_{N-1}(t)$ are basis functions which are orthogonal on a certain interval and f_i are constant coefficients. The following integral property of basis vectors is also exploited for differential equation solution:

$$\int_{\alpha}^{t} \ldots \int_{\alpha}^{t} \Phi(\tau)\,(d\tau)^k \cong P^k \Phi(t) \tag{2}$$

where P is a square constant matrix and

$$\Phi^T(t) = \left[\, \phi_0(t) \;\; \phi_1(t) \;\; \ldots \;\; \phi_{N-1}(t) \,\right]$$

Clearly, the form of P depends on the particular choice of the basis vector $\Phi(t)$.

The Legendre polynomials may have advantages over other orthogonal functions. This was shown by way of examples (Paraskevopoulos, 1985) where Legendre polynomials converge to the exact solution of a differential equation faster than the other types of orthogonal functions, as, for example Walsh functions, Hermite and Laguerre polynomials.

3. SHIFTED LEGENDRE POLYNOMIALS

3.1 Legendre polynomials

The Legendre polynomials are defined for the time interval $x \in [-1, 1]$ and they have the following analytical form given by the Olinde-Rodrigues formula (Bell, 1968):

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n} \tag{3}$$

Using the above expression for $L_n(x)$, one may readily determine the first few Legendre polynomials : $L_0(x) = 1$, $L_1(x) = x$, ...
The Legendre polynomials are also given by the recursive formula (Gradshteyn and Ryzhik, 1979) :

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x) \tag{4}$$

The polynomials $L_i(x)$ form a complete set and are orthogonal with

$$\int_{-1}^{1} L_i(x) L_j(x) dx = \frac{2}{2i+1} \delta_{ij} \tag{5}$$

where δ_{ij} is the Kronecker delta.

3.2 Shifted Legendre polynomials

For practical use of Legendre polynomials in the time interval $t \in [0, t_f]$, it is necessary to shift the defining domain of Legendre polynomials from the interval $[-1, 1]$ to $[0, t_f]$ through the variable transformation:

$$x = \frac{2t}{t_f} - 1, \quad 0 \le t \le t_f \tag{6}$$

The shifted Legendre polynomials $s_i(t)$ $(i = 0, 1, 2, \ldots)$ for $0 \le t \le t_f$ are thus given by:

$$s_{n+1}(t) = \frac{2n+1}{n+1}\,\frac{2t - t_f}{t_f}\, s_n(t) - \frac{n}{n+1} s_{n-1}(t) \tag{7}$$

with $s_0(t) = 1$ and $s_1(t) = \dfrac{2t}{t_f} - 1$

It is apparent that polynomials $s_n(t)$ also constitute a complete set and are orthogonal with:

$$\int_0^{t_f} s_i(t) s_j(t)\, dt = \frac{t_f}{2i+1}\delta_{ij} \tag{8}$$

Any time function $f(t)$ that is absolutely integrable on the time interval $[0, t_f]$ may be expanded into shifted Legendre series as follows:

$$f(t) = \sum_{i=0}^{\infty} f_i s_i(t) \tag{9}$$

where (Hwang and Guo, 1984)

$$f_i = \frac{2i+1}{t_f} \int_0^{t_f} f(t) s_i(t)\, dt \tag{10}$$

If equation (9) is truncated up to its first N terms, then it may be written as:

$$f(t) \cong \sum_{i=0}^{N-1} f_i s_i(t) = F_N^T S_N(t) \tag{11}$$

with $F_N = \left[\, f_0 \;\, f_1 \;\, \ldots \;\, f_{N-1} \,\right]^T$,
and $S_N(t) = \left[\, s_0(t) \;\, s_1(t) \;\, \ldots \;\, s_{N-1}(t) \,\right]^T$

The shifted Legendre polynomials and coefficients f_i, $(i = 0, 1, \ldots, N-1)$ have the particularity to minimise the integral squared-error:

$$\varepsilon = \int_0^{t_f} \left(f(t) - \sum_{i=0}^{N-1} f_i s_i(t) \right)^2 dt \tag{12}$$

3.3 Operational matrix of integration

Since the shifted Legendre polynomials $s_i(t)$, $(i = 0, 1, \ldots)$ satisfy (Gradshteyn and Ryzhik, 1979) the differential equation:

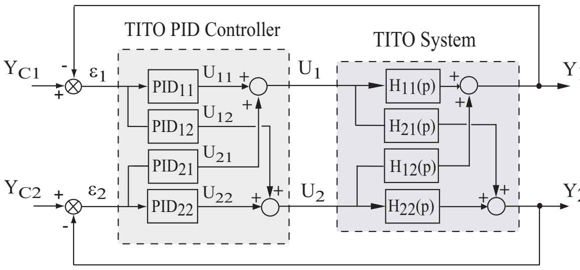

Fig. 1. TITO system with TITO PID controller

$$s_i(t) = \frac{t_f}{2(2i+1)} \left[\frac{ds_{i+1}}{dt} - \frac{ds_{i-1}}{dt} \right] \tag{13}$$

and $s_i(0) = (-1)^i$, it can be easily shown that the integrals of $s_i(t)$, $(i = 0, 1, \ldots)$ are given by:

$$\int_0^t s_i(\tau) d\tau = \begin{cases} \dfrac{t_f}{2} \left[s_1(t) - s_0(t) \right] & ,\mathrm{i} = 0 \\ \dfrac{t_f}{2(2i+1)} \left[s_{i+1}(t) - s_{i-1}(t) \right] & \end{cases} \tag{14}$$

From equation (14) we can obtain the integral of truncated shifted Legendre vector

$$\int_0^t S_N(\tau) d\tau \cong P_N S_N(t) \tag{15}$$

where P_N is the operational matrix of integration (Hwang and Shih, 1982).

4. PROPOSED METHOD FOR MIMO PID CONTROLLERS SYNTHESIS

4.1 Problem formulation

The class of systems considered are the multi-input multi-output (MIMO) linear time-invariant systems described by the state equations:

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX \end{cases} \tag{16}$$

A, B and C are constant matrices with respective dimensions $(n \times n)$, $(n \times m)$ and $(m \times n)$. The structure of the proposed MIMO PID controller is associated to the considered system in closed loop.

For presenting the proposed method, consider a TITO (Two Inputs -Two Outputs) system. The TITO PID controller structure is shown by Figure 1. The control vector can be written as follows:

$$U = \begin{bmatrix} U_1 \\ \\ U_2 \end{bmatrix} = \begin{bmatrix} U_{11} + U_{21} \\ \\ U_{12} + U_{22} \end{bmatrix} \tag{17}$$

$$U = \begin{bmatrix} K_{p11}(Y_{C1} - Y_1) + K_{i11}\int (Y_{C1} - Y_1)dt + \\ +K_{d11}\dfrac{d(Y_{C1} - Y_1)}{dt} + K_{p21}(Y_{C2} - Y_2) + \\ +K_{i21}\int (Y_{C2} - Y_2)dt + K_{d21}\dfrac{d(Y_{C2} - Y_2)}{dt} \\ \\ K_{p12}(Y_{C1} - Y_1) + K_{i12}\int (Y_{C1} - Y_1)dt + \\ +K_{d12}\dfrac{d(Y_{C1} - Y_1)}{dt} + K_{p22}(Y_{C2} - Y_2) + \\ +K_{i22}\int (Y_{C2} - Y_2)dt + K_{d22}\dfrac{d(Y_{C2} - Y_2)}{dt} \end{bmatrix}$$

where K_{pjk}, K_{ijk} and K_{djk} $(jk = 11, 12, 21, 22)$ are the parameters of the Proportional, Integral and Derivative blocks of the standard PID controllers.

Then, the control vector can be written under the following form:

$$U = \begin{bmatrix} K_{p11} & K_{p21} \\ K_{p12} & K_{p22} \end{bmatrix} (Y_C - Y) + \begin{bmatrix} K_{i11} & K_{i21} \\ K_{i12} & K_{i22} \end{bmatrix} \int (Y_C - Y)dt + \begin{bmatrix} K_{d11} & K_{d21} \\ K_{d12} & K_{d22} \end{bmatrix} (\dot{Y}_C - \dot{Y}) \quad (18)$$

where $\mathrm{Y_C} = \begin{bmatrix} Y_{C1} \\ Y_{C2} \end{bmatrix}$ and $\mathrm{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

The definition of proportional $\bar{K}_p$, integral $\bar{K}_i$ and derivative $\bar{K}_d$ gain matrices given by:

$$\bar{K}_p = \begin{bmatrix} K_{p11} & K_{p21} \\ K_{p12} & K_{p22} \end{bmatrix}$$
$$\bar{K}_i = \begin{bmatrix} K_{i11} & K_{i21} \\ K_{i12} & K_{i22} \end{bmatrix}$$
$$\bar{K}_d = \begin{bmatrix} K_{d11} & K_{d21} \\ K_{d12} & K_{d22} \end{bmatrix}$$

leads to the equation:

$$U = \bar{K}_p(Y_C - Y) + \bar{K}_i \int (Y_C - Y)\, dt + \bar{K}_d(\dot{Y}_C - \dot{Y}) \quad (19)$$

With $Y = CX$, the state equation will be written as follows:

$$\dot{X} = AX + B\bar{K}_p(Y_C - CX) + B\bar{K}_i \int (Y_C - CX)\, dt + B\bar{K}_d(\dot{Y}_C - C\dot{X}) \quad (20)$$

Integration of equation (20) yields:

$$(B\bar{K}_pC - A)\int Xdt + B\bar{K}_iC \iint Xdt + (I_n + B\bar{K}_dC)X = B\bar{K}_p \int Y_Cdt + B\bar{K}_i \iint Y_Cdt + B\bar{K}_dY_C \quad (21)$$

The expansion of the state vector X and the input Vector Y_C into truncated shifted Legendre polynomials as follows:

$$X(t) \cong \sum_{i=0}^{N-1} X_{N,i}s_i(t) = X_NS_N(t) \quad (22)$$

$$Y_C(t) \cong \sum_{i=0}^{N-1} Y_{CN,i}s_i(t) = Y_{CN}S_N(t) \quad (23)$$

The single integration of the shifted Legendre basis vector can be approximated by equation (15). The double integration is approximated by:

$$\int_0^t\int_0^t S_N(\tau)d\tau^2 \cong P_N^2S_N(t) \quad (24)$$

The equation (21) can then be approached by the following one:

$$[(B\bar{K}_pC - A)X_NP_N + B\bar{K}_iCX_NP_N^2 + (I_n + B\bar{K}_dC)X_N]S_N = \left[B\bar{K}_pY_{CN}P_N + B\bar{K}_iY_{CN}P_N^2 + B\bar{K}_dY_{CN}\right] S_N \quad (25)$$

By using the Vec operator and the property (Brewer, 1978):

$$Vec(ABC) = (C^T \otimes A)Vec(B)$$

(for any matrices A, B and C having appropriate dimensions, where $\otimes$ is the Kronecker product), the equation (25) yields to (Ayadi, 2004):

$$M(\bar{K}_p, \bar{K}_i, \bar{K}_d)Vec(X_N) = T(\bar{K}_p, \bar{K}_i, \bar{K}_d)Vec(Y_{CN}) \quad (26)$$

where

$$M(\bar{K}_p, \bar{K}_i, \bar{K}_d) = P_N^T \otimes (B\bar{K}_pC - A) + P_N^{2T} \otimes B\bar{K}_iC + I_N \otimes (I_n + B\bar{K}_dC) \quad (27)$$

$$T(\bar{K}_p, \bar{K}_i, \bar{K}_d) = P_N^T \otimes B\bar{K}_p + P_N^{2T} \otimes B\bar{K}_i + I_N \otimes B\bar{K}_d \quad (28)$$

4.2 Reference model

The chosen reference is represented by its state equations:

$$\begin{cases} \dot{Z} = EZ + FY_C \\ Y_{ref} = GZ \end{cases} \quad (29)$$

with E a square $(r \times r)$ matrix. An analogue development (integration of the state equation, expansion in shifted Legendre series, operator Vec...) gives:

$$Vec(Z_N) = WVec(Y_{CN}) \quad (30)$$

where

$$W = \left(I_{r\times N} - P_N^T \otimes E\right)^{-1} (P_N^T \otimes F)$$

4.3 Proposed approach for problem solution

It is desired that the considered system with the TITO PID controller has an analogue dynamic behaviour to the reference model for all inputs Y_C (Y_{C1} and Y_{C2}). This condition yields

$$Y = Y_{ref} \Leftrightarrow CX = GZ \Leftrightarrow CX_N = GZ_N \quad (31)$$

With the Vec operator, this condition becomes

$$(I_N \otimes C)Vec(X_N) = (I_N \otimes G)Vec(Z_N) \quad (32)$$

Finally, the obtained equations system is composed by equations (26), (30) and (32). By substitution, it gives:

$$\begin{aligned}(I_N \otimes C)\ M^{-1}T\ Vec(Y_{CN}) = \\ (I_N \otimes G)WVec(Y_{CN})\end{aligned} \quad (33)$$

This equation is verified for all inputs Y_C, so it gives a set of algebraic equations verified by the matrices of proportional, integral and derivative gains respectively $\bar{K}_p$, $\bar{K}_i$ and $\bar{K}_d$:

$$(I_N \otimes C)\, M^{-1}\, T = (I_N \otimes G)\, W \quad (34)$$

where M and T are square matrices depending on $\bar{K}_p$, $\bar{K}_i$ and $\bar{K}_d$ and are given by equations (27) and (28).

The parameters of the TITO PID controller (gain matrices) are derived by minimising the norm:

$$\xi = \left\| (I_N \otimes C)\ \ M^{-1}\, T - (I_N \otimes G)\, W \right\| \quad (35)$$

representing the norm of the difference between both parts of the equation (34). This constrained minimisation can be leaded by using the function "fmincon" of the software "MATLAB".

5. SIMULATION EXAMPLE

Consider a TITO linear process defined by its transfer matrix given by:

$$H(p) = \begin{bmatrix} \dfrac{0.01p^2 + p + 1}{p^3 + 6p^2 + 6p + 1} & \dfrac{0.095p + 0.5}{p^3 + 6p^2 + 6p + 1} \\ \\ \dfrac{0.25}{p^3 + 6p^2 + 6p + 1} & \dfrac{1}{p^3 + 6p^2 + 6p + 1} \end{bmatrix}$$

where p denotes the Laplace operator.

The reference model is an uncoupled TITO system with desired performances. This model is defined by the transfer matrix:

$$H_r(p) = \begin{bmatrix} \dfrac{0.723}{p^2 + 1.53p + 0.723} & 0 \\ \\ 0 & \dfrac{0.723}{p^2 + 1.53p + 0.723} \end{bmatrix}$$

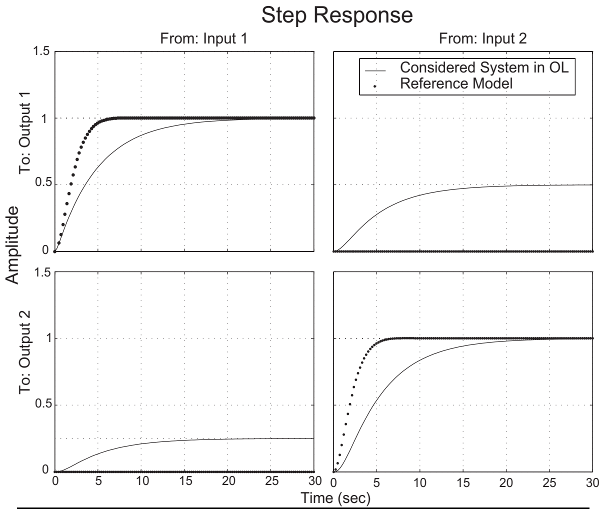

Fig. 2. Step response in open-loop of the TITO considered system and the reference model

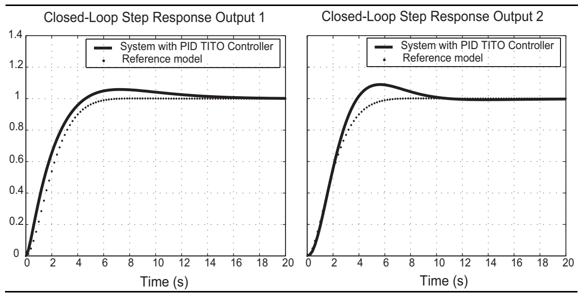

Fig. 3. Step response in closed-loop of the TITO considered system with the TITO PID controller and the reference model

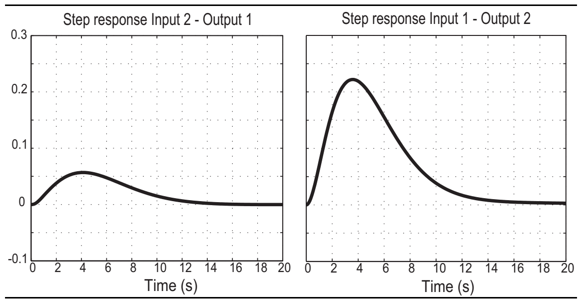

Fig. 4. Closed-loop step responses (Input 2 - Output1 and Input 1 - Output 2)

The step responses of the considered system in open loop and the reference model are shown by Figure 2.

Figure 3 shows the process closed-loop response with the TITO PID controller obtained by the proposed method for both outputs.

The uncoupling of both outputs can be seen with the cross-step responses: input 1- output 2 and input 2- output 1 in Figure 4.

6. CONCLUSION

In this paper, a new analytical method was introduced for MIMO PID controllers synthesis by using orthogonal functions as a tool of approximation. The presented method was applied to a TITO interconnected process but can be easily extended to MIMO systems. The use of operational matrix of integration has permitted the transformation of differential equations into algebraic ones depending on MIMO PID Controller parameters. This technique allows the synthesis of MIMO PID controller with a chosen reference model.

The shifted Legendre polynomials have been used as an orthogonal function basis but the method still effective with any other basis such as : Walsh and Block-pulse functions or Chebychev, Laguerre, Hemite polynomials.

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