RELIABILITY AND EFFICIENCY OF EXTENDED LINEARIZATION ALGORITHMS FOR GENERAL ROBUST CONTROL PROBLEMS

Tsuyoshi KIYAMA * **Takumi SAKAMOTO** *

* *Graduate School of Engineering, Osaka University, 2-1, Yamada-oka, Suita, Osaka, 565-0871, JAPAN*
{tsuyoshi, sakamoto}@watt.mech.eng.osaka-u.ac.jp

Abstract: This paper proves that a certain class of nonconvex matrix inequalities is equivalent to linear matrix inequalities (LMIs) plus a nonconvex rank constraint. From the equivalence, this paper proposes two heuristic algorithms, that is extended linearization algorithms, to solve LMIs with a rank constraint using LMI-based approach. Reliability and efficiency of the algorithms are investigated statistically, and then extensive numerical experiments will indicate that the algorithms have decent performances from the viewpoint of computation in comparison with the existing method: the standard alternating projection method. It is also important that our approaches can be applied to a large number of other rank-minimization problems over LMIs, for example, the robust well-posedness problem which is an extension of general robust control problems.

Keywords: Ranks, Relaxation, Statistical analysis, Computer-aided design, PI controllers, Stabilizing controllers, Convex optimisation, Multipliers.

1. INTRODUCTION

In the last decade, the usefulness of linear matrix inequalities (LMIs) for control systems analysis and synthesis has been recognized with the aid of powerful computational algorithms to solve optimization/feasibility problems involving LMIs. Moreover, not only problems in terms of LMIs but those of LMIs plus a nonconvex rank constraint can be solved to some level via LMI-based approaches (Grigoriadis and Skelton 1996, Ghaoui *et al*. 1997, Iwasaki 1999, Apkarian and Tuan 2000, Fazel *et al*. 2003, Kiyama and Nishio 2004). Among these, the linearization (Ghaoui *et al*. 1997) and the dual iteration algorithm (Iwasaki 1999) produce excellent results for the problems with a rank constraint. Unfortunately, these approaches restrict the structures of the matrix with the rank constraint, equivalently control problems and the concave minimization algorithm (Apkarian and Tuan 2000) requires the linearizing change of variable, and then the general case of the structures remains open problem for these three approaches. On the other hand, there exist the standard alternating projection method (Grigoriadis and Skelton 1996), Log-det heuristics method (Fazel *et al*. 2003), the augmented Lagrangian method (Fares *et al*. 2001) and so on which do not restrict the matrix structures, however, these approaches require more computational complexity and new adding variables for the computations.

First, this paper proves that a certain larger class of nonconvex matrix inequalities is equivalent to LMIs plus a nonconvex rank constraint than that of (Ghaoui *et al*. 1997, Iwasaki 1999). Two types of the convex relaxation conditions without the rank constraints lead to our algorithms, and then this paper proposes two heuristic algorithms: an extension of the linearization algorithm to solve LMIs plus a rank constraint using LMI-based approaches. Note that it is the exclusive point in our approaches that the adding variables like (Fazel *et al*. 2003, Fares *et al*. 2001) and the linearizing change of variable like (Apkarian and Tuan 2000, Fares *et al*. 2001) are not necessary.

Next, this paper derives new solvability conditions of the robust well-posedness problem (Skelton *et al*. 1997, Iwasaki and Hara 1998): an extension of general robust control problems in order that our approaches can be applied to a large number of other problems.

Finally, reliability and efficiency of our algorithms are investigated statistically. That is, by numerical experiments, we show how likely it is for our algorithms to stabilize the systems via PI controllers that are known to be stabilizable, with how much computational burden. The extensive numerical experiments will indicate that our algorithms have a decent performance from the viewpoint of computation in comparison with the existing method: the standard alternating projection method (Grigoriadis and Skelton 1996). Moreover, we will clarify comparative merits and demerits of our proposed algorithms and the other one through the extensive numerical experiments.

Notation: The sets of $n\times m$ real and complex matrices are denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$. The $n\times n$-identity matrix is denoted by I_n. For a matrix $M=[m_{ij}]$, m_{ij} denotes the element of (i,j), M' denotes the transpose and $\sigma_{min}(M)$ denotes the smallest singular value. $M^{\perp}\in\mathbb{R}^{(n-r)\times n}$ satisfies $r=$ rank of M, $M^{\perp}M=0$ and $M^{\perp}M^{\perp\prime}>0$. For a symmetric matrix X, $X>0$ ($X\geq 0$) means that X is positive (semi)definite. $\mathtt{diag}(M_1,M_2)$ means a block diagonal matrix composed of a matrix M_1 and a matrix M_2.

2. BASIC RESULT

We first prove the following basic lemma (Kiyama and Nishio 2004) which will be used to derive extended linearization algorithms in this paper.

Lemma 1. A real symmetric matrix Π is given by

$$\Pi := \begin{bmatrix} R & S \\ S' & Q \end{bmatrix},$$

where $Q>0$. Then the following four statements are equivalent.

(i) $R-SQ^{-1}S'<0$.

(ii) $[\,S'\ \ Q'\,]'^{\perp}\,\Pi\,[\,S'\ \ Q'\,]'^{\perp\prime}<0$.

(iii) There exists a real symmetric matrix W satisfying

$$\Pi<W,\quad W\geq 0,\quad \mathrm{rank}(W)=\mathrm{rank}(Q). \quad (1)$$

(iv) There exists a real symmetric matrix U satisfying

$$\Upsilon := \begin{bmatrix} R+U & S \\ S' & Q \end{bmatrix}\geq 0,\quad U>0,$$
$$\mathrm{rank}(\Upsilon)=\mathrm{rank}(Q). \quad (2)$$

Proof. First, we prove that statement (i) $\Leftrightarrow$ statement (ii). From $Q>0$, there exists Q^{-1}. Then

$$R-SQ^{-1}S' = [\,-I\ \ SQ^{-1}\,]\,\Pi\,[\,-I\ \ SQ^{-1}\,]' = [\,S'\ \ Q'\,]'^{\perp}\,\Pi\,[\,S'\ \ Q'\,]'^{\perp\prime}<0.$$

Next, we prove that statement (ii) $\Rightarrow$ statement (iii). Suppose statement (ii) holds. From the equivalence of statement (i) and statement (ii), and Finsler's theorem (Skelton *et al.* 1997),

$$\begin{aligned} &R-SQ^{-1}S'<0 \quad \Leftrightarrow \\ &{}^{\exists}\mu>0 \ \text{ s.t. } \ \Pi<\mu\,[\,S'\ \ Q'\,]'\,[\,S'\ \ Q'\,], \\ &W:=\mu\,[\,S'\ \ Q'\,]'\,[\,S'\ \ Q'\,]\geq 0. \end{aligned}$$

Then statement (iii) holds. Conversely, we prove that statement (iii) $\Rightarrow$ statement (ii). Suppose statement (iii) holds. If there exists W satisfying statement (iii), then, from the following full-column-rank decomposed representation of W:

$$W =: [\,F'\ \ J'\,]'\,[\,F'\ \ J'\,]\geq 0,\quad JJ'>Q>0$$

and Finsler's theorem,

$$\begin{aligned} &[\,-I\ \ FJ^{-1}\,]\,\Pi\,[\,-I\ \ FJ^{-1}\,]' \\ &= R-SQ^{-1}S' \\ &\quad +(SQ^{-1}-FJ^{-1})Q(SQ^{-1}-FJ^{-1})'<0 \\ &\Rightarrow R-SQ^{-1}S'<0, \end{aligned}$$

and then statement (i), equivalently, statement (ii) holds. Finally, we prove that statement (iv) $\Leftrightarrow$ statement (i). Suppose statement (iv) holds. If there exists $U>0$ satisfying statement (iv), then

$$\Upsilon = \begin{bmatrix} R+U & S \\ S' & Q \end{bmatrix} = \begin{bmatrix} I & SQ^{-1} \\ 0 & I \end{bmatrix} \times \begin{bmatrix} R+U-SQ^{-1}S' & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} I & 0 \\ Q^{-1}S' & I \end{bmatrix}\geq 0$$

holds, and then, from the rank constraint (2),

$$R+U-SQ^{-1}S'=0,$$
$$R-SQ^{-1}S'=-U<0$$

and statement (i) hold. Similarly, from the above reverse operation, statement (i) $\Rightarrow$ statement (iv). □

When we utilize the following condition:

$$\Pi<W,\quad W\geq 0$$

except for the rank constraint from (1), statement (iii) becomes a convex relaxation condition of statement (i), equivalently, statement (ii). Similarly, when we utilize the following condition:

$$\Upsilon = \begin{bmatrix} R+U & S \\ S' & Q \end{bmatrix}\geq 0,\quad U>0$$

except for the rank constraint (2), statement (iv) becomes a convex relaxation condition of statement (i), equivalently, statement (ii).

3. GENERAL PROBLEM AND ALGORITHMS

In this section, we describe two simple and effective algorithms in order to solve the conditions in Lemma 1, the robust well-posedness problem (Skelton *et al.* 1997, Iwasaki and Hara 1998) and so on.

3.1 General Problem

Here we clarify the above problem in order to solve the conditions in Lemma 1 as follows:

Original Problem: Find a scalar $\mu > 0$, a real symmetric matrix R and a real matrix S satisfying

$$R - S(\mu I_n)^{-1}S' < 0.$$

This original problem is transformed to the following two equivalent ones: Problem 1 and Problem 2 from Lemma 1.

Problem 1: Find a scalar $\mu > 0$, real symmetric matrices $W \geq 0$ and R, and a real matrix S satisfying

$$\tilde{\Pi} := \begin{bmatrix} R & S \\ S' & \mu I_n \end{bmatrix} < W := \begin{bmatrix} W_{11} & W_{12} \\ W_{12}' & wI_n \end{bmatrix}, \tag{3}$$

$$\text{rank } (W) = \text{rank } (\mu I_n) = n. \tag{4}$$

Problem 2: Find a scalar $\mu > 0$, real symmetric matrices $U > 0$ and R, and a real matrix S satisfying

$$\tilde{\Upsilon} := \begin{bmatrix} R+U & S \\ S' & \mu I_n \end{bmatrix} \geq 0, \tag{5}$$

$$\text{rank } (\tilde{\Upsilon}) = \text{rank } (\mu I_n) = n. \tag{6}$$

3.2 Extended linearization algorithms

Two algorithms are described below. These approaches are extensions of the linearization algorithm in the reference (Ghaoui *et al.* 1997). Hence, we call the each approach an extended linearization algorithm.

Algorithm 1

1. Solve $\tilde{\Pi}$, W s.t. $\tilde{\Pi} < W$, $W \geq 0$ and $\mu > 0$.
2. $W_{11}^0 := W_{11}$, $W_{12}^0 := W_{12}$, $w^0 := w$ and let $j = 1$.
3. Fix $\mathscr{W}_{11} := W_{11}^{j-1}$, $\mathscr{W}_{12} := W_{12}^{j-1}$, $\tilde{w} := w^{j-1}$. Solve $\lambda_j := \min_{\mu>0, W\geq 0, R, S}$

$$\begin{cases} \mathrm{tr}[(\tilde{w}W_{11} + w\mathscr{W}_{11} - \tilde{w}\mathscr{W}_{11}) \\ -(W_{12}\mathscr{W}_{12}' + \mathscr{W}_{12}W_{12}' - \mathscr{W}_{12}\mathscr{W}_{12}')] \end{cases} \quad \text{s.t.} \quad (3)$$

   and let $W_{11}^j := W_{11}$, $W_{12}^j := W_{12}$, $w^j := w$.
4. If $\left|\frac{\lambda_j - \lambda_{j-1}}{\lambda_j}\right| \leq \tau$ for sufficiently small $\tau > 0$, then stop. Otherwise let $j \to j-1$ and go to **3.**.

This algorithm is based on a convex relaxation approach which satisfies the rank constraint (4) step by step from the convex relaxation condition without the rank constraint. Refer to (Kiyama and Nishio 2004) for further information.

This paper proposes another convex relaxation approach to compare with Algorithm 1 as follows.

Algorithm 2

1. Solve $\tilde{\Upsilon}$ s.t. $\tilde{\Upsilon} \geq 0$, $U > 0$ and $\mu > 0$.
2. $R^0 := R$, $S^0 := S$, $U^0 := U$, $\mu^0 := \mu$ and let $j = 1$.
3. Fix $\mathscr{R} := R^{j-1}$, $\mathscr{S} := S^{j-1}$, $\mathscr{U} := U^{j-1}$, $\tilde{\mu} := \mu^{j-1}$. Solve $\lambda_j := \min_{U>0, \mu>0, R, S}$

$$\begin{cases} \mathrm{tr}[\tilde{\mu}(R+U) + \mu(\mathscr{R}+\mathscr{U}) - \tilde{\mu}(\mathscr{R}+\mathscr{U}) \\ -(S\mathscr{S}' + \mathscr{S}S' - \mathscr{S}\mathscr{S}')] \end{cases} \tag{7}$$
$$\text{s.t.} \quad (5)$$

   and let $R^j := R$, $S^j := S$, $U^j := U$, $\mu^j := \mu$.
4. If $\left|\frac{\lambda_j - \lambda_{j-1}}{\lambda_j}\right| \leq \tau$ for sufficiently small $\tau > 0$, then stop. Otherwise let $j \to j-1$ and go to **3.**.

We explain Algorithm 2 in the following explanation 1 and 2 similar to (Kiyama and Nishio 2004). First, we explain the numerical computational method to satisfy the rank constraint (6) step by step from the convex relaxation condition without the rank constraint.

Explanation 1 of Algorithm 2

The following equation

$$\Upsilon = \begin{bmatrix} I & S/\mu \\ 0 & I_n \end{bmatrix} \begin{bmatrix} R+U-SS'/\mu & 0 \\ 0 & \mu I_n \end{bmatrix} \times \begin{bmatrix} I & 0 \\ S'/\mu & I_n \end{bmatrix}$$

holds. If $R+U-SS'/\mu = 0$, then the rank constraint (6) holds. We can choose

$$\mathrm{tr}[\mu(R+U) - SS'] \tag{8}$$

as the objective function to be approached to 0 subject to LMIs and $\mathrm{tr}[\mu(R+U) - SS'] \geq 0$ due to $\Upsilon \geq 0$ and $\mu > 0$. This is a simple and effective key idea of our method.

Explanation 2 of Algorithm 2

Next, a linear approximated function of a trace function of a matrix product term $F(X,Y) = \mathrm{tr}[XY]$ defined by real matrices X and Y is considered as the following lemma.

Lemma 2. Let a trace function of a matrix product term $F(X,Y) = \mathrm{tr}[XY]$ defined by real matrices $X = [x_{ij}] \in \mathbb{R}^{p\times q}$ and $Y = [y_{ij}] \in \mathbb{R}^{q\times p}$, and a fixed point $(X_0, Y_0) = ([x_{ij}^0], [y_{ij}^0])$ on the matrices (X,Y) be given. Then a linear approximated function of $F(X,Y) = \mathrm{tr}[XY]$ on the point $(X,Y) = (X_0, Y_0)$ is

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\Big\{\frac{\partial}{\partial x_{ij}} x_{ij}y_{ji}|_{y_{ji}=y_{ji}^0}(x_{ij} - x_{ij}^0) + \frac{\partial}{\partial y_{ji}} x_{ij}y_{ji}|_{x_{ij}=x_{ij}^0}(y_{ji} - y_{ji}^0) + x_{ij}^0 y_{ji}^0\Big\}$$
$$= \mathrm{tr}[XY_0 + X_0Y - X_0Y_0]. \tag{9}$$

Proof. A linear approximated function of a function $x_{ij}y_{ji}$ on a fixed point $(x_{ij}, y_{ji}) = (x_{ij}^0, y_{ji}^0)$ is

$$
\frac{\partial}{\partial x_{ij}} x_{ij} y_{ji}|_{y_{ji}=y_{ji}^0} (x_{ij} - x_{ij}^0)
$$

$$
+ \frac{\partial}{\partial y_{ji}} x_{ij} y_{ji}|_{x_{ij}=x_{ij}^0} (y_{ji} - y_{ji}^0) + x_{ij}^0 y_{ji}^0.
$$

From $\mathrm{tr}[XY] = \sum_{i=1}^{p} \sum_{j=1}^{q} x_{ij} y_{ji}$, we can see that (9) becomes the linear approximated function of $F(X,Y)$ on the point $(X,Y) = (X_0, Y_0)$. $\square$

Consequently, from the above lemma , we can understand that (7) is the linear approximated function of (8) at the fixed point $(\mu, R, S, U) = (\tilde{\mu}, \mathscr{R}, \mathscr{S}, \mathscr{U})$.

4. ROBUST WELL-POSEDNESS PROBLEM

This section is concerned with the robust well-posedness problem (Skelton *et al.* 1997, Iwasaki and Hara 1998) which is an extension of the $\mathscr{H}_\infty$ control problem with constant scaling matrices and various control system synthesis problems.

Here we consider the feedback control system consists of constant matrices which are connected each other as shown in Fig. 1, where $\mathscr{M}$ is a given real matrix and ∇ is an element of matrices in a known complex subset $\boldsymbol{\nabla}$. This uncertain system is robustly stable if and only if there exists a real number ε satisfying

$$
\sigma_{min} \begin{pmatrix} I & -\mathscr{M} \\ -\nabla & I \end{pmatrix} \geq \varepsilon > 0, \quad {}^\forall \nabla \in \boldsymbol{\nabla}.
$$

We call this characterization of the feedback system in Fig. 1 the robust well-posedness (Skelton *et al.* 1997, Iwasaki and Hara 1998).

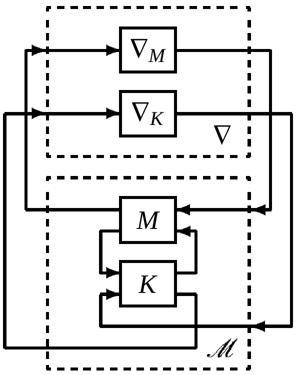

Fig. 1. Uncertain system.

Then we can define the robust well-posedness problem based on the robust well-posedness. We consider the feedback control system depicted in Fig. 1, where M is a given real matrix and ∇_M and ∇_K are elements of matrices in a known complex subset $\boldsymbol{\nabla}$. That is $\nabla := \mathrm{diag}(\nabla_M, \nabla_K)$. We are now ready to define the robust well-posedness problem.

> **Robust well-posedness problem**: When let a real matrix M and a complex subset $\boldsymbol{\nabla}$ be given, find a real matrix K such that a feedback system becomes well posed.

Here we divide the matrix M from 2 inputs to 2 outputs in Fig. 1 and M is represented by

$$
M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.
$$

Moreover, a matrix L is defined by

$$
L := \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} := \left[\begin{array}{cc|cc} M_{11} & 0 & M_{12} & 0 \\ 0 & 0 & 0 & I \\ \hline M_{21} & 0 & M_{22} & 0 \\ 0 & I & 0 & 0 \end{array}\right], \qquad (10)
$$

where, for technical simplicity, we assume $M_{22} = 0$ and choose an appropriate size of the matrix L such that

$$
\mathscr{M} = L_{11} + L_{12} K L_{21}
$$

holds.

Using Lemma 1, we have the following theorem similar to (Kiyama *et al.* 2002):

Theorem 1. Let a real matrix M and a complex subset $\boldsymbol{\nabla}$ be given. L_{11}, L_{12} and L_{21} are defined by (10), and a real symmetric matrix Θ is defined by

$$
\Theta := \begin{bmatrix} \mathscr{R} & \mathscr{S} \\ \mathscr{S}' & \mathscr{Q} \end{bmatrix}.
$$

Then the robust well-posedness problem is solvable if and only if there exist matrices $\mathscr{Q} > 0$, Θ, K, $\mathscr{W}$ and a real number $\mu > 0$ satisfying

$$
\begin{bmatrix} \mathscr{R} & \mathscr{S} & \mathscr{M} \\ \mathscr{S}' & \mathscr{Q} & -I \\ \mathscr{M}' & -I & \mu I_k \end{bmatrix} < \mathscr{W}, \quad \mathscr{W} \geq 0, \quad \mathrm{rank}(\mathscr{W}) = k,
$$

$$
[\nabla \ I] \Theta [\nabla \ I]^* \geq 0, \quad {}^\forall \mathrm{diag}(\nabla_M, \nabla_K) \in \boldsymbol{\nabla},
$$

equivalently, if and only if there exist matrices $\mathscr{Q} > 0$, Θ, K, U and a real number $\mu > 0$ satisfying

$$
\Upsilon := \begin{bmatrix} \mathscr{R} + U_{11} & \mathscr{S} + U_{12} & \mathscr{M} \\ \mathscr{S}' + U_{12}' & \mathscr{Q} + U_{22} & -I \\ \mathscr{M} & -I & \mu I_k \end{bmatrix} \geq 0,
$$

$$
U := \begin{bmatrix} U_{11} & U_{12} \\ U_{12}' & U_{22} \end{bmatrix} > 0, \quad \mathrm{rank}(\Upsilon) = k,
$$

$$
[\nabla \ I] \Theta [\nabla \ I]^* \geq 0, \quad {}^\forall \mathrm{diag}(\nabla_M, \nabla_K) \in \boldsymbol{\nabla}.
$$

Proof. The proof is straightforward from Lemma 1. $\square$

It is well known that the existing solvability conditions (Skelton *et al.* 1997, Scherer 1996, Fares *et al.* 2001) include both the matrix variable Θ and Θ^{-1} for the robust well-posedness problem. On the other hand, Theorem 1 does not include Θ^{-1} in the conditions. At the sacrifice without Θ^{-1}, the rank constraints appear in Theorem 1 and they cause the nonconvexity in the solvability conditions. However, we can easily see that these new nonconvex conditions can be checked approximately with our extended linearization algorithms, the alternating projection method and so on. This is contrast with the existing results, where we introduce a new variable X which corresponds to Θ^{-1}, a nonconvex constraint of $\Theta X = I$ is added to the original solvability conditions, and these conditions with the constraint of $\Theta X = I$ must be checked by a numerical method. In this meaning of easy computation, our result of solvability conditions is very important.

5. NUMERICAL EXPERIMENTS

We will investigate reliability and efficiency of the two extended linearization algorithms through numerical experiments. For simplicity, this section considers the following stabilization problem in the special case of the robust well-posedness problem.

5.1 Stabilization problem

Consider the feedback system depicted in Fig. 2

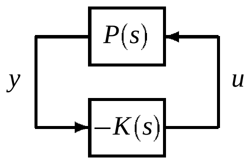

Fig. 2. Feedback system.

where $P(s)$ is a given single-input, single-output (SISO) linear time-invariant (LTI) nominal plant represented by

$$\begin{bmatrix} \dot{x}_p \\ \hline y \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} x_p \\ \hline u \end{bmatrix}, \qquad (11)$$

and $K(s)$ is SISO PI controller given by

$$\begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1 \\ \hline c_0 & c_1 \end{array}\right] \begin{bmatrix} x_c \\ y \end{bmatrix}.$$

Then the closed loop system in Fig. 2 is denoted by

$$\begin{aligned} \begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} &= \begin{bmatrix} A - BD_cC & B(I - D_cD) \\ -A_cC & -A_cD \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} \\ &=: \begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} =: \mathscr{A} \begin{bmatrix} x_p \\ x_c \end{bmatrix}, \\ A_c := \frac{c_0}{1 + c_1 D}&, \quad D_c := \frac{c_1}{1 + c_1 D}. \end{aligned}$$

Hence the stabilization problem can be recast as the following problem:

> **Problem:** Find scalars $\mu > 0$, A_c, D_c, and a matrix $P > 0$ satisfying
>
> $$\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} - \begin{bmatrix} \mathscr{A}' \\ -I \end{bmatrix} (\mu I)^{-1} \begin{bmatrix} \mathscr{A}' \\ -I \end{bmatrix}' < 0. \qquad (12)$$

Note that the class of this problem is equivalent to that of Original Problem in Subsection 3.1. After solving the above problem, the controller parameters are obtained as follows:

$$c_0 = \frac{A_c}{1 - D_c D}, \quad c_1 = \frac{D_c}{1 - D_c D}.$$

5.1.1. *Random system*

Reliability and efficiency of the two extended linearization algorithms are investigated statistically. That is, by numerical experiments, we show how likely it is for the two extended linearization algorithms to stabilize the systems that are known to be stabilizable, with how much computational burden.

A number of systems, stabilizable via SISO PI controller, are randomly generated where the systems have one unstable pole and zero at least. The procedure is as follows:

Procedure for $P(s)$

1. Scalars c_0, c_1 and $\varepsilon > 0$, and a matrix A are randomly generated.
2. If A is stable, then $\mathscr{A} :=$ A. Otherwise $\mathscr{A} :=$ A $- \lambda I - \varepsilon I$ where λ means the maximum real part of eigenvalues of A.
3. If $c_0 + c_1 \mathscr{A}_{22} = 0$, then go to **1.**. Otherwise compute the state-space matrices of the plant transfer function $P(s)$ in (11):

$$\begin{aligned} A &:= \mathscr{A}_{11} - \mathscr{A}_{12}\mathscr{A}_{21}c_1/(c_0 + c_1\mathscr{A}_{22}), \\ B &:= \mathscr{A}_{12}c_0/(c_0 + c_1\mathscr{A}_{22}), \\ C &:= -\mathscr{A}_{21}/(c_0 + c_1\mathscr{A}_{22}), \\ D &:= -\mathscr{A}_{22}/(c_0 + c_1\mathscr{A}_{22}). \end{aligned}$$

4. If $P(s)$ has one unstable pole and zero at least, then stop. Otherwise go to **1.**.

5.2 Results

This subsection describes results of numerical experiments. First, the two extended linearization algorithms are applied for the 2-nd order 600 systems thus generated by the procedure for $P(s)$. The algorithms are stopped when the matrix inequality (12) becomes negative definite. That is, we do not optimize the stability degree but just try to stabilize the system. If the number of iteration exceeds 2000 while the matrix inequality (12) is non-negative definite, then the algorithm is stopped and we conclude that the algorithm fails to stabilize the system.

For the same 600 systems, another typical existing method, for example, the standard alternating projection method (APM) (Grigoriadis and Skelton 1996) for output feedback stabilization via PI controller is also applied for comparison.

Table 1. Comparison with algorithms.

	Algorithm 1	Algorithm 2	APM
number of successful times	585	500	489
success rate [%]	97.5	83.3	81.5
average CPU time	5.4	39.9	670.0
average iteration	7.6	83.1	438.5

Table 1 summarizes the results of comparison with the algorithms. The success rate is computed by dividing the number of successfully stabilized systems by the number of sample systems (= 600), while the average CPU time is the average of those for the 600 sample systems. The success rate for the extended linearization algorithm 1 is 97.5% (=585/600), while those for the extended linearization algorithm 2 and APM decrease. The computational complexity (measured by the CPU time and the iteration) of the extended linearization algorithm 1 seems to be always most rapid extremely.

Next, for each different number (1, ..., 10) of the system order, 50 random systems are generated by the procedure for $P(s)$. The two extended linearization algorithms (replacing $P > 0$ with $P > 0.1$ for a numerical standardization) are applied for output feedback stabilization via PI controller for comparison.

Table 2. Results of Algorithm 1.

plant order	number of successful times	average CPU time
1	50	1.0
2	50	4.1
3	48	16.6
4	50	49.5
5	48	57.8
6	50	193.1
7	49	345.4
8	50	1112.4
9	49	1582.1
10	50	1060.4

Table 3. Results of Algorithm 2.

plant order	number of successful times	average CPU time
1	50	2.1
2	50	17.3
3	48	41.3
4	46	89.4
5	46	160.6
6	46	243.0
7	42	494.0
8	41	124.6
9	45	229.0
10	47	866.2

The numbers of successful times and the required CPU times of Algorithm 1 and 2 are summarized in Table 2 and 3, respectively. The computational complexity (measured by the CPU time) of the two extended linearization algorithms seems to grow rapidly (exponentially) with the system order, equivalently, the matrix inequality sizes of (3) and (5). It should be noted that the success rates for the extended linearization algorithm 1 are always higher than those for the extended linearization algorithm 2. Correspondingly, the required CPU times are always shorter for Algorithm 2, respectively. However, there are the cases where the required average CPU time is not shorter for Algorithm 2 since the calculation of the average CPU time does not include the unsuccessful cases. The point is that Algorithm 1 obtains better solutions with less computation than Algorithm 2 and APM.

6. CONCLUSION

First, we have proved that a certain class of nonconvex matrix inequalities is equivalent to LMIs with a rank constraint. Two computational algorithms: the two extended linearization algorithms based on the basic lemma: Lemma 1 and two LMI optimization approaches have been proposed to solve the problems of LMIs with a rank constraint. Since reliability and efficiency of the two extended linearization algorithms are investigated statistically, this paper has considered the stabilization problems via PI controller having the restriction of order and structure. As a result, it is pointed out that Algorithm 1 obtains better solutions with less computation than Algorithm 2 and APM. Our approaches can be applied to a large number of other rank-minimization problems over LMIs that arise in control theory, for example, the well-posedness problem of feedback control systems. In this meaning, our approaches are very important. At present, more intensive investigation is being developed to compare with other existing methods.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge helpful discussions with Professor M. Ikeda at Osaka University. This research is supported by The Grant-in-Aid: No. 16760342 for Young Scientists (B) by The Ministry of Education, Culture, Sports, Science and Technology in Japan.

REFERENCES

Apkarian, P. and H. D. Tuan (2000). Robust control via concave minimization local and global algorithms. *IEEE Trans. Auto. Contr.* **45**(2), 299–305.

Fares, B., P. Apkarian and D. Noll (2001). An augmented lagrangian method for a class of LMI-constrained problems in robust control theory. *Int. J. Contr.* **74**(4), 348–360.

Fazel, M., H. Hindi and S. P. Boyd (2003). Log-det heuristic for matrix rank minimization with applications to Hankel and Euclidean distance matrices. *Proc. American Contr. Conf.* pp. 2156–2162.

Ghaoui, L. El, F. Oustry and M. AitRami (1997). A cone complementarity linearization algorithm for static output-feedback and related problems. *IEEE Trans. Auto. Contr.* **42**(8), 1171–1176.

Grigoriadis, K. M. and R. E. Skelton (1996). Low-order control design for LMI problems using alternating projection methods. *Automatica* **32**, 1117–1125.

Iwasaki, T. (1999). The dual iteration for fixed order control. *IEEE Trans. Auto. Contr.* **44**(4), 783–788.

Iwasaki, T. and S. Hara (1998). Well-posedness of feedback systems: insights into exact robustness analysis and approximate computations. *IEEE Trans. Auto. Contr.* **43**(5), 619–630.

Kiyama, T. and E. Nishio (2004). Finite frequency property-based robust control analysis and synthesis. *Proc. American Contr. Conf.* pp. 3962–3967.

Kiyama, T., S. Toyora and S. Hara (2002). A solvability condition of general robust control problems. *Proc. SICE Annual Conf.* pp. 677–678.

Scherer, C. (1996). Robust generalized H_2 control for uncertain and LPV systems with general scalings. *IEEE Conf. Decision Contr.* pp. 3970–3975.

Skelton, R. E., T. Iwasaki and K. M. Grigoriadis (1997). *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis.