STOCHASTIC ROBUST REDUCED ORDER H-INFINITY OBSERVER-BASED CONTROL

M. Zasadzinski, S. Halabi, H. Rafaralahy, H. Souley Ali, M. Darouach

Université Henri Poincaré – Nancy I,
CRAN UMR 7039 – CNRS
IUT de Longwy,
186, rue de Lorraine, 54400 Cosnes et Romain, FRANCE
e-mail : {mzasad,souley}@iut-longwy.uhp-nancy.fr

Abstract: This paper considers robust reduced order $\mathcal{H}_\infty$ observer-based control for stochastic systems. The approach is divided into two steps. First, we search for a linear control law to ensure $\mathcal{H}_\infty$ specification. Then, the obtained linear combination of the state is used to solve an "unbiasedness" (decoupling) condition on the drift part of the closed-loop system. Finally, deterministic functional filtering techniques are applied to obtain the observer-based controller.

Keywords: Linear stochastic systems, Uncertain systems, Observer-based control, Reduced-order controller, H-infinity optimization, LMI.

1. INTRODUCTION

We are usually used the deterministic differential equations to model physical systems. But this approach can be inadequate, particularly when the model is not exactly known. Then a stochastic differential equations (SDE) may be used to represent these kind of systems. The advantage of SDE is that they contain a random term which represents the randomness within the systems to model. Thus, the systems understudy are composed by two parts : the drift one which represents the dominant action of the system and the diffusion one representing randomness along the dominant curve. In such situation the deterministic approach does not work, so we consider in the present paper a stochastic differential system in the Itô form. The observer-based control (Hsu *et al.*, 1994; Juan *et al.*, 1994; Stoorvogel *et al.*, 1994; Iwasaki and Skelton, 1995), (Mita *et al.*, 1998; Alazard and Apkarian, 1999; Jun'e and Zhaolin, 2002) is usually applied when we do not have access to all the states of a system. But to our knowledge there are only few results on the stochastic observer-based control theory (see (Chen and Zhang, 2004)).

In this paper, we propose a method to design a control law and the observer into two steps. First, we search for a linear control law which ensures $\mathcal{H}_\infty$ specification. Then, the obtained linear combination of the state is used to solve the unbiasedness (decoupling) condition on the drift part of the closed-loop system.

We finally apply functional filtering techniques developed for deterministic systems in (Darouach *et al.*, 2001) to determine the observer-based controller matrices.

Note that in many practical situations, there are uncertainties which affect the system. In this paper, the stochastic system is also subjected to norm-bounded uncertainties.

2. PRELIMINARIES AND NOTATIONS

Throughout the paper, $\mathbf{E}$ represents expectation operator with respect to some probability measure $\mathcal{P}$. In the sequel $\mathrm{herm}(A)$ stands for $A + A^T$.

Notice also that, given a matrix M, the general-

ized inverse of M is $M^\dagger$ satisfying $M = MM^\dagger M$ (Lancaster and Tismenetsky, 1985).
$L_2\left(\Omega,\mathbb{R}^k\right)$ is the space of square-integrable $\mathbb{R}^k$-valued functions on the probability space $(\Omega,\mathcal{F},\mathcal{P})$ where Ω is the sample space, $\mathcal{F}$ is a σ-algebra of subsets of the sample space called events and $\mathcal{P}$ is the probability measure on $\mathcal{F}$. $(\mathcal{F}_t)_{t\geqslant 0}$ denote an increasing family of σ-algebras $(\mathcal{F}_t)\in\mathcal{F}$. We also denote by $\widehat{L}_2\left([0,\infty);\mathbb{R}^k\right)$ the space of non-anticipatory square-integrable stochastic process $f(.) = (f(t))_{t\in[0,\infty)}$ in $\mathbb{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,\infty)}$ satisfying

$$\|f\|^2_{\widehat{L}_2} = \mathbf{E}\{\int_0^\infty \|f(t)\|^2 \,\mathrm{d}\,t\} < \infty$$

where $\|.\|$ is the well-known Euclidean norm.

3. PROBLEM STATEMENT

Let us consider the following uncertain stochastic system

$$\begin{cases} \mathrm{d}\,x(t) = ((A+\Delta A(t))\,x(t) \\ \quad +(B_1+\Delta B_1(t))\,v(t)+B_2u(t))\,\mathrm{d}\,t \\ \quad +((A_0+\Delta A_0(t))\,x(t)+(B_0+\Delta B_0(t))\,v(t))\,\mathrm{d}\,w(t) \\ z(t) = C_1x(t) + D_{11}v(t) + D_{12}u(t) \\ y(t) = (C_2+\Delta C_2(t))\,x(t) + (D_{21}+\Delta D_{21}(t))\,v(t) \end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $z(t)\in\mathbb{R}^q$ is the controlled output, $y(t)\in\mathbb{R}^p$ is the output vector, $u(t)\in\mathbb{R}^m$ is the input and $v(t)\in\mathbb{R}^r$ is the disturbance vector. $w(t)$ is a zero mean scalar Wiener process verifying (Has'minskii, 1980)

$$\mathbf{E}(\mathrm{d}\,w(t)) = 0 \text{ and } \mathbf{E}(\mathrm{d}\,w(t)^2) = \mathrm{d}\,t. \tag{2}$$

The matrices $\Delta A(t)$, $\Delta B_1(t)$, $\Delta A_0(t)$, $\Delta B_0(t)$, $\Delta C_2(t)$ and $\Delta D_{21}(t)$ represent the parametric uncertainties and satisfy the following relation

$$\begin{bmatrix} \Delta A(t) & \Delta B_1(t) \\ \Delta A_0(t) & \Delta B_0(t) \\ \Delta C_2(t) & \Delta D_{21}(t) \end{bmatrix} = \begin{bmatrix} M_x \\ M_{x_0} \\ M_y \end{bmatrix} \Delta(t) \begin{bmatrix} N_x & N_v \end{bmatrix} \tag{3}$$

with $\Delta^T(t)\Delta(t)\leqslant I_k$ and $\Delta(t)\in\mathbb{R}^{\ell\times k}$.
In the sequel, all the variables will be given without referring to time t explicitly (for example x instead of $x(t)$). We aim to design an observer-based controller with the following structure :

$$\mathrm{d}\,\eta = H\eta\,\mathrm{d}\,t + J_1y\,\mathrm{d}\,t + J_2u\,\mathrm{d}\,t \tag{4a}$$
$$u = \eta + Ey, \tag{4b}$$

where $\eta(t)\in\mathbb{R}^m$, $m\leqslant n$ is the observer state and H, J_1, J_2 and E are to be designed.

Definition 1. (Hinrichsen and Pritchard, 1998; Xu and Chen, 2003) The nominal stochastic system (1) (without uncertainties i.e. $\Delta(t) = 0$) is said to be mean-square asymptotically stable if all initial states $x(0)$, subject to $v(t) = 0$, yield

$$\lim_{t\to\infty}\mathbf{E}\left(\|x(t)\|^2\right) = 0. \tag{5}$$

■

Definition 2. (Xu and Chen, 2003) The uncertain stochastic system (1) is said to be robustly stochastically stable if it is mean-square asymptotically stable for all admissible uncertainties $\Delta A(t)$, $\Delta A_0(t)$, $\Delta B_1(t)$, $\Delta B_0(t)$, $\Delta C_2(t)$ and $\Delta D_{21}(t)$. ■

Definition 3. (Lin *et al.*, 2001) The uncertain system (1) is said to be robustly stochastically stabilizable based on functional observer if there exist a gain matrix L, a functional observer $\mathrm{d}\,\eta = H\eta\,\mathrm{d}\,t + J_1y\,\mathrm{d}\,t + J_2u\,\mathrm{d}\,t$ and a control law $u = \eta + Ey$ such that

(i) $\lim_{t\to\infty}\mathbf{E}\left(\|u - Lx\|^2\right) = 0$ if $v\equiv 0$,
(ii) the closed-loop system (1)-(4) is robustly stochastically stable. ■

Problem 4. (Lin *et al.*, 2001) The objective is to establish a stochastic functional observer (4a) and a control law (4b) such that

(i) $\lim_{t\to\infty}\mathbf{E}\left(\|u - Lx\|^2\right) = 0$ if $v\equiv 0$,
(ii) the resulting closed-loop system (1)-(4) is robustly stochastically stable with the following prescribed disturbance attenuation

$$\|z\|^2_{\widehat{L}_2} \leqslant \gamma\|v\|^2_{\widehat{L}_2} \tag{6}$$

for every $v(t)\in\widehat{L}_2\left([0,\infty];\mathbb{R}^m\right)$. ■

4. FIRST STEP: DESIGN OF THE FEEDBACK GAIN

First, consider the nominal system (system (1) without the uincertainties) and define an observation error signal as

$$e = Lx - u = \varepsilon - ED_{21}v \tag{7}$$

$$\text{with} \quad \varepsilon = \Psi x - \eta \quad \text{and} \quad \Psi = L - EC_2 \tag{8}$$

From (8), the dynamics of $\varepsilon(t)$ is given by

$$\mathrm{d}\,\varepsilon = (H\varepsilon+(\Psi A-H\Psi-J_1C_2)x+(\Psi B_1-J_1D_{21})v+ (\Psi B_2 - J_2)u)\,\mathrm{d}\,t + (\Psi A_0x + \Psi B_0v)\,\mathrm{d}\,w \tag{9}$$

In order to ensure that the dynamics error is asymptotically mean-square stable independently of the state x and the perturbation v in the drift part (in the diffusion part this is due to (2)), the following two conditions are posed

$$0 = \Psi A - H\Psi - J_1C_2 \tag{10a}$$
$$J_2 = \Psi B_2. \tag{10b}$$

This is "equivalent" to the unbiasedness condition in the deterministic case.

Now, after having given the motivation of conditions (10a)-(10b), return to the uncertain system (1).

From (7), the output $z(t)$ of system (1) is given by

$$
\begin{aligned}
z = &\begin{bmatrix} C_1 + D_{12}L & -D_{12} \end{bmatrix} \xi \\
&+ (D_{11} + D_{12}E(M_y \Delta(t) N_v + D_{21}))\, v. \qquad (11)
\end{aligned}
$$

where $\xi = \begin{bmatrix} x^T & \varepsilon^T \end{bmatrix}^T$.

Before proceeding, let us give the following assumption which will be cleared later.

Assumption 5. Assume that $E\begin{bmatrix} M_y & D_{21} \end{bmatrix} = 0$. ■

Then, using the previous developments, assumption 5, relation $u(t) = Lx(t)$ and conditions (10a)-(10b), the closed-loop (1)-(4) system is given by

$$
\begin{cases}
\mathrm{d}\,\xi = \begin{bmatrix} A+B_2L & -B_2 \\ 0 & H \end{bmatrix} \xi\, \mathrm{d}\,t + \underbrace{\begin{bmatrix} \Delta A(t) & 0 \\ \Psi\Delta A(t) - J_1\Delta C_2(t) & 0 \end{bmatrix}}_{\Delta\mathbb{A}(t)} \xi\, \mathrm{d}\,t \\
\quad + \begin{bmatrix} B_1 \\ \Psi B_1 - J_1 D_{21} \end{bmatrix} v\, \mathrm{d}\,t + \underbrace{\begin{bmatrix} \Delta B_1(t) \\ \Psi\Delta B_1(t) - J_1\Delta D_{21}(t) \end{bmatrix}}_{\Delta\mathbb{B}(t)} v\, \mathrm{d}\,t \\
\quad + \begin{bmatrix} A_0 & 0 \\ \Psi A_0 & 0 \end{bmatrix} \xi\, \mathrm{d}\,w + \underbrace{\begin{bmatrix} \Delta A_0(t) & 0 \\ \Psi\Delta A_0(t) & 0 \end{bmatrix}}_{\Delta\mathbb{A}_0(t)} \xi\, \mathrm{d}\,w \\
+ \begin{bmatrix} B_0 \\ \Psi B_0 \end{bmatrix} v\, \mathrm{d}\,w + \underbrace{\begin{bmatrix} \Delta B_0(t) \\ \Psi\Delta B_0(t) \end{bmatrix}}_{\Delta\mathbb{B}_0(t)} v\, \mathrm{d}\,w \\
z = \begin{bmatrix} C_1 + D_{12}L & -D_{12} \end{bmatrix} \xi + D_{11} v
\end{cases}
\qquad (12)
$$

Note that from (12), matrix L can be initially determined, then the observer's matrices afterwords. In fact, L is the state feedback given by the following lemma.

Lemma 6. The system (1) is robustly stochastically stabilizable by $u(t) = Lx(t)$ and $\|z\|^2_{\widehat{L}_2} \leqslant \gamma\|v\|^2_{\widehat{L}_2}$ if there exit matrices $Q = Q^T > 0, Q \in \mathbb{R}^{n\times n}, Y \in \mathbb{R}^{m\times n}$ and two scalars $\mu_1 > 0$ and $\mu_2 > 0$ such that

$$
\begin{bmatrix}
(1{,}1) & B_1 & (1{,}3) & M_x & QA_0^T & 0 & QN_x^T \\
B_1^T & -\gamma^2 I_r & D_{11}^T & 0 & B_0^T & 0 & N_v^T \\
(1{,}3)^T & D_{11} & -I_q & 0 & 0 & 0 & 0 \\
M_x^T & 0 & 0 & -\mu_1 I_\ell & 0 & 0 & 0 \\
A_0 Q & B_0 & 0 & 0 & -Q & M_{x_0} & 0 \\
0 & 0 & 0 & 0 & M_{x_0}^T & -\mu_2 I_\ell & 0 \\
N_x Q & N_v & 0 & 0 & 0 & 0 & -(\mu_1+\mu_2)^{-1} I_k
\end{bmatrix} < 0 \qquad (13)
$$

$$
\text{with} \quad \begin{aligned} (1,1) &= AQ + QA^T + B_2 Y + Y^T B_2^T, \\ (1,3) &= QC_1^T + Y^T D_{12}^T. \end{aligned} \qquad (14)
$$

The gain L is then given by

$$
L = YQ^{-1} \qquad (15)
$$

■

Proof. Consider the uncertain system (1) where we replace $u(t)$ by $Lx(t)$. Then applying the first LMI of theorem 1 in (Xu and Chen, 2003) we have the LMI (13). ❑

5. SECOND STEP: DESIGN OF THE OBSERVER-BASED CONTROLLER

Here we will use the gain L calculated in section 4 to derive an "unbiasedness" condition on the drift part of the estimation error (see (Souley Ali *et al.*, 2004)).

Matrix L is assumed to be full row rank without loss of generality. Notice that if it is not the case for the solution given by (15), we may perturb L a little to fulfill the full row rank condition.

Since L is a full row rank matrix, the equation (10a) is equivalent to

$$
(\Psi A - H\Psi - J_1 C_2) \begin{bmatrix} L^\dagger & I_n - L^\dagger L \end{bmatrix} = 0 \qquad (16)
$$

where $L^\dagger$ is a generalized inverse of matrix L satisfying $L = LL^\dagger L$ (Lancaster and Tismenetsky, 1985) (as $\operatorname{rank} L = m$, we have $LL^\dagger = I_m$).
From (16) the following relations hold (Darouach *et al.*, 2001)

$$
0 = \Psi A L^\dagger - H\Psi L^\dagger - J_1 C_2 L^\dagger \qquad (17a)
$$

$$
0 = \Psi\overline{A} + HE\overline{C} - J_1\overline{C} \qquad (17b)
$$

$$
\text{where } \overline{A} = A(I_n - L^\dagger L),\ \overline{C} = C_2(I_n - L^\dagger L). \quad (18)
$$

Defining $K = J_1 - HE$ and using the definition of Ψ, (17a) gives

$$
H = \overline{\overline{A}} - \mathcal{K}\overline{\overline{C}} \qquad (19)
$$

where

$$
\overline{\overline{A}} = LAL^\dagger, \overline{\overline{C}} = \begin{bmatrix} C_2 A L^\dagger \\ C_2 L^\dagger \end{bmatrix}, \mathcal{K} = \begin{bmatrix} E & K \end{bmatrix} \qquad (20)
$$

$$
\begin{bmatrix} E & K \end{bmatrix} \overline{\Sigma} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix}, \overline{\Sigma} = \begin{bmatrix} M_y & D_{21} & C_2\overline{A} \\ 0 & 0 & \overline{C} \end{bmatrix}, \quad (21)
$$

$$
\begin{bmatrix} E & K \end{bmatrix} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger + \overline{Z}(I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) \qquad (22)
$$

respectively, where $\overline{Z}$ is an arbitrary matrix of appropriate dimensions.

Notice that the uncertain terms combinations $\Psi\Delta A(t) - J_1\Delta C_2(t)$ and $\Psi\Delta B_1(t) - J_1\Delta D_{21}(t)$ are easily calculated by means of the system matrices. For example, using (7) and (22), the first term is given by

$$
\begin{aligned}
&\Psi\Delta A(t) - J_1\Delta C_2(t) = \\
&\left(LM_x - \begin{bmatrix} E & K \end{bmatrix} \begin{bmatrix} C_2 M_x \\ M_y \end{bmatrix} \right) \Delta(t) N_x + HEM_y\Delta(t) N_x.
\end{aligned} \qquad (23)
$$

In the second term in (23), one has the expression HE. Using (19), (20) and (22), one can see that this expression is bilinear in the gain $\overline{Z}$. Similarly we have the same bilinear expressin while computing the term $\Psi B_1 - J_1 D_{21}$. In order to avoid this bilinearity the assumption 5 must be used.

$$
\begin{aligned}
&\Psi\Delta A(t) - J_1\Delta C_2(t) = \\
&\left(\left(LM_x - C^L_{\{2,M\}} \right) - \overline{Z} \left((I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) C_{\{2,M\}} \right) \right) \Delta(t) N_x
\end{aligned}
$$

with matrix $C_{\{2,M\}}$ and $C^L_{\{2,M\}}$ are given by

$$C_{\{2,M\}} = \begin{bmatrix} C_2 M_x \\ M_y \end{bmatrix}, C^L_{\{2,M\}} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger C_{\{2,M\}}.$$

The matrix $\Psi \Delta B_1(t) - J_1 \Delta D_{21}(t)$ is calculated in a similar way and the closed-loop system (12) becomes

$$\begin{cases} \mathrm{d}\,\xi = \left(\widehat{\mathbb{A}} + \Delta\mathbb{A}(t)\right) \xi \,\mathrm{d}\,t + \left(\widehat{\mathbb{B}} + \Delta\mathbb{B}(t)\right) v \,\mathrm{d}\,t \\ \qquad + \left(\widehat{\mathbb{A}}_0 + \Delta\mathbb{A}_0(t)\right) \xi \,\mathrm{d}\,w + \left(\widehat{\mathbb{B}}_0 + \Delta\mathbb{B}_0(t)\right) v \,\mathrm{d}\,w \\ z = \widehat{\mathbb{C}}\xi + D_{11} v. \end{cases} \tag{24}$$

where

$$\begin{bmatrix} \Delta\mathbb{A}(t) & \Delta\mathbb{B}(t) \\ \Delta\mathbb{A}_0(t) & \Delta\mathbb{B}_0(t) \end{bmatrix} = \begin{bmatrix} \mathbb{M}_x \\ \mathbb{M}_{x_0} \end{bmatrix} \Delta(t) \begin{bmatrix} \mathbb{N}_x & \mathbb{N}_v \end{bmatrix}, \tag{25}$$

with

$$\widehat{\mathbb{A}} = \begin{bmatrix} A + B_2 L & -B_2 \\ 0 & \widehat{A} - \overline{Z}\widehat{C} \end{bmatrix}, \quad \widehat{\mathbb{B}} = \begin{bmatrix} B_1 \\ \widehat{B}_{1,1} - \overline{Z}\widehat{B}_{1,2} \end{bmatrix}$$

$$\widehat{\mathbb{A}}_0 = \begin{bmatrix} A_0 & 0 \\ \widehat{A}_{0,1} - \overline{Z}\widehat{A}_{0,2} & 0 \end{bmatrix}, \quad \widehat{\mathbb{B}}_0 = \begin{bmatrix} B_0 \\ \widehat{B}_{0,1} - \overline{Z}\widehat{B}_{0,2} \end{bmatrix}$$

$$\widehat{\mathbb{C}} = \begin{bmatrix} C_1 + D_{12} L & -D_{12} \end{bmatrix},$$

$$\mathbb{M}_x = \begin{bmatrix} M_x \\ \widehat{M}_{x_{1,1}} - \overline{Z}\widehat{M}_{x_{1,2}} \end{bmatrix}, \mathbb{M}_{x_0} = \begin{bmatrix} M_x \\ \widehat{M}_{x_{0,1}} - \overline{Z}\widehat{M}_{x_{0,2}} \end{bmatrix}$$

$$\widehat{A} = LAL^\dagger - C^L_{\{1,A\}}, \widehat{C} = (I_{2p} - \Sigma\,\Sigma^\dagger) C_{\{1,A\}}$$

$$\widehat{B}_{1,1} = LB_1 - C^L_{\{1,B\}}, \widehat{B}_{1,2} = (I_{2p} - \Sigma\,\Sigma^\dagger) C_{\{1,B\}}$$

$$\widehat{A}_{0,1} = LA_0 - C^L_{\{2,A\}}, \widehat{A}_{0,2} = (I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) C_{\{2,A\}}$$

$$\widehat{B}_{0,1} = LB_0 - C^L_{\{2,B\}}, \widehat{B}_{0,2} = (I_{2p} - \Sigma\,\Sigma^\dagger) C_{\{2,B\}}$$

$$\widehat{M}_{x_{1,1}} = LM_x - C^L_{\{2,M\}}, \widehat{M}_{x_{1,2}} = (I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) C_{\{2,M\}}$$

$$\widehat{M}_{x_{0,1}} = LM_{x_0} - C^L_{\{2,M_0\}}, \widehat{M}_{x_{0,2}} = (I_{2p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger) C_{\{2,M_0\}}$$

$$\mathbb{N}_x = \begin{bmatrix} N_x & 0 \end{bmatrix}, \mathbb{N}_v = N_v,$$

and

$$C_{\{1,A\}} = \begin{bmatrix} C_2 A L^\dagger \\ C_2 L^\dagger \end{bmatrix}, C^L_{\{1,A\}} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger C_{\{1,A\}}$$

$$C_{\{1,B\}} = \begin{bmatrix} C_2 B_1 \\ D_{21} \end{bmatrix}, C^L_{\{1,B\}} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger C_{\{1,B\}}$$

$$C_{\{2,A\}} = \begin{bmatrix} C_2 A_0 \\ 0 \end{bmatrix}, C^L_{\{2,A\}} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger C_{\{2,A\}}$$

$$C_{\{2,B\}} = \begin{bmatrix} C_2 B_0 \\ 0 \end{bmatrix}, C^L_{\{2,B\}} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger C_{\{2,B\}}$$

$$C_{\{2,M_0\}} = \begin{bmatrix} C_2 M_{x0} \\ 0 \end{bmatrix}, C^L_{\{2,M_0\}} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger C_{\{2,M_0\}}$$

We have now the next theorem which solves the robust $\mathcal{H}_\infty$ observed-based controller design problem.

Theorem 7. The robust $\mathcal{H}_\infty$ observer-based unbiased controller design (problem 4) is solved under $E\,[\, M_y \quad D_{21}\,] = 0$ and where L is given by (13)-(15) if, for some two scalars $\mu_1 > 0$ and $\mu_2 > 0$, there exist matrices $\mathbb{P} = \mathbb{P}^T > 0$ and $\mathbb{Q} = \mathbb{Q}^T > 0$, $\mathbb{P}, \mathbb{Q} \in \mathbb{R}^{(n+m)\times(m+n)}$ such that

$$\begin{bmatrix} \mathbb{K}_u & 0 \\ 0 & I_{r+2\ell} \end{bmatrix}^T \begin{bmatrix} \overline{\mathbb{A}}\mathbb{P} + \mathbb{P}\overline{\mathbb{A}}^T & \mathbb{P}\widehat{\mathbb{C}}^T & \mathbb{P}\overline{\mathbb{A}}_0^T & (\mu_1+\mu_2)\mathbb{P}N_x^T & \overline{\mathbb{B}} & \overline{\mathbb{M}}_x & 0 \\ \widehat{\mathbb{C}}\mathbb{P} & -I_q & 0 & 0 & D_{11} & 0 & 0 \\ \overline{\mathbb{A}}_0\mathbb{P} & 0 & -\mathbb{P} & 0 & \overline{\mathbb{B}}_0 & 0 & \overline{\mathbb{M}}_{x_0} \\ (\mu_1+\mu_2)N_x\mathbb{P} & 0 & 0 & -I_k & (\mu_1+\mu_2)N_v & 0 & 0 \\ \overline{\mathbb{B}}^T & D_{11}^T & \overline{\mathbb{B}}_0^T & (\mu_1+\mu_2)N_v^T & -\gamma^2 I_r & 0 & 0 \\ \overline{\mathbb{M}}_x^T & 0 & 0 & 0 & 0 & -\mu_1 I_\ell & 0 \\ 0 & 0 & \overline{\mathbb{M}}_{x_0}^T & 0 & 0 & 0 & -\mu_2 I_\ell \end{bmatrix} \begin{bmatrix} \mathbb{K}_u & 0 \\ 0 & I_{r+2\ell} \end{bmatrix} < 0, \tag{28a}$$

$$\begin{bmatrix} \mathbb{K}_y & 0 \\ 0 & I_{q+k} \end{bmatrix}^T \begin{bmatrix} \mathbb{Q}\overline{\mathbb{A}} + \overline{\mathbb{A}}^T\mathbb{Q} & \mathbb{Q}\overline{\mathbb{B}} & \mathbb{Q}\overline{\mathbb{M}}_x & \overline{\mathbb{A}}_0^T\mathbb{Q} & 0 & \widehat{\mathbb{C}}^T & (\mu_1+\mu_2)N_x^T \\ \overline{\mathbb{B}}^T\mathbb{Q} & -\gamma^2 I_r & 0 & \overline{\mathbb{B}}_0^T\mathbb{Q} & 0 & D_{11}^T & (\mu_1+\mu_2)N_v^T \\ \overline{\mathbb{M}}_x^T\mathbb{Q} & 0 & -\mu_1 I_\ell & 0 & 0 & 0 & 0 \\ \mathbb{Q}\overline{\mathbb{A}}_0 & \mathbb{Q}\overline{\mathbb{B}}_0 & 0 & -\mathbb{Q} & \mathbb{Q}\overline{\mathbb{M}}_{x_0} & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathbb{M}}_{x_0}^T\mathbb{Q} & -\mu_2 I_\ell & 0 & 0 \\ \widehat{\mathbb{C}} & D_{11} & 0 & 0 & 0 & -I_q & 0 \\ (\mu_1+\mu_2)N_x & (\mu_1+\mu_2)N_v & 0 & 0 & 0 & 0 & -I_k \end{bmatrix} \begin{bmatrix} \mathbb{K}_y & 0 \\ 0 & I_{q+k} \end{bmatrix} < 0, \tag{28b}$$

$$I_{n+m} = \mathbb{P}\mathbb{Q}, \tag{28c}$$

where $\mathbb{K}_y$ and $\mathbb{K}_u$ are two matrices whose columns span the null spaces of $\begin{bmatrix} \widetilde{\mathbb{C}} & \widehat{B}_{1,2} & \widehat{M}_{x_{1,2}} & 0 & 0 \\ \widetilde{\mathbb{C}}_0 & \widehat{B}_{0,2} & 0 & 0 & \widehat{M}_{x_{0,2}} \end{bmatrix}$ and $\begin{bmatrix} -\widetilde{\mathbb{B}}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\widetilde{\mathbb{B}}^T & 0 \end{bmatrix}$, respectively, and

$$\overline{\mathbb{A}} = \begin{bmatrix} A + B_2 L & -B_2 \\ 0 & \widehat{A} \end{bmatrix}, \overline{\mathbb{B}} = \begin{bmatrix} B_1 \\ \widehat{B}_{1,1} \end{bmatrix}, \overline{\mathbb{A}}_0 = \begin{bmatrix} A_0 & 0 \\ \widehat{A}_{0,1} & 0 \end{bmatrix}, \widetilde{\mathbb{B}} = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

$$\overline{\mathbb{B}}_0 = \begin{bmatrix} B_0 \\ \widehat{B}_{0,1} \end{bmatrix}, \overline{\mathbb{M}}_x = \begin{bmatrix} M_x \\ \widehat{M}_{x_{1,1}} \end{bmatrix}, \overline{\mathbb{M}}_{x_0} = \begin{bmatrix} M_x \\ \widehat{M}_{x_{0,1}} \end{bmatrix},$$

$$\text{and} \quad \widetilde{\mathbb{C}} = \begin{bmatrix} 0 & \widehat{C} \end{bmatrix}, \widetilde{\mathbb{C}}_0 = \begin{bmatrix} \widehat{A}_{0,2} & 0 \end{bmatrix}. \tag{29}$$

All gains $\overline{Z}$ are given by

$$\overline{Z} = \overline{\mathbb{H}}_R^\dagger \mathbb{K} \overline{\mathbb{G}}_L^\dagger + \mathbb{Z} - \overline{\mathbb{H}}_R^\dagger \overline{\mathbb{H}}_R \mathbb{Z} \overline{\mathbb{G}}_L \overline{\mathbb{G}}_L^\dagger \tag{30}$$

where

$$\mathbb{K} = -\mathbb{R}_1^{-1} \overline{\mathbb{H}}_L^T \mathbb{S}_1 \overline{\mathbb{G}}_R^T \left(\overline{\mathbb{G}}_R \mathbb{S}_1 \overline{\mathbb{G}}_R^T\right)^{-1} + \mathbb{R}_1^{-1} \mathbb{S}_2^{1/2} \mathbb{R}_2 \left(\overline{\mathbb{G}}_R \mathbb{S}_1 \overline{\mathbb{G}}_R^T\right)^{-1/2},$$

$$\mathbb{S}_1 = \left(\overline{\mathbb{H}}_L \mathbb{R}_1^{-1} \overline{\mathbb{H}}_L^\dagger - \overline{\mathbb{Q}}\right)^{-1} > 0,$$

$$\mathbb{S}_2 = \mathbb{R}_1 - \overline{\mathbb{H}}_L^T \left(\mathbb{S}_1 - \mathbb{S}_1 \overline{\mathbb{G}}_R^T \left(\overline{\mathbb{G}}_R \mathbb{S}_1 \overline{\mathbb{G}}_R^T\right)^{-1} \overline{\mathbb{G}}_R \mathbb{S}_1\right) \overline{\mathbb{H}}_L,$$

$$\overline{\mathbb{Q}}=\begin{bmatrix} \mathbb{Q}\overline{\mathbb{A}}+\overline{\mathbb{A}}^T\mathbb{Q} & \mathbb{Q}\overline{\mathbb{B}} & \mathbb{Q}\overline{\mathbb{M}}_x & \overline{\mathbb{A}}_0^T\mathbb{Q} & 0 & \widehat{\mathbb{C}}^T & (1,7)^T \\ \overline{\mathbb{B}}^T\mathbb{Q} & -\gamma^2 I_r & 0 & \overline{\mathbb{B}}_0^T\mathbb{Q} & 0 & D_{11}^T & (2,7)^T \\ \overline{\mathbb{M}}_x^T\mathbb{Q} & 0 & -\mu_1 I_\ell & 0 & 0 & 0 & 0 \\ \mathbb{Q}\overline{\mathbb{A}}_0 & \mathbb{Q}\overline{\mathbb{B}}_0 & 0 & -\mathbb{Q} & \mathbb{Q}\overline{\mathbb{M}}_{x_0} & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathbb{M}}_{x_0}^T\mathbb{Q} & -\mu_2 I_\ell & 0 & 0 \\ \widehat{\mathbb{C}} & D_{11} & 0 & 0 & 0 & -I_q & 0 \\ (1,7) & (2,7) & 0 & 0 & 0 & 0 & -(\mu_1+\mu_2)I_k \end{bmatrix}$$

with $(1,7) = (\mu_1 + \mu_2)N_x$ and $(2,7) = (\mu_1 + \mu_2)N_v$, $\mathbb{R}_1$, $\mathbb{R}_2$ and $\mathbb{Z}$ are arbitrary matrices of appropriate dimensions satisfying $\mathbb{R}_1 = \mathbb{R}_1^T > 0$ and $\|\mathbb{R}_2\| < 1$. Matrices $\overline{\mathbb{H}}_L$, $\overline{\mathbb{H}}_R$, $\overline{\mathbb{G}}_L$ and $\overline{\mathbb{G}}_R$ are any full rank matrices such that $\overline{\mathbb{H}} = \overline{\mathbb{H}}_L\overline{\mathbb{H}}_R$ and $\overline{\mathbb{G}} = \overline{\mathbb{G}}_L\overline{\mathbb{G}}_R$ with

$$\overline{\mathbb{H}} = \begin{bmatrix} -\widetilde{\mathbb{B}}^T\mathbb{Q}^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\widetilde{\mathbb{B}}^T\mathbb{Q}^T & 0 & 0 & 0 \end{bmatrix}^T$$

$$\overline{\mathbb{G}} = \begin{bmatrix} \widetilde{\mathbb{C}} & \widehat{B}_{1,2} & \widehat{M}_{x_{1,2}} & 0 & 0 & 0 & 0 \\ \widetilde{\mathbb{C}}_0 & \widehat{B}_{0,2} & 0 & 0 & \widehat{M}_{x_{0,2}} & 0 & 0 \end{bmatrix}$$

■

Proof. The uncertain system (24) is said to be robustly stochastically mean-square stable with (6) if, for some two scalars $\mu_1 > 0$ and $\mu_2 > 0$, there exists $\mathbb{Q} = \mathbb{Q}^T > 0$ such that (see lemma 6)

$$\overline{\mathbb{Q}} + \text{herm}\left(\overline{\mathbb{H}} \begin{bmatrix} \overline{Z} & 0 \\ 0 & \overline{Z} \end{bmatrix} \overline{\mathbb{G}}\right) < 0 \qquad (31)$$

or, from the projection lemma (Iwasaki and Skelton, 1994), if and only if there exist matrices $\mathbb{P} = \mathbb{P}^T > 0$ and $\mathbb{Q} = \mathbb{Q}^T > 0$ with $\mathbb{P} = \mathbb{Q}^{-1}$ such that (28a), (28b) and (28c) hold.

The LMI (28a) and (28b) are then obtained by applying the projection lemma (Iwasaki and Skelton, 1994) to the above inequality, where equation (30) is deduced from relation (22) in (Iwasaki and Skelton, 1994). The theorem is proved. ❑

The cope complementary linearization technique (El Ghaoui *et al.*, 1997) can be used to solve the non convex constraint (28c) to obtain the matrices $\mathbb{P}$ and $\mathbb{Q}$.

6. NUMERICAL EXAMPLE

The matrices of the system (1) are given by

$$A = \begin{bmatrix} -3.5 & -0.5 & 0 & 0 \\ 0 & -1.5 & 0.1 & 0 \\ 0.1 & 0 & -2 & 0 \\ 0.1 & 0 & 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.1 & 0.3 \\ -1 & 0.2 \\ 0.6 & 0.5 \\ 0 & -0.2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0.2 & -0.1 & 0 & -0.3 \\ -1 & 1 & 0 & 1 \\ -0.6 & 0.1 & 0 & -0.5 \\ 0.5 & 0.1 & -0.5 & -0.4 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} -0.1 & 0.1 \\ 0.5 & 0 \\ -0.6 & 0.2 \\ 0.5 & -0.1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D_{11} = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix},$$

$$D_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

and the uncertainties matrices (3) are given by

$$M_x = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.2 \\ 0.2 \end{bmatrix}, M_{x_0} = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.2 \\ 0.2 \end{bmatrix}, \; M_y = \begin{bmatrix} -0.01 \\ 0.01 \end{bmatrix},$$

$$N_x = \begin{bmatrix} 0.1 & 0.1 & 0.2 & 0.2 \end{bmatrix}, \; N_v = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix},$$

$$\xi(0) = [\; -2 \; 1 \; 2 \; 1 \; -0.1424 \; -4.6981]^T$$

where $\xi(0)$ is the initial state $[\; x^T(0) \; \varepsilon^T(0)]^T$. The first step gives $\gamma = 2.9$, $\mu_1 = 6.7645$, $\mu_2 = 6.8004$ and the feedback gain

$$L = \begin{bmatrix} -0.0256 & -0.0831 & 0.0243 & -0.0295 \\ 1.7818 & -0.6466 & -0.2400 & -0.8256 \end{bmatrix}$$

the second step gives the gain

$$\overline{Z} = \begin{bmatrix} 180.8 & 142.3 & 274 & -664.8 \\ -2274.2 & -1902 & -3753.2 & 8958.3 \end{bmatrix}.$$

The matrices of the observer (4) are then

$$H = \begin{bmatrix} -1.8846 & 0.0316 \\ 5.2478 & -1.1874 \end{bmatrix}, J_1 = \begin{bmatrix} -0.0714 & -0.0018 \\ -4.2074 & -0.4131 \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 1.0353 & 1.0353 \\ 0.9695 & 0.9695 \end{bmatrix}, E = \begin{bmatrix} -0.0353 & -0.0353 \\ 0.0305 & 0.0305 \end{bmatrix}.$$

The following figures show the simulation of the augmented system (24) (the state x and the observation error ε for two values of $\Delta(t) = 0.7$ and $\Delta(t) = -0.7$. In these figures the disturbance signal v is presented only in the error plots.

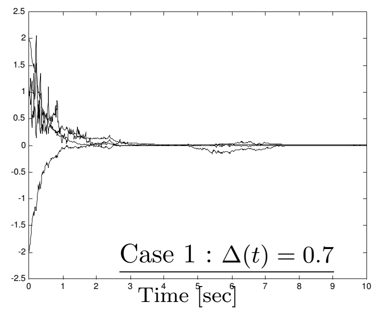

Fig. 1. The state $x(t)$.

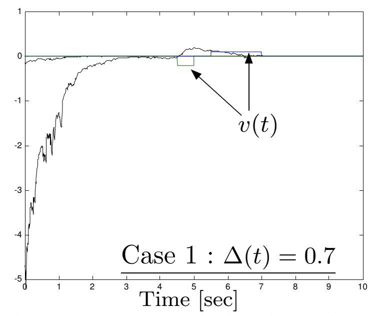

Fig. 2. The error $\varepsilon(t)$ and the disturbance $v(t)$.

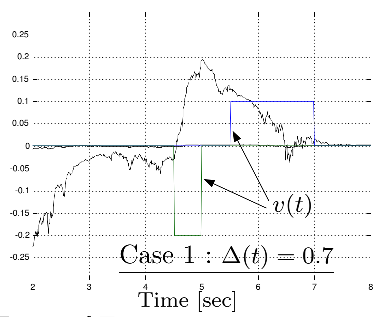

Fig. 3. Zoom of Fig 2.

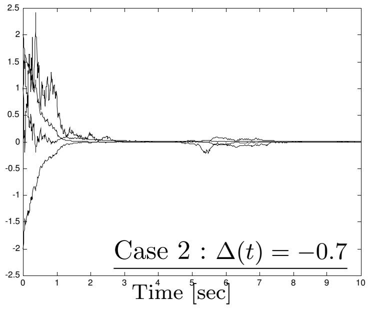

Fig. 4. The state $x(t)$.

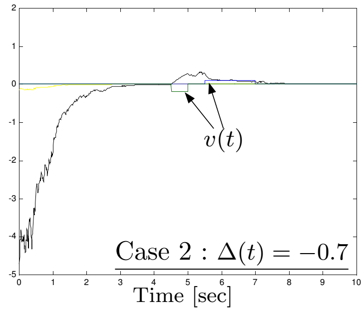

Fig. 5. The error $\varepsilon(t)$ and the disturbance $v(t)$.

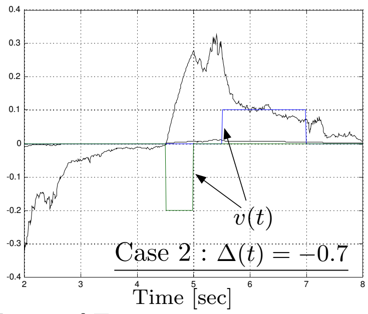

Fig. 6. Zoom of Fig 5.

7. CONCLUSION

This paper has presented a simple solution to the stochastic robust $\mathcal{H}_\infty$ reduced order observer-based control problem via LMI methods. First, a suitable feedback gain is calculated to achieve the desired performance. Second, unbiased functional filtering techniques for determinitic systems are used to derive the observer-based controller.

8. REFERENCES

Alazard, D. and P. Apkarian (1999). Exact observer-based structures for arbitrary compensators. *Int. J. Robust & Nonlinear Contr.* **9**, 101–118.

Chen, B.S. and W. Zhang (2004). Stochastic $\mathcal{H}_2/\mathcal{H}_\infty$ control with state-dependent noise. *IEEE Trans. Aut. Contr.* **49**, 45–57.

Darouach, M., M. Zasadzinski and H. Souley Ali (2001). Robust reduced order unbiased filtering via LMI. In: *Proc. European Contr. Conf.*. Porto, Portugal.

El Ghaoui, L., F. Oustry and M. Aït Rami (1997). A cone complementary linearization algorithm for static output-feedback and related problems. *IEEE Trans. Aut. Contr.* **42**, 1171–1176.

Has'minskii, R.Z. (1980). *Stochastic Stability of Differential Equations*. Siijthoff and Noordhoff. Aplhen aan den Rijn, The Netherlands.

Hinrichsen, D. and A.J. Pritchard (1998). Stochastic $\mathcal{H}_\infty$. *SIAM J. Contr. Opt.* **36**, 1504–1538.

Hsu, C.S., X. Yu, H.H. Yeh and S.S. Banda (1994). $\mathcal{H}_\infty$ compensator design with minimal order observers. *IEEE Trans. Aut. Contr.* **39**, 1679–1681.

Iwasaki, T. and R.E. Skelton (1994). All controllers for the general $\mathcal{H}_\infty$ control problems : LMI existence conditions and state space formulas. *Automatica* **30**, 1307–1317.

Iwasaki, T. and R.E. Skelton (1995). All fixed order $\mathcal{H}_\infty$ controllers : observer-based structure and covariance bounds. *IEEE Trans. Aut. Contr.* **40**, 512–516.

Juan, J.C., H.H. Yeh and S.S. Banda (1994). Observer-based compensators for fixed-order $\mathcal{H}_\infty$ control. In: *Proc. IEEE Conf. Decision & Contr.*. Lake Buena Vista, USA.

Jun'e, F. and C. Zhaolin (2002). Observer-based stability controller design for linear systems with delayed state. In: *Proc. Triennal IFAC World Congress*. Barcelona, Spain.

Lancaster, P. and M. Tismenetsky (1985). *The Theory of Matrices*. 2nd ed.. Academic Press. Orlando, USA.

Lin, Z., X. Guan, Y. Liu and P. Shi (2001). Observer-based robust control for uncertain systems with time-varying delay. *IMA J. of Mathematical Control and Information* **18**, 439–450.

Mita, T., M. Hirita, K. Murata and H. Zhang (1998). $\mathcal{H}_\infty$ control versus disturbance-observer-based control. *IEEE Trans. Ind. Elect.* **45**, 488–495.

Souley Ali, H., S. Halabi, H. Rafaralahy, M. Zasadzinski and M. Darouach (2004). Stochastic reduced-order unbiased filtering. In: *Proc. Mathematical Theory of Networks and Systems Symposium*. Leuven, Belgium.

Stoorvogel, A.A., A. Saberi and B.M. Chen (1994). A reduced order observer based controller design for $\mathcal{H}_\infty$ optimization. *IEEE Trans. Aut. Contr.* **39**, 355–360.

Xu, S. and T. Chen (2003). Robust $\mathcal{H}_\infty$ filtering for uncertain impulsive stochastic systems under sampled measurements. *Automatica* **39**, 500–516.