CRITERIA FOR SYSTEM IDENTIFICATION WITH QUANTIZED DATA AND THE OPTIMAL QUANTIZATION SCHEMES

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Abstract: In this paper, we first examine several criteria for system identification with quantized output data and show that the ordinary parameter estimator for quantization-free case is still reasonable according to those criteria. Then, we give the optimal quantization schemes for minimizing the estimation errors under a constraint on the number of the quantized subsections of the output signals or the expectation of the optimal code length when the quantized data is encoded.

Keywords: identification, quantization, least-squares method, maximum likelihood principle

1. INTRODUCTION

In the last few years, quantization problem of signals to reduce the information of the transmitted signals in controlled systems has been discussed actively by several research groups and interesting results have been achieved (Wong and Brockett, 1997, 1999; Brockett and Liberzon, 2000; Nair and Evans, 2000; Elia and Mitter, 2001; Nair and Evans, 2002; Tsumura and Maciejowski, 2003). Compared to this activity in the stabilization or estimation problem, the quantization problem for system identification has not been adequately considered, although it is also an important subject when networked control systems are partially or totally unknown.

From such view point, this problem was considered in (Tsumura and Maciejowski, 2002) and an optimal quantization scheme for minimizing estimation errors under a constraint on the number of levels of the quantized signals was given. The optimal quantization is not uniform and the profile of the distribution of the quantized subsections was shown. Moreover, its extension to more general case with high resolution quantizers was also given in (Tsumura, 2004).

In those previous researches, the optimal quantizers were given on a condition of applying the ordinary parameter estimation; least squares method for normal continuous I/O data. This problem setting arouses a question on the reasonability of such ordinary parameter estimation for the quantized data.

In this paper, we first examine this reasonability and show that the ordinary parameter estimation for the quantization-free data is still optimal for several criteria in an asymptotic situation with high resolution quantizater. Then, we give optimal analytic quantizations of signals for such criteria. The solutions are simple functions of the distribution density of input signals and we can easily figure out the profile of the density of the number of quantized subsections.

2. CRITERIA AND OPTIMAL ESTIMATORS

In this paper, we consider to derive optimal quantizers in analytically simple forms for intuitive understanding the essential property of optimal quantization and consider system identification for a simple MA model. The plant is:

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$$\overline{y}(i) = q(y_o(i) + w(i)),\ y_o(i) = \phi(i)\theta, \tag{1}$$
$$\phi(i) := [\, u(i) \quad u(i-1) \quad \cdots \quad u(i-n+1) \,],$$
$$\theta := [\, \theta_1 \quad \theta_2 \quad \cdots \quad \theta_n \,]^{\mathrm{T}},$$

where w is a noise and q is a quantizer defined by

$$q(y) := \mathrm{sgn}(y)\bar{y}_j,\ y \in S_j,\ \bar{y}_j \geq 0,\ S_0 := \{y = 0\},$$
$$S_j := \{y : d_{j-1} < y \leq d_j\},\ j > 0,$$
$$S_j := \{y : d_{j-1} \leq y < d_j\},\ j < 0,$$
$$d_0 = 0 < d_1 < d_2 \cdots,\ d_{-j} = -d_j. \tag{2}$$

In the case of quantization-free, the estimated parameter $\hat{\theta}$ using the least squares method with I/O data $u(i)$ and $y(i) = y_o(i) + w(i)$ is given by

$$\hat{\theta} = (U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}Y, \tag{3}$$
$$U := [\, \phi(1)^{\mathrm{T}} \quad \phi(2)^{\mathrm{T}} \quad \cdots \quad \phi(N)^{\mathrm{T}} \,]^{\mathrm{T}},$$
$$Y := [\, y(1) \quad y(2) \quad \cdots \quad y(N) \,]^{\mathrm{T}}.$$

The estimation (3) is also reasonable according to the criterion of the maximum likelihood with Gaussian noise. With respect to this fact, a question on such property for the case of quantized data arises. In the followings of this section, we focus on showing the reasonability of (3) when $Y = \overline{Y} := [\, \overline{y}(1) \quad \overline{y}(2) \quad \cdots \quad \overline{y}(N) \,]^{\mathrm{T}}$.

2.1 Least Squares Method

In this subsection, we will show the reasonability of (3) with respect to the criterion of the least squares method. A quantized $\overline{y}$ means $y \in S_j$ for some j, then we introduce a one-parameter representation of $\overline{y}$ as

$$[\overline{y}](s) := (1-s)\overline{y}^l + s\overline{y}^u,\ 0 \leq s \leq 1, \tag{4}$$

by using a parameter s. Even if $\overline{y}$ represents a point y as a special case, the representation (4) is still valid with setting $\overline{y}^l = \overline{y}^u = y$.

Next we define operations on the sets as

$$[\overline{a}] + [\overline{b}]$$
$$:= \left((1-s_{\overline{a}})\overline{a}^l + s_{\overline{a}}\overline{a}^u\right) + \left((1-s_{\overline{b}})\overline{b}^l + s_{\overline{b}}\overline{b}^u\right),$$
$$[\overline{a}] \cdot [\overline{b}]$$
$$:= \left((1-s_{\overline{a}})\overline{a}^l + s_{\overline{a}}\overline{a}^u\right) \cdot \left((1-s_{\overline{b}})\overline{b}^l + s_{\overline{b}}\overline{b}^u\right).$$

We also define an inner by

$$\langle [\overline{a}], [\overline{b}] \rangle := \int_0^1\!\!\int_0^1 [\overline{a}] \cdot [\overline{b}] ds_{\overline{a}} ds_{\overline{b}}. \tag{5}$$

The 2-norm on the sets is defined by

$$\|[\overline{x}]\|_2 := \left(\langle \overline{x}, \overline{x} \rangle\right)^{\frac{1}{2}}.$$

When $[\overline{x}]$ is a sets vector, we can also define its 2-norm in a similar way.

Assume the quantized observations $\overline{y}(1)$, $\overline{y}(2)$, ... , $\overline{y}(N)$ are given, then, we define

$$[\overline{Y}] := \begin{bmatrix} (1-s_{\overline{y}(1)})\overline{y}^l(1) + s_{\overline{y}(1)}\overline{y}^u(1) \\ (1-s_{\overline{y}(2)})\overline{y}^l(2) + s_{\overline{y}(2)}\overline{y}^u(2) \\ \vdots \\ (1-s_{\overline{y}(N)})\overline{y}^l(N) + s_{\overline{y}(N)}\overline{y}^u(N) \end{bmatrix}.$$

Let

$$\left\| [\overline{Y}] - [U\tilde{\theta}] \right\|_2^2 \tag{6}$$

be the criterion for estimation and it is calculated by

$$\left\| [\overline{Y}] - [U\tilde{\theta}] \right\|_2^2$$
$$= \langle [\overline{Y}], [\overline{Y}] \rangle - 2\langle [U\tilde{\theta}], [\overline{Y}] \rangle + \langle [U\tilde{\theta}], [U\tilde{\theta}] \rangle,$$

where

$$\langle [\overline{Y}], [\overline{Y}] \rangle = \sum_i \frac{1}{3}\left((\overline{y}^l(i))^2 + \overline{y}^l(i)\overline{y}^u(i) + (\overline{y}^u(i))^2\right),$$
$$\langle [U\tilde{\theta}], [\overline{Y}] \rangle = (U\tilde{\theta})^{\mathrm{T}}\overline{Y},$$
$$\langle [U\tilde{\theta}], [U\tilde{\theta}] \rangle = (U\tilde{\theta})^{\mathrm{T}}(U\tilde{\theta}).$$

Then, (6) is reduced to

$$\langle [\overline{Y}], [\overline{Y}] \rangle - 2(U\tilde{\theta})^{\mathrm{T}}\overline{Y} + (U\tilde{\theta})^{\mathrm{T}}(U\tilde{\theta})$$

and (3) with $Y = \overline{Y}$ is known to be its optimal estimator.

The same conclusion with another least square criterion is also derived. The reasonability of the normal least squares method can be explained as follows: at first, define the evaluated criterion for the estimator $\tilde{\theta}$ by

$$\|y_o - \tilde{y}\|_2^2, \tag{7}$$

where $\tilde{y} := U\tilde{\theta}$ and y_o is the noise-free true output. An observation y means that the most possible true output y' is given by

$$y' = \arg\max_{y''} f_w(y - y''),$$

where f_w is the appropriate probability density of the noise w. Under a reasonable assumption: $\arg\max_w f_w(w) = 0$, we get $y' = y$. Therefore,

$$\|y - \tilde{y}\|_2^2 \tag{8}$$

is the reasonable evaluated criterion and its minimizer $\hat{\theta}$ is given by (3).

Next we follow this procedure for the case of quantized data $\overline{y}$. The probability that the true output is y_o, is given by

$$P(y_o|\overline{y}) = \int_{[\overline{y}]} f_w(y - y_o)dy.$$

If the width of the section $[\overline{y}]$ is enough small with respect to the variance of the noise w, $f_w(y - y_o)$ is nearly constant in the section $[\overline{y}]$. Then, we get an approximation

$$P(y_o|\overline{y}) \sim g^{-1}(\overline{y}) f_w(\overline{y} - y_o).$$

With an assumption: $\arg\max_w f_w(w) = 0$, the maximizer y_o is $y_o = \overline{y}$, therefore, with respect to the criterion (7), (8) with $y = \overline{y}$ is the reasonable criterion and (3) with the substitution $Y = \overline{Y}$ is also derived as the optimal estimator.

2.2 Maximum Likelihood Method

Maximum likelihood method is an essential criterion for parameter estimation in the probabilistic approach. In this subsection, we investigate this criterion for the parameter estimation with quantized data.

When $\overline{y}(i)$ is observed, the likelihood of the parameter θ with the MA model (1) is given by

$$P([\overline{y}](i)|\theta) = \int_{[\overline{y}]} f_w(y - (U\theta)_i)dy. \qquad (9)$$

When the quantizer is enough dense, let $\overline{y}$ be the center of $[\overline{y}](i)$, then, (9) is approximated by

$$P([\overline{y}](i)|\theta) \sim g^{-1}(\overline{y}(i)) f_w(\overline{y}(i) - (U\theta)_i).$$

If $w(i)$ ($i = 1, 2, \ldots$) are mutually independent random signals, the probability of the observations $\overline{y}(1), \overline{y}(2), \ldots, \overline{y}(N)$ is given by

$$\prod_{i=1}^{N} P([\overline{y}](i)|\theta). \qquad (10)$$

Therefore, the maximum likelihood of θ is given as the maximizer of (10). With a calculation:

$$\log \prod_{i=1}^{N} P([\overline{y}](i)|\theta) = \sum_i \log g^{-1}(\overline{y}(i)) + \sum_i \log f_w(\overline{y}(i) - (U\theta)_i), \quad (11)$$

we know that the maximizer of (10) coincides with the maximum likelihood estimator with $y = \overline{y}$. From its direct consequence, when f_w is a normal distribution, then (3) with the substitution $Y = \overline{Y}$ is the reasonable estimator even for the quantized data from the maximum likelihood criterion.

From the above discussions, we conclude that (3) with $Y = \overline{Y}$ is the reasonable estimator according to the criterion (6), (7), or (10) (case of Gaussian of noise).

3. OPTIMAL QUANTIZATION

3.1 Preliminaries

In Section 2, we derive (3) with $Y = \overline{Y}$ is still a reasonable estimator even in the case of quantized data. With this result, in this section, we next consider to derive the optimal quantizer by using the derived $\hat{\theta}$. In (Tsumura and Maciejowski, 2002; Tsumura, 2004), minimization of the variance of $U^{\mathrm{T}}E$ with this estimator was considered, where E is the quantization error vector defined by $\overline{Y} = Y_o + E + W$. In this paper, from the consistency of the criteria discussed in Section 2, we consider to derive the optimal quantizers with respect to

$$\mathrm{E}\left[\|Y_o - U\hat{\theta}\|_2^2\right] = \mathrm{E}\left[W^{\mathrm{T}}UU^{\mathrm{T}}W + E^{\mathrm{T}}UU^{\mathrm{T}}E + 2W^{\mathrm{T}}UU^{\mathrm{T}}E\right]. \qquad (12)$$

The quantizer affects only the second and the third term in (12) and moreover when the quantization is enough dense, the expectation of the second term can be approximated as

$$\mathrm{E}\left[E^{\mathrm{T}}UU^{\mathrm{T}}E\right] = \mathrm{E}\left[\sum_{k=0}^{n-1}\left(\sum_{i=1}^{N} u(i-k)e(i)\right)^2\right] \sim N\mathrm{E}\left[\sum_{k=1}^{n} \phi_k^2(i)e^2(i)\right]. \qquad (13)$$

The right hand side of (13) except for N is written by

$$\mathrm{E}\left[\sum_{k=1}^{n} \phi_k^2(i)e^2(i)\right] = \int \left(\sum_{k=1}^{n} \phi_k^2(i)\right) e^2(i) \times f_\phi(\phi_1(i), \ldots, \phi_n(i)) f_w(w(i)) \times d\phi_1(i) \cdots d\phi_n(i) dw(i) \qquad (14)$$

where f_ϕ is the joint density of $\phi_1, \phi_2, \ldots, \phi_n$.

Now we define subsets Φ_j of the regression vector ϕ associated with the subsection S_j by

$$\Phi_j := \{\phi : y = \phi\theta \in S_j\}, \qquad (15)$$

and consider the following variable transformation (Tsumura and Maciejowski, 2002):

$$y = \phi\theta = \phi T \cdot T^{-1}\theta =: \tilde{\phi}\tilde{\theta}, \; T^{-1}\theta = \begin{bmatrix} \tilde{\theta}_1 \\ O \end{bmatrix},$$

where T is an orthogonal matrix. Then, Φ_j is represented as

$$\Phi_j = \left\{ \phi : \tilde{\phi}_1 \tilde{\theta}_1 \in (d_{j-1},\ d_j] \right\},\ j > 0.$$

We also define subsections on the space of $\tilde{\phi}_1$:

$$I_j := \left\{ \tilde{\phi}_1 : \tilde{\phi}_1 \tilde{\theta}_1 \in (d_{j-1},\ d_j] \right\},\ j > 0,$$

then, the subsections S_j, Φ_j, and I_j correspond to each other, and the probability distribution of y depends only on that of $\tilde{\phi}_1$. Therefore, in order to analyse the probability distribution of y and the error e, the variable $\tilde{\phi}_1$ and its subsection I_j are convenient to deal with.

Owing to the orthogonal transformation of ϕ, (14) is also given by

$$\begin{aligned}
&\int \left(\sum_{k=1}^{n} \phi_k^2(i) \right) e^2(i) f_\phi(\phi_1(i), \ldots, \phi_n(i)) \\
&\quad \times f_w(w(i)) d\phi_1(i) \cdots d\phi_n(i) dw(i) \\
&= \int \left(\sum_{k=1}^{n} \tilde{\phi}_k^2(i) \right) e^2(i) f_{\tilde{\phi}}(\tilde{\phi}_1(i), \ldots, \tilde{\phi}_n(i)) \\
&\quad \times f_w(\tilde{w}(i)) d\tilde{\phi}_1(i) \cdots d\tilde{\phi}_n(i) d\tilde{w}(i),
\end{aligned} \tag{16}$$

where $f_{\tilde{\phi}}(\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_n)$ is the joint density of $\tilde{\phi}_1$, $\tilde{\phi}_2, \ldots, \tilde{\phi}_n$. Here let $\overline{\phi}$ denote

$$\overline{\phi} := [\, \tilde{\phi}_2 \quad \tilde{\phi}_3 \quad \cdots \quad \tilde{\phi}_n \,]^{\mathrm{T}},$$

then, the marginal distribution density $f_{\tilde{\phi}_1}(\tilde{\phi}_1)$ on the space of $\tilde{\phi}_1$ is defined by

$$f_{\tilde{\phi}_1}(\tilde{\phi}_1) := \int f_{\tilde{\phi}}([\, \tilde{\phi}_1 \ \ \overline{\phi}^{\mathrm{T}} \,]^{\mathrm{T}}) d\overline{\phi}.$$

With the fact that the distribution of e is only given by that of $\tilde{\phi}_1$, then (16) is written by

$$\begin{aligned}
&\int e^2 \left(\sum_{k=1}^{n} \tilde{\phi}_k^2 \right) f_{\tilde{\phi}}(\tilde{\phi}_1, \ldots, \tilde{\phi}_n) f_{\tilde{w}}(\tilde{w}) \\
&\quad \times d\tilde{\phi}_1 \cdots d\tilde{\phi}_n d\tilde{w} \\
&= \int e^2 \cdot \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) f_{\tilde{w}}(\tilde{w}) d\tilde{\phi}_1 d\tilde{w},
\end{aligned} \tag{17}$$

where $\sigma(\tilde{\phi}_1)$ is defined by

$$\begin{aligned}
&\int \left(\sum_{k=1}^{n} \tilde{\phi}_k^2 \right) f_{\tilde{\phi}}(\tilde{\phi}_1, \ldots, \tilde{\phi}_n) d\tilde{\phi}_2 \cdots d\tilde{\phi}_n \\
&=: \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1).
\end{aligned} \tag{18}$$

On the other hand, the expectation of $U^{\mathrm{T}}E$ should be zero, therefore,

$$\begin{aligned}
&\mathrm{E}\left[\sum_{i=1}^{N} (\phi_k(i) \cdot e(i)) \right] \\
&=: N \sum_j \mathrm{E}_{\Phi_j}(\phi_k(i) \cdot e(i)) = 0.
\end{aligned} \tag{19}$$

We can also approximate $\mathrm{E}[W^{\mathrm{T}}UU^{\mathrm{T}}E]$ as

$$\begin{aligned}
&\mathrm{E}[W^{\mathrm{T}}UU^{\mathrm{T}}E] \\
&\sim N \int w \cdot e \cdot \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) f_{\tilde{w}}(\tilde{w}) d\tilde{\phi}_1 d\tilde{w}.
\end{aligned}$$

From them, the essential part for the minimization of (12) by the quantization is

$$\begin{aligned}
&\mathrm{E}\left[E^{\mathrm{T}}UU^{\mathrm{T}}E + 2W^{\mathrm{T}}UU^{\mathrm{T}}E \right] \\
&\sim N \int (e^2 + 2w \cdot e) \cdot \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) f_{\tilde{w}}(\tilde{w}) d\tilde{\phi}_1 d\tilde{w}.
\end{aligned} \tag{20}$$

With this in mind, we next consider the minimization of (20) such that $\mathrm{E}_{I_j}(\phi_k(i) \cdot e(i)) = 0\ (\forall j)$ (or $\mathrm{E}_{I_j}(\tilde{\phi}_k(i) \cdot e(i)) = 0$) under constraints of the total number of the quantized subsections or the code length when the quantized data is optimally encoded. The former case is called "fixed-rate quantization" and the latter case is called "variable-rate quantization." The difficulty to solve these problems is the calculation of (20) or the quantities in the constraints in analytic forms. In the following subsections, we solve this difficulty for high resolution case.

3.2 Fixed-rate Quantization

The key idea to solve these problems is introducing the following quantity on the distribution of quantization subsections.

Definition 3.1. The quantity $g(\tilde{\phi}_1)$ which satisfies $g(\tilde{\phi}_1)d\tilde{\phi}_1$ is equal to the number of quantized subsections in $d\tilde{\phi}_1$, is called distribution density of the number of quantized subsections.

Note that this quantity is the same introduced in (Bennett, 1948; Lloyd, 1982) and from this definition, $g^{-1}(\tilde{\phi}_1)$ represents the width of the quantization step at $\tilde{\phi}_1$.

By using this g, the minimization problem is written as

$$\min_g \mathrm{E}\left[\|Y_o - U\hat{\theta}\|_2^2 \right]. \tag{21}$$

We next assume "smoothness" of the density $g(\tilde{\phi}_1)$ and $f_{\tilde{\phi}_1}(\tilde{\phi}_1)$ such that we can select the mean values

$g_i \sim g(\tilde{\phi}_1)$ and $f_i \sim f_{\tilde{\phi}_1}(\tilde{\phi}_1)$ for the subsection I_i and

$$p_i := \int_{I_i} f_{\tilde{\phi}_1}(\tilde{\phi}_1) d\tilde{\phi}_1 =: f_i g_i^{-1}.$$

Moreover, by using $\sigma(\tilde{\phi}_1)$ of $f_{\tilde{\phi}}$ at $\tilde{\phi}_1$ defined in (18), an approximation of (14) (or (16), (17)) can be given by

$$\begin{aligned} &\int e^2 \left(\sum_{k=1}^{n} \tilde{\phi}_k^2 \right) f_{\tilde{\phi}}(\tilde{\phi}_1, \ldots, \tilde{\phi}_n) f_{\tilde{w}}(\tilde{w}) \\ &\quad \times d\tilde{\phi}_1 \cdots d\tilde{\phi}_n d\tilde{w} \\ &= \int e^2 \cdot \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) f_{\tilde{w}}(\tilde{w}) d\tilde{\phi}_1 d\tilde{w} \\ &\sim \int \frac{1}{12} g(\tilde{\phi}_1)^{-2} \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) d\tilde{\phi}_1. \end{aligned} \tag{22}$$

Now we define the followings.

$$\overline{w} := \int_0^{\infty} \tilde{w} \cdot f_{\tilde{w}}(\tilde{w}) d\tilde{w} \tag{23}$$

$$w_o := w' | \int_0^{\infty} (\tilde{w} - w') \cdot f_{\tilde{w}}(\tilde{w}) d\tilde{w} = 0 \tag{24}$$

$$F(\tilde{\phi}_1) := \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) \tag{25}$$

As similar to (22),

$$\begin{aligned} &\int e \cdot w \cdot \left(\sum_{k=1}^{n} \tilde{\phi}_k^2 \right) f_{\tilde{\phi}_1}(\tilde{\phi}_1, \ldots, \tilde{\phi}_n) f_{\tilde{w}}(\tilde{w}) \\ &\quad \times d\tilde{\phi}_1 \cdots d\tilde{\phi}_n d\tilde{w} \\ &= 2\overline{w} w_o \int \frac{1}{12} g(\tilde{\phi}_1)^{-2} \left(\sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) \right)'' d\tilde{\phi}_1 \end{aligned} \tag{26}$$

is derived. Therefore,

$$\begin{aligned} &\mathrm{E}\left[E^{\mathrm{T}} U U^{\mathrm{T}} E + 2 W^{\mathrm{T}} U U^{\mathrm{T}} E \right] \\ &\sim N \int \frac{1}{12} g(\tilde{\phi}_1)^{-2} R(\tilde{\phi}_1) d\tilde{\phi}_1, \end{aligned} \tag{27}$$

$$\begin{aligned} R(\tilde{\phi}_1) &:= \sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) \\ &\quad + 4\overline{w} w_o \left(\sigma^2(\tilde{\phi}_1) f_{\tilde{\phi}_1}(\tilde{\phi}_1) \right)''. \end{aligned} \tag{28}$$

From the above approximation, the original minimization problem at $N \to \infty$ and high resolution case can be replaced by the following:

$$g_{\mathrm{f}}(\tilde{\phi}_1) := \arg\min_{g} \int \mathcal{F}(g(\tilde{\phi}_1),\ G(\tilde{\phi}_1)) d\tilde{\phi}_1 \tag{29}$$

$$\text{s.t. } G(\tilde{\phi}_1^{\min}) = 0,\ G(\tilde{\phi}_1^{\max}) = M, \tag{30}$$

where

$$\mathcal{F}(g(\tilde{\phi}_1),\ G(\tilde{\phi}_1)) := \left(\frac{1}{g(\tilde{\phi}_1)} \right)^2 R(\tilde{\phi}_1) \tag{31}$$

$$\frac{d}{d\tilde{\phi}_1} G(\tilde{\phi}_1) = g(\tilde{\phi}_1). \tag{32}$$

By employing the similar process in (Bennett, 1948; Lloyd, 1982), we can derive the next result.

Proposition 3.1. The solution of (29) is:

$$g_{\mathrm{f}}(\tilde{\phi}_1) = K R(\tilde{\phi}_1)^{\frac{1}{3}} \tag{33}$$

$$K = D^{-1} M \tag{34}$$

$$D = \int R(\tilde{\phi}_1)^{\frac{1}{3}} d\tilde{\phi}_1. \tag{35}$$

Moreover, the optimized value is given by

$$\int \mathcal{F}(g_{\mathrm{f}}(\tilde{\phi}_1),\ G_{\mathrm{f}}(\tilde{\phi}_1)) d\tilde{\phi}_1 = D^3 M^{-2}. \tag{36}$$

From this result, the asymptotic optimal quantization at high resolution case is easily calculated if the marginal distribution of the regressor vector $f_{\tilde{\phi}_1}(\tilde{\phi}_1)$ is known.

3.3 *Variable-rate Quantization*

In the previous subsection, we derived an optimal quantizer to minimize the identification error under the constraint of the number of quantization steps, i.e., fixed-rate quantization, in the case of high resolution. On the other hand, for the purpose to reduce the information of the observed data, it is more reasonable to apply variable-rate coding for the quantized signals and measure the mean code length as the quantity of the information. According to this observation, we consider the minimization of (21) under constraint of the expectation of the optimal code length, that is, variable-rate quantization, in high resolution case.

Let $C(\cdot)$ be an encoder which is a mapping from source alphabets to code alphabets and $l(\cdot)$ the code length. We regard $q(\tilde{\phi}_1)$ as the corresponding source alphabets, then, $l(C(q(\tilde{\phi}_1)))$ represents the code length of the code alphabets. The expectation of the variable-rate optimal code length for a quantized signal has relation with entropy of the source from the following well-known proposition.

Proposition 3.2. (Shannon, 1948),
(Cover and Thomas, 1991) Let x be source alphabets, then,

$$\mathrm{E}[l(C(x))] \geq H(x), \tag{39}$$

where $H(x)$ represents the entropy of x.

With this proposition, the optimization problem of the quantizer for the code length is reduced to the same problem under constraint of entropy of the quantized signals.

On the other hand, the basic idea and tools to represent the quantity (21) or the quantizer in high resolution case are the same of the previous subsection. Then, we can get the asymptotic approximation of the entropy of the quantized signal:

$$
\begin{aligned}
&H(f,g) := \sum_i -p_i \log p_i \\
&= \sum_i - \int_{I_i} f_{\tilde{\phi}_1}(\tilde{\phi}_1) d\tilde{\phi}_1 \log f_i g_i^{-1} \\
&\sim H_{\mathrm{d}}(f_{\tilde{\phi}_1}) \\
&\quad + \int -f_{\tilde{\phi}_1}(\tilde{\phi}_1) \log \left(g(\tilde{\phi}_1)^{-1} \right) d\tilde{\phi}_1, \qquad (40)
\end{aligned}
$$

where $H_{\mathrm{d}}(f) := \int -\log f dF$. By using (40), we consider the following problem for high resolution case.

$$
g_{\mathrm{v}}(\tilde{\phi}_1) := \arg\min_g \int \mathcal{F}(g(\tilde{\phi}_1),\ G(\tilde{\phi}_1)) d\tilde{\phi}_1 \qquad (41)
$$

$$
\text{s.t. } H(f,g) = \log M \qquad (42)
$$

Note that M is an expected number of quantization steps in the sense of (42).

By employing the similar process in (Gish and Pierce, 1968; Berger, 1972), we can derive the next result.

Proposition 3.3. The solution of (41) is:

$$
\begin{aligned}
g_{\mathrm{v}}(\tilde{\phi}_1) &= KMR(\tilde{\phi}_1)^{\frac{1}{2}} f_{\tilde{\phi}_1}(\tilde{\phi}_1)^{-\frac{1}{2}} && (43) \\
K &= \exp L && (44) \\
L &:= -\frac{3}{2} H(f) - \frac{1}{2} \int f_{\tilde{\phi}_1} \log R(\tilde{\phi}_1) d\tilde{\phi}_1 \\
&= \int f_{\tilde{\phi}_1}(\tilde{\phi}_1) \log \frac{f_{\tilde{\phi}_1}(\tilde{\phi}_1)^{\frac{3}{2}}}{R(\tilde{\phi}_1)^{\frac{1}{2}}} d\tilde{\phi}_1 && (45)
\end{aligned}
$$

Moreover, the optimized value is given by

$$
\int \mathcal{F}(g_{\mathrm{v}}(\tilde{\phi}_1),\ G_{\mathrm{v}}(\tilde{\phi}_1)) d\tilde{\phi}_1 = K^{-2} M^{-2}. \qquad (46)
$$

As the previous proposition, the optimal quantizer for the code length can be explicitly given by using $f(\tilde{\phi}_1)$.

4. CONCLUSION

In this paper, we investigated the reasonability of the ordinary parameter estimator for the quantized data. We showed that it is also reasonable for the criteria of least squares errors or maximum likelihood. Then, we extended the results of optimal quantization problem for system identification given by (Tsumura and Maciejowski, 2002) and (Tsumura, 2004) for a criterion given in this paper. We consider two cases of the optimization: fixed-rate quantization and variable-rate quantization. For high resolution case, we explicitly derived the optimal quantizations and the minimized quantization errors for these two cases.

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