# A ROBUST VERSION OF THE ELIMINATION LEMMA

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AbstractIn this paper we extend the elimination lemma to hold for a set of inequalities. The elimination lemma is a linear algebra result which has been successfully used to solve a large number of filtering and control problems. We show that the obtained extension can be used to provide alternative solutions to robust state feedback control and robust filtering problems when the uncertainty is described by a convex polytope. We also investigate some conjectures which, if true, would be able to solve an open problem in dynamic output feedback robust control. We provide counter-examples to these conjectures.


Keywords: Elimination Lemma, Robust Filtering, Robust Control.

## 1. INTRODUCTION

The following linear algebra result, known as *Elimination Lemma*, has been extensively used in systems and control.

*Lemma 1.* (Elimination). Let $Q \in \mathbb{S}^n$, $B \in \mathbb{R}^{m\times n}$, and $C \in \mathbb{R}^{p\times n}$ be given matrices. The following statements are equivalent:

i) $({B^{\perp}}^T Q B^{\perp} < 0$ or $B^T B > 0)$
and $({C^{\perp}}^T Q C^{\perp} < 0$ or $C^T C > 0)$,
ii) $\exists\, K \in \mathbb{R}^{p\times m} : \; Q + C^T K B + B^T K^T C < 0.$

See, for instance, (Boyd *et al.*, 1994; Skelton *et al.*, 1997) for a proof and applications.

Several linear filtering and control design problems can be stated in the form *ii*), where $K$ represents the parameters of the filter or controller to be designed and $Q$, $B$ and $C$ are functions of an instrumental symmetric and positive definite matrix coming from Lyapunov stability theory. In many cases, it is possible to show that even if the matrix inequality in *ii*) is not jointly convex on $K$ and the instrumental Lyapunov matrix, the set of solutions for the inequalities given in *i*) will be convex. In fact, this is the idea behind the entire book (Skelton *et al.*, 1997), which shows that more than twenty relevant control problems can be solved with the help of Lemma 1, producing inequalities in the form of item *i*) which are linear functions of the instrumental variables. This particular class of inequalities has been known in the control literature as Linear Matrix Inequalities (LMI). Other widely referenced results that make use of the elimination lemma, to name a few, are (Gahinet and Apkarian, 1994; Iwasaki and Skelton, 1994; Apkarian and Gahinet, 1995; Scherer, 1995). Recently, this result has also been used in converse form, starting from inequalities in the form *i*) and producing an inequality in the form *ii*) affine on an extra variable $K$ (multiplier), as in (de Oliveira and Skelton, 2001).

Unfortunately, Lemma 1 has several limitations which prevent its application to many robust filtering and control design problems. In this context, it would be desirable to have a "robust version" of this lemma, which would be able to eliminate the variable $K$ from the set of inequalities

$$\exists\, K \in \mathbb{R}^{p\times m} :\ Q_i + C_i^T K B_i + B_i^T K^T C_i < 0 \quad (1)$$

for all $i = 1, \ldots, N$. The challenge here is the fact that a single $K$ should simultaneously satisfies all $N$ inequalities. For instance, in the context of robust control design, $K$ could represent a robust controller, which is able to stabilize a family of $N$ given plants or, under some extra mild assumptions on the matrices $Q_i$, $B_i$, and $C_i$, to stabilize all convex combinations of the given $N$ plants.

This paper investigates whether appropriate versions of Lemma 1 can be constructed for inequalities in the form (1) or some variations. In particular, we establish a necessary and sufficient elimination under the assumption that $B_i = B$, $C_i = C$, for all $i = 1, \ldots, N$. We show that this result is already enough to provide alternative solutions to robust control and filtering problems addressed in the literature. We also prove that several conjectures that can be constructed from straightforward variations of Lemma 1 to handle sets of inequalities in the form (1) are not correct.

## 2. A ROBUST ELIMINATION RESULT

On of the results of this paper is to establish the following extension to Lemma 1.

*Theorem 1.* (Robust Elimination). Let $Q_i \in \mathbb{S}^n$, $i = 1, \ldots, N$, $B \in \mathbb{R}^{m\times n}$, and $C \in \mathbb{R}^{p\times n}$ be given matrices. The following statements are equivalent:

i) $\exists\, Q_0 \in \mathbb{S}^n :\ ({B^{\perp}}^T Q_0 B^{\perp} < 0 \text{ or } B^T B > 0)$, $({C^{\perp}}^T Q_0 C^{\perp} < 0 \text{ or } C^T C > 0)$, and $Q_0 \geq Q_i, \quad \forall i = 1, \ldots, N$,
ii) $\exists\, K \in \mathbb{R}^{p\times m} :\ Q_i + C^T K B + B^T K^T C < 0$, $\forall i = 1, \ldots, N$.

A proof of this theorem is given in Appendix A. In the above theorem, the variable $K$ has been indeed eliminated from the inequalities in item *i*). However, another variable $Q_0$ had to be introduced. We will discuss some issues related to this fact later in Section 5.

The above result can be applied to provide alternative solutions to many robust state feedback control and robust filtering design problems available in the literature. We illustrate this here by revisiting the problem of robust filtering.

### 2.1 Robust filtering

Consider the uncertain linear time-invariant system

$$\begin{pmatrix} \dot{x} \\ z \end{pmatrix} = M \begin{pmatrix} x \\ w \end{pmatrix}, \quad (2)$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^r$, $z \in \mathbb{R}^p$, and the matrix

$$M := \begin{bmatrix} F & G \\ H_z & J_z \end{bmatrix} \quad (3)$$

is unknown but lies in the convex hull of the finite set of vertices $M_i$, $i = 1, \ldots, N$. All measurements taken from the above system are represented in the measurement equation

$$y = H_y x + J_y w, \quad (4)$$

where $y \in \mathbb{R}^q$. As in (Geromel and de Oliveira, 2001), we connect to the above measurement a linear filter with structure

$$\begin{pmatrix} \dot{\hat{x}} \\ \hat{z} \end{pmatrix} = \hat{M} \begin{pmatrix} \hat{x} \\ y \end{pmatrix}, \quad (5)$$

where $\hat{x} \in \mathbb{R}^n$ and $\hat{z} \in \mathbb{R}^p$ and

$$\hat{M} := \begin{bmatrix} \hat{F} & \hat{G} \\ \hat{H} & \hat{J} \end{bmatrix}. \quad (6)$$

The particular robust filtering problem we will address here is the computation of a robust filter such that an upper bound to the $H_\infty$ norm of the transfer function from the input $w$ to the filtering error $e := z - \hat{z}$ is less than a prespecified value $\mu$. The same technique can however be applied to similar problems, such as robust $H_2$ filtering, as well. We assume that the filter has the same order as the plant.

The filtering error $e$ can be described by the uncertain linear system

$$\begin{pmatrix} \dot{\bar{x}} \\ e \end{pmatrix} = \mathcal{M} \begin{pmatrix} \bar{x} \\ w \end{pmatrix} \quad (7)$$

where

$$\mathcal{M} = \begin{bmatrix} \mathcal{F} & \mathcal{G} \\ \mathcal{H} & \mathcal{J} \end{bmatrix}, \quad (8)$$

is unknown but lies in the convex hull of the finite set of vertices defined by

$$\mathcal{F}_i = \begin{bmatrix} F_i & 0 \\ \hat{G} H_y & \hat{F} \end{bmatrix}, \qquad \mathcal{G}_i = \begin{bmatrix} G_i \\ \hat{G} J_y \end{bmatrix}, \quad (9)$$

$$\mathcal{H}_i = \begin{bmatrix} H_{zi} - \hat{J} H_y & -\hat{H} \end{bmatrix}, \quad \mathcal{J}_i = \begin{bmatrix} J_{zi} - \hat{J} J_y \end{bmatrix}, \quad (10)$$

for $i = 1, \ldots, N$.

The $H_\infty$ norm of the transfer function from the input $w$ to the filtering error $e$ is less than $\mu$ if and only if the inequalities (from the Bounded Real Lemma (Skelton *et al.*, 1997))

$$\begin{bmatrix} \mathcal{F}_i^T \mathcal{P} + \mathcal{P} \mathcal{F}_i & \mathcal{P} \mathcal{G}_i & \mathcal{H}_i^T \\ \mathcal{G}_i^T \mathcal{P} & -\mu & \mathcal{J}_i^T \\ \mathcal{H}_i & \mathcal{J}_i & -\mu \end{bmatrix} < 0, \quad (11)$$

have a solution for some symmetric matrix $\mathcal{P} > 0$ and all $i = 1, \ldots, N$. To simplify our discussion we assume that $\mathcal{P}$ has the particular partitioning structure

$$\mathcal{P} := \begin{bmatrix} Z & Y \\ Y & Y \end{bmatrix}.$$

This assumption can be imposed without loss of generality (see (Skelton *et al.*, 1997), p. 143, for instance). One can then show that (11) can be decomposed in the form of the inequalities in item *ii*) of Theorem 1 where

$$Q_i = \begin{bmatrix} ZF_i + F_i^T Z & F_i^T Y & ZG_i & H_{zi}^T \\ YF_i & 0 & YG_i & 0 \\ G_i^T Z & G_i^T Y & -\mu & J_{zi}^T \\ H_{zi} & 0 & J_{zi} & -\mu \end{bmatrix}, \quad (12)$$

$$B = \begin{bmatrix} 0 & I & 0 & 0 \\ H_y & 0 & J_y & 0 \end{bmatrix}, \quad C = \begin{bmatrix} Y & Y & 0 & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}. \quad (13)$$

If we notice now that all matrices in $B$ are known such that $B^\perp$ is also a known constant matrix and that

$$C^\perp = \begin{bmatrix} 0 & I \\ 0 & -I \\ I & 0 \\ 0 & 0 \end{bmatrix} \quad (14)$$

is also a constant matrix, even though $C$ depends on the nonsingular matrix $Y$. Therefore, as $B^\perp$ and $C^\perp$ are constants, and $Q_i$, $i = 1, \ldots, N$ are affine functions of variables $Z$ and $Y$, the conclusion is that the inequalities in item $i$) of Theorem 1 are also affine, hence LMI. The solution of these LMI along with the linear constraint

$$Z > 0, \qquad Y > 0, \qquad Z > Y, \quad (15)$$

which comes from $\mathcal{P} > 0$, can be used to verify whether there exists a feasible robust $H_\infty$ filter. This result is equivalent to the one obtained in (Geromel and de Oliveira, 2001) with the help of a change of variable.

## 3. CONJECTURES ON ROBUST ELIMINATION

An interesting aspect of using Lemma 1 to move from *ii*) to *i*) is that the inequalities in *i*) are simpler to solve, having less variables and being of lower dimension than the one in *ii*). That is not necessarily true in Theorem 1, in which an extra variable had to be introduced. In this section we will investigate whether there exists simpler versions of Theorem 1 which preserves this interesting property of Lemma 1. We start by multiplying (1) on the right by $B_i^\perp$ and on the left by its transpose and repeating the same procedure for $C_i^\perp$ to produce

$${B_i^\perp}^T Q_i B_i^\perp < 0, \text{ and } {C_i^\perp}^T Q_i C_i^\perp < 0, \quad (16)$$

for all $i = 1, \ldots, N$. This shows that (1) implies (16). As we will see next, the converse is unfortunately not true in general. We show this by presenting a collection of counter-examples to some conjectures. The first conjecture we consider is the following.

*Conjecture 1.* Let $Q_i \in \mathbb{S}^n$, $i = 1, \ldots, N$, $B \in \mathbb{R}^{m \times n}$, and $C \in \mathbb{R}^{p \times n}$ be given matrices. The following statements are equivalent:

i) $({B^\perp}^T Q_i B^\perp < 0$ or $B^T B > 0)$ and $({C^\perp}^T Q_i C^\perp < 0$ or $C^T C > 0)$, $\forall i = 1, \ldots, N$,
ii) $\exists K \in \mathbb{R}^{p \times m}: \; Q_i + C^T K B + B^T K^T C < 0$, $\forall i = 1, \ldots, N$.

The interested reader can verify that if this particular version of the elimination lemma is true, than it is possible to show that some open problems in control theory such as, for instance, the design of robust dynamic output feedback controllers for plants with structured polytopic uncertainty (Geromel *et al.*, 1999; de Oliveira *et al.*, 2000) can be given an LMI formulation. Unfortunately, we show in Section 4 that this conjecture is *false* when $N > 1$. A possible variation of Conjecture 1 is the following.

*Conjecture 2.* Let $Q \in \mathbb{S}^n$, $B_i \in \mathbb{R}^{m \times n}$, and $C_i \in \mathbb{R}^{p \times n}$, $i = 1, \ldots, N$, be given matrices. The following statements are equivalent:

i) $({B_i^\perp}^T Q B_i^\perp < 0$ or $B_i^T B_i > 0)$ and $({C_i^\perp}^T Q C_i^\perp < 0$ or $C_i^T C_i > 0)$, $\forall i = 1, \ldots, N$,
ii) $\exists K \in \mathbb{R}^{p \times m}: \; Q + C_i^T K B_i + B_i^T K^T C_i < 0$, $\forall i = 1, \ldots, N$.

We also show a counter-example in Section 4 that proves this is false. Indeed, we show that all possible variations of these conjecture that check exclusively inequalities in the form (16) can not be correct when $N > 1$.

Interestingly enough, the particular case of the above conjectures given in the following lemma is indeed true.

*Lemma 2.* Let $Q_i \in \mathbb{S}^n$, $i = 1, \ldots, N$, and $B \in \mathbb{R}^{m \times n}$ be given matrices. The following statements are equivalent:

i) ${B^\perp}^T Q_i B^\perp < 0, \qquad \forall i = 1, \ldots, N$,
ii) $\exists K \in \mathbb{R}^{n \times m}: \; Q_i + K B + B^T K^T < 0$, $\forall i = 1, \ldots, N$.

A formal proof is provided in Appendix A.

## 4. COUNTER-EXAMPLES OF THE ROBUST ELIMINATION CONJECTURES

The following is a counter-example for Conjecture 1. Consider the matrices

$$Q_1 = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -1 & -4 \\ -4 & -1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

For these matrices $B^\perp = C^T$, $C^\perp = B^T$ and

$$B^{\perp T} Q_i B^\perp = C^{\perp T} Q_i C^\perp = -1 < 0, \qquad \forall i = 1, 2,$$

which implies that the inequalities in item $i$) of Conjecture 1 are feasible. However,

$$Q_1 + C^T K B + B^T K^T C = \begin{bmatrix} -1 & K+4 \\ K+4 & -1 \end{bmatrix} < 0,$$
$$Q_2 + C^T K B + B^T K^T C = \begin{bmatrix} -1 & K-4 \\ K-4 & -1 \end{bmatrix} < 0,$$

which implies that all values of $K$ that make the above inequalities feasible are in the set

$$\{|K+4| < 1\} \cap \{|K-4| < 1\} = \emptyset.$$

The following is a counter-example for Conjecture 2. For the choice of matrices

$$Q = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix},$$
$$C_1 = C_2 = B_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

we have $B_1^\perp = C_1^\perp = C_2^\perp = B_2^T$, $B_2^\perp = B_1^T$ and

$$B_i^{\perp T} Q B_i^\perp = C_i^{\perp T} Q C_i^\perp = -1 < 0, \qquad i = 1, 2.$$

Once more the inequalities in item $i$) of Conjecture 2 are feasible while

$$Q + C_1^T K B_1 + B_1^T K^T C_1 = \begin{bmatrix} 2K-1 & 4 \\ 4 & -1 \end{bmatrix} < 0,$$
$$Q + C_2^T K B_2 + B_2^T K^T C_2 = \begin{bmatrix} -1 & K+4 \\ K+4 & -1 \end{bmatrix} < 0,$$

which implies that all values of $K$ that make the above inequalities feasible are in the set

$$\{K < -15/2\} \cap \{|K+4| < 1\} = \emptyset.$$

The above counter-example also rules out the last possible variation which would be to let $Q$ and $B$ constant and have distinct $C_i$ for all $i = 1, \ldots, N$ (or $C$ constant and $B_i$, $i = 1, \ldots, N$). These counter-examples show that no possible robust version of the elimination lemma that can be build by checking exclusively the inequalities in the form of (16) is correct when formulated for $N > 1$ inequalities.

## 5. DISCUSSION

The most striking difference between the robust elimination result in Theorem 1 and Lemma 1 is the need for an extra variable $Q_0$. In fact, this is related to the fact that we can only establish a partial ordering for symmetric matrices (see for instance (Jarre, 2000)). Indeed, consider the inequalities

$$Q_i < R, \quad Q_i < S, \qquad \forall i = 1, \ldots, N.$$

There might not exist $Q_0$ such that

$$Q_i < Q_0 < R, \quad Q_i < Q_0 < S, \qquad \forall i = 1, \ldots, N.$$

Indeed, using Finsler's Lemma (see Appendix A) one can rewrite item $i$) of Conjecture 1 as

$$\exists\, \alpha \in \mathbb{R}^+ : Q_i < \alpha B^T B \quad \text{and} \quad Q_i < \alpha C^T C,$$

for all $i = 1, \ldots, N$. Using the same reasoning, the item $i$) of Theorem 1 can be rewritten as

$$\exists\, Q_0 \in \mathbb{S}^n, \alpha \in \mathbb{R}^+ :$$
$$Q_i < Q_0 < \alpha B^T B \quad \text{and} \quad Q_i < Q_0 < \alpha C^T C,$$

for all $i = 1, \ldots, N$, from where it is clear that the role of $Q_0$ is to ensure the existence of a matrix situated "between" all the $Q_i$, $i = 1, \ldots, N$ and the matrices $\alpha B^T B$ and $\alpha C^T C$. Notice that when $N = 1$, if

$$Q_1 < R, \quad Q_1 < S,$$

then there always exists a sufficiently small $\epsilon > 0$ such that $Q_0 = Q_1 + \epsilon I$ and

$$Q_1 < Q_0 < R, \quad Q_1 < Q_0 < S,$$

therefore, there is no need to search explicitly for such $Q_0$.

The same fact is true when considering

$$Q_i < R, \qquad \forall i = 1, \ldots, N,$$

for which there always exists a sufficiently small $\epsilon > 0$ such that $Q_0 = R - \epsilon I$ and

$$Q_i < Q_0 < R, \qquad \forall i = 1, \ldots, N.$$

This explains why there is no need to include the variable $Q_0$ in the robust result of Lemma 2.

We hope that the results shown in this paper can provide some insight on the difficulties of solving some robust control problems and be used in the future to solve open problems in systems, filtering and control theory.

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## Appendix A. PROOFS OF THE ROBUST ELIMINATION RESULTS

We need the following well known result (see (Skelton *et al.*, 1997) for a proof).

*Lemma 3.* (Finsler). Let $Q \in \mathbb{S}^n$, and $B \in \mathbb{R}^{m\times n}$ be given matrices with $\operatorname{rank}(B) < n$. The following statements are equivalent:

i) ${B^\perp}^T Q B^\perp < 0$,
ii) $\exists\, \alpha \in \mathbb{R}^+,\ \alpha < \infty : \quad Q - \alpha B^T B < 0.$

### *A.1 Proof of Lemma 2*

$ii) \Rightarrow i)$ : Multiply the inequality in $ii)$ on the right by $B^\perp$ and on the left by ${B^\perp}^T$ to obtain the inequalities in $i)$ for each $i = 1, \dots, N$.

$i) \Rightarrow ii)$ : From Lemma 3, if the inequalities given in item $i)$ of Lemma 2 have a feasible solution then there exist $\alpha_i \in \mathbb{R}^+$, $\alpha_i < \infty$, $i = 1, \dots, N$ such that

$$Q_i - \alpha_i B^T B < 0, \qquad \forall i = 1, \dots, N.$$

Therefore, for $\alpha \in \mathbb{R}^+$, $\alpha := \max_i(\alpha_i) < \infty$,

$$Q_i - \alpha B^T B < 0, \qquad \forall i = 1, \dots, N.$$

Hence, defining $K := -(1/2)\alpha B^T$ we have that

$$Q_i + KB + B^T K^T < 0, \qquad \forall i = 1, \dots, N.$$

### *A.2 Proof of Theorem 1*

$i) \Rightarrow ii)$ : If the inequalities given in item $i)$ of Theorem 1 have feasible solutions then, from Lemma 1, there exists $K$ such that

$$Q_0 + C^T K B + B^T K^T C < 0.$$

But as $Q_0 \geq Q_i$, $i = 1, \dots, N$, we have that

$$\begin{aligned} 0 &> Q_0 + C^T K B + B^T K^T C, \\ &\geq Q_i + C^T K B + B^T K^T C, \end{aligned}$$

for all $i = 1, \dots, N$.

$ii) \Rightarrow i)$ : If the inequalities in item $ii)$ of Theorem 1 have feasible solutions then there exists a sufficiently small $\epsilon > 0$ such that

$$Q_i + C^T K B + B^T K^T C + \epsilon \mathbf{I} < 0,$$

for all $i = 1, \dots, N$. Hence, defining

$$\begin{aligned} &Q_0 := -(C^T K B + B^T K^T C + \epsilon \mathbf{I}), \\ &\qquad \Rightarrow \quad Q_0 \geq Q_i, \quad \forall i = 1, \dots, N. \end{aligned}$$

Also notice that

$$Q_0 + C^T K B + B^T K^T C = -\epsilon \mathbf{I} < 0,$$

which, using Lemma 1, implies that ${B^\perp}^T Q_0 B^\perp < 0$, and ${C^\perp}^T Q_0 C^\perp < 0$.