OUTPUT FEEDBACK CONTROL VIA ADAPTIVE OBSERVERS WITH PERSISTENCY OF EXCITATION

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Abstract: We address the problem of adaptive observer design for nonlinear time-varying systems which can be transformed in the so-called output feedback form (linear in the unmeasured variables). The observer design follows up previous work on adaptive observers for linear systems and has the form of the classical Luenberger observers for linear systems except that the observer gain is time-varying. A specific form of persistency of excitation is imposed to guarantee the convergence of the (state and parameter) estimation errors. As for the output feedback loop, we proceed using a cascade approach, i.e., we impose the appropriate conditions so that the closed loop system has a cascaded structure. Uniform global asymptotic stability may then be concluded based on cascaded systems theory.

Key Words— persistency of excitation, observers, time-varying systems, output feedback control.

1 Introduction

We address the problem of adaptive observer-based output feedback for (certain) nonlinear time-varying systems. We restrict our attention to control problems leading to nonlinear time-varying systems (e.g. non-autonomous stabilization and tracking) that may be transformed into the so-called output feedback form. See for instance (Besançon *et al.* 1998) and some of the references in (Nijmeijer and eds. 1999). Besides observer design, we consider the problem of parameter identification under the assumption that parameters also appear linearly in the model. Such problem has been studied exhaustively for linear systems and for many classes of nonlinear systems (see e.g. (Marino and Tomei 1993, Krstić *et al.* 1995) and references therein). Thus, we are concerned with the problem of *adaptive observer* design and output feedback control.

Even though the condition of linearity in the unmeasured states is restrictive, it has been extensively used earlier (cf. (Marino and Tomei 1993, Krstić *et al.* 1995)). In more recent references, this condition is relaxed, for instance, via high gain observers; see (Praly and Jiang 2004) and some of the references therein, allowing for partially-ISS systems in triangular form. Other works, as (Arcak and Kokotović 2001, Aamo *et al.* 2000), address systems with sector nonlinearities and/or globally Lipschitz functions of the unmeasured variables (e.g. (Praly 2003)).

The results that we present here are inspired from (Zhang 2002) and follow up (Besançon *et al.* 1996, Besançon and de L. Morales 2003, Loría and de León Morales 2003). As in the latter reference, we study the problem of output feedback observer-based control problem from a cascades view-point: we see the closed-loop system as a cascade of two inner loops. The first is given by the plant dynamics with the adaptive observer and the second, by the plant with a state feedback controller, interconnected by nonlinearities generated by the implementation of the certainty equivalence controller. Even though we assume that the system is transformable into the output form, it shall be apparent that, as in (Loría and de León Morales 2003) our results apply to systems with globally Lipschitz nonlinearities.

Notation. We say that a function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathcal{A}$ with $\mathcal{A}$ a closed, not necessarily compact set, satisfies the basic regularity assumption (BRA) if $\phi(t, \cdot)$ is locally Lipschitz and $\phi(\cdot, x)$ is measurable. We denote the usual Euclidean norm of vectors by $|\cdot|$ and use the same symbol for the matrix induced norm. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_\infty$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is nondecreasing, $\beta(s, \cdot)$ is non-increasing and $\lim_{s \to 0^+} \beta(s, t) = \lim_{t \to \infty} \beta(s, t) = 0$. We denote the solution of a differential equation $\dot{x} = f(t, x)$ starting at $x_\circ$ at time $t_\circ$ by $x(\cdot, t_\circ, x_\circ)$. We say that such a system described is forward complete if all the solutions starting at $t_\circ \geq 0$, $x_\circ = x(t_\circ, t_\circ, x_\circ)$ exist for each $t \geq t_\circ$.

Definition 1 (Uniform global stability) The origin of

$$\dot{x} = f(t, x) \tag{1}$$

where $f(\cdot, \cdot)$ satisfies the BRA, is said to be uniformly globally stable (UGS) if there exists $\gamma \in \mathcal{K}_\infty$ such that, for each $(t_\circ, x_\circ) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, each solution $x(\cdot, t_\circ, x_\circ)$ of (1) satisfies

$$|x(t, t_\circ, x_\circ)| \leq \gamma(|x_\circ|) \qquad \forall\, t \geq t_\circ\ . \tag{2}$$

Definition 2 (Uniform global asymptotic stability) The origin of (1) is said to be uniformly globally asymptotically stable (UGAS) if it is UGS and uniformly globally attractive, i.e., for each pair of strictly positive real numbers (r, σ), there exists $T > 0$ such that for each solution

$$\|x_\circ\| \leq r \implies \|x(t, t_\circ, x_\circ)\| \leq \sigma \qquad \forall\, t \geq t_\circ + T\ .$$

We also need to study the behavior of nonlinear time-varying systems whose dynamics depend on a parameter λ, taken from a closed, not necessarily compact set $\mathcal{D}$. That is, systems of the form

$$\dot{x} = f(t, \lambda, x) \tag{3}$$

where $f(\cdot, \lambda, \cdot)$ satisfies the BRA, and $f(t, \cdot, x)$ is continuous.

Definition 3 The origin of the system $\dot{x} = f(t, \lambda, x)$ is said to be λ-uniformly globally asymptotically stable (λ-UGAS) if all the conditions of Definition 2 are met with T and $\gamma(\cdot)$ independent of λ.

Definition 4 (λ-UGES) The origin of the system $\dot{x} = f(t, \lambda, x)$ is said to be λ-uniformly globally exponentially stable (λ-UGES) if there exist two constants k and $\gamma > 0$ such that, for all $t \geq t_\circ \geq 0$, all $x_\circ \in \mathbb{R}^n$ and all $\lambda \in \mathcal{D}$,

$$|x(t, \lambda, t_\circ, x_\circ)| \leq k\, |x_\circ|\, e^{-\gamma(t - t_\circ)}\,. \tag{4}$$

Such definition is useful, for instance, when studying stability of nonlinear (possibly time-varying) systems by regarding them as linear time-varying, along trajectories. For the sake of illustration, consider the system $\dot{x} = -x^3$ with initial conditions $(t_\circ, x_\circ)$ and the linear time-varying parameterized system $\dot{z} = -a(t, \lambda) z$ with initial conditions $(\tau_\circ, z_\circ) = (t_\circ, x_\circ)$ and $a(t, \lambda) := x(t, t_\circ, x_\circ)^2$ i.e., $\lambda := (t_\circ, x_\circ)$. Since the trajectories of both systems coincide we can establish the stability of $\dot{x} = -x^3$ via conditions imposed on $a(t, \lambda)$. The advantage of this approach is to analyze a linear system instead of a nonlinear one; however, the price paid for such analysis is to impose conditions *along the trajectories* of the nonlinear system. While such technique may appear surprising at first sight it has been used in numerous publications (see e.g. (Khalil 1996*a*, Ortega and Fradkov 1993, Janković 1996, Khalil 1996*b*, Loría *et al.* 2002*a*). In particular, in (Loría and Panteley 2002) we established rigorous conditions for stability of parameterized linear time-varying systems, that serve in the analysis of model-reference adaptive control schemes (MRAC). Such results apply, for instance, to the systems considered in the previously cited references.

The observer design and stability analysis carried out in this paper follows this approach. The conditions that we impose take the form of a specific property of so-called persistency of excitation introduced in (Loría and Panteley 2002) for parameterized systems and that we remind here for convenience.

Definition 5 (λ-uniform persistency of excitation) Let the function $\phi : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}^{n \times m}$, be continuous. We say that $\phi(\cdot, \cdot)$ is λ-uniformly persistently exciting (λ-uPE) if there exist two parameters μ and $T > 0$ such that, for all $\lambda \in \mathcal{D}$,

$$\int_t^{t+T} \phi(\tau, \lambda) \phi(\tau, \lambda)^\top d\tau \geq \mu I \qquad \forall\, t \geq 0\,. \tag{5}$$

2 Main results

2.1 Basic assumptions

We consider the problem of observer design and adaptive output feedback control for nonlinear time-varying systems of the form

$$\begin{aligned} \dot{x} &= f_x(t, x, \theta) + g_x(t, x, \theta) u && \text{(6a)} \\ y &= C(t) x\,, && \text{(6b)} \end{aligned}$$

where $\theta \in \mathbb{R}^m$ denotes a vector of unknown constant parameters, $f_x(\cdot, \cdot, \theta)$, $g_x(\cdot, \cdot, \theta)$ satisfy the

BRA for all $\theta \in \Theta \subset \mathbb{R}^m$, g is uniformly bounded in $t \geq 0$ for all $\theta \in \Theta$, and $C(\cdot)$ is continuous and bounded. We make the standing assumptions that:
1) there exists a state feedback control law $u = k(t, x, \theta)$, uniformly bounded in $t \geq 0$ for all $\theta \in \Theta$, such that the closed loop system

$$\dot{x} = f_x(t, x, \theta) + g_x(t, x, \theta)k(t, x, \theta) \tag{7}$$

is UGAS;
2) there exists a map $\Pi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ such that, defining

$$\xi := \Pi(t, x)\,,$$

the system (6) is transformable into

$$\begin{aligned} \dot{\xi} &= A_\xi(t, u, y)\xi + B_\xi(t, u, y) + \Psi(t, u, y)\theta && \text{(8a)}\\ y &= C(t)\xi && \text{(8b)} \end{aligned}$$

where A_ξ, B_ξ and Ψ are continuous bounded functions from $\mathbb{R}_{\geq 0} \times \mathbb{R}^p \times \mathbb{R}^m$ to $\mathbb{R}^n$. On occasions, to compact the notation we may also write A_ξ instead of $A_\xi(t, u, y)$ and similarly for other variables.

2.2 Observer design

The observer design is based on (Hammouri and de León Morales 1991, Besançon and de L. Morales 2003, Loría and de León Morales 2003) and is inspired by (Zhang 2002). Essentially, as in (Loría and de León Morales 2003), it consists in constructing an observer for (8) with a time-varying gain having the fundamental property of persistency of excitation (see e.g. (Narendra and Annaswamy 1989) or (Sastry and Bodson 1989) for a definition), which guarantees exponential convergence of the estimation error. The choice of such a gain follows (Hammouri and de León Morales 1991, Besançon and de L. Morales 2003). Let us define the extended state of system (8) together with $\dot{\theta} = 0$, its estimate and the respective estimation error as

$$\zeta := \begin{pmatrix} \xi \\ \theta \end{pmatrix}, \quad \hat{\zeta} := \begin{pmatrix} \hat{\xi} \\ \hat{\theta} \end{pmatrix}, \quad \text{and} \quad \bar{\zeta} := \hat{\zeta} - \zeta\,.$$

Then, the estimation goal is to define an observer[1] for the system

$$\dot{\zeta} = A_\zeta(t, u, y)\zeta + B_\zeta(t, u, y)\,. \tag{9}$$

where

$$\begin{aligned} A_\zeta(t, u, y) &:= \begin{pmatrix} A_\xi(t, u, y) & \Psi(t, u, y) \\ 0 & 0 \end{pmatrix},\\ B_\zeta(t, u, y) &:= \begin{pmatrix} B_\xi(t, u, y) \\ 0 \end{pmatrix},\\ C_\zeta(t) &:= \begin{pmatrix} C(t) & 0 \end{pmatrix}, \end{aligned}$$

such that $\bar{\zeta}$ tends to 0 exponentially. Let such observer take the form

$$\dot{\hat{\zeta}} = A_\zeta\hat{\zeta} + B_\zeta - L_\zeta C_\zeta\bar{\zeta}\,, \tag{10}$$

where the estimation gain

$$L_\zeta(t, u, y) := \begin{pmatrix} L_\xi(t, u, y) \\ L_\theta(t, u, y) \end{pmatrix}$$

is to be defined so that the estimation error dynamics,

$$\dot{\bar{\zeta}} = \big(A_\zeta(t, u, y) - L_\zeta(t, u, y)C_\zeta(t)\big)\bar{\zeta}\,, \tag{11}$$

obtained from subtracting (9) to (10), is uniformly exponentially stable. We stress that, here, "uniformly" refers to the initial conditions $(t_\circ, \zeta_\circ)$ and the input and output trajectories $u(t, \hat{\zeta}(t, t_\circ, \hat{\zeta}_\circ))$ and $C(t)x(t, t_\circ, x_\circ)$, which depend in their turn, on the initial conditions of the overall control system. To avoid cumbersome notation and for further analysis, we define $\lambda := [t_\circ, \hat{\zeta}_\circ, x_\circ]$ belonging to $\mathcal{D} := \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+m} \times \mathbb{R}^n$ and the signals $\tilde{u}(t, \lambda) := u(t, \hat{\zeta}(t, t_\circ, \hat{\zeta}_\circ))$ and $\tilde{y}(t, \lambda) := C(t)x(t, t_\circ, x_\circ)$. Then, following the discussion from Section 1, we shall analyse the stability of the nonlinear system (11) by analysing the linear time-varying differential equation

$$\dot{\bar{\zeta}} = \big(\tilde{A}_\zeta(t, \lambda) - \tilde{L}_\zeta(t, \lambda)C_\zeta(t)\big)\bar{\zeta}\,, \tag{12}$$

where we also defined

$$\begin{aligned} \tilde{A}_\zeta(t, \lambda) &:= A_\zeta(t, \tilde{u}(t, \lambda), \tilde{y}(t, \lambda))\\ \tilde{L}_\zeta(t, \lambda) &:= \tilde{L}_\zeta(t, \tilde{u}(t, \lambda), \tilde{y}(t, \lambda))\,. \end{aligned}$$

Similarly, we define

$$\widetilde{\Psi}(t, \lambda) := \Psi(t, \tilde{u}(t, \lambda), \tilde{y}(t, \lambda))\,.$$

We are ready to present our PE observer. For the sake of well-posedness, we assume *for the time-being*, that the system is forward complete uniformly in λ; in particular, all the functions involved in the definition of $\tilde{A}_\xi$, $\tilde{L}_\xi$ exist for all $t \geq t_\circ$, all $t_\circ \geq 0$ and all $\lambda \in \mathcal{D}$.

Proposition 1 For any $\lambda \in \mathcal{D}$, let the estimation gain be given by

$$\tilde{L}_\zeta := \begin{pmatrix} \Lambda P_\theta^{-1}\Lambda^\top C^\top + P_x^{-1}C^\top \\ P_\theta^{-1}\Lambda^\top C^\top \end{pmatrix},$$

Λ, P_x, P_θ being $n \times n$-matrices, solutions of

$$\begin{cases} \dot{\Lambda} = \big(\tilde{A}_\xi - P_x^{-1}C^\top C\big)\Lambda + \widetilde{\Psi} \\ \Lambda(t_\circ, \lambda) = \Lambda_\circ \end{cases} \tag{13}$$

$$\begin{cases} \dot{P}_x = -\rho_x P_x - \tilde{A}_\xi^\top P_x - P_x\tilde{A}_\xi + C^\top C \\ P_x(t, \lambda) = P_{x\circ} > 0 \quad \forall\, t \in [t_\circ, T_x] \end{cases} \tag{14}$$

$$\begin{cases} \dot{P}_\theta = -\rho_\theta P_\theta + \Lambda^\top C^\top C\Lambda \\ P_\theta(t_\circ, \lambda) = P_{\theta\circ} > 0 \quad \forall\, t \in [t_\circ, T_\theta]\,. \end{cases} \tag{15}$$

where ρ_x and ρ_θ are positive numbers and T_x, T_θ are defined below.

[1] With an abuse of terminology we include in the word "observer" the adaptive estimation law for θ.

Assumption 1 Let $\Phi_\xi(t,t_\circ,\lambda)$ denote the transition matrix associated to $\tilde{A}_\xi(t,\lambda)$, i.e., the solution of

$$\left\{ \begin{array}{l} \dot{\Phi}_\xi(t,t_\circ,\lambda) = \tilde{A}_\xi(t,\lambda)\Phi_\xi(t,t_\circ,\lambda)\,, \\ \Phi_\xi(t_\circ,t_\circ,\lambda) = I\,. \end{array} \right.$$

We assume that there exist some positive numbers T_x, μ_x, T_θ and μ_θ such that, for all $t \geq 0$ and all $\lambda \in \mathcal{D}$,

$$\int_t^{t+T_x} \Phi_\xi(\tau,t,\lambda)^\top C(\tau)^\top C(\tau)\Phi_\xi(\tau,t,\lambda)d\tau \geq \mu_x$$

$$\int_t^{t+T_\theta} \Lambda(\tau,\lambda)^\top C(\tau)^\top C(\tau)\Lambda(\tau,\lambda)d\tau \geq \mu_\theta\,.$$

Under these assumptions the origin of the estimation error dynamics given by (12) is λ-UGES and, consequently, the origin of the system (11) is UGES. □

The choice of the estimation gains as well as the λ-PE conditions imposed in Assumption 1 ensure that the estimation error dynamics is excited in the sense of Definition 5. In (Loría and de León Morales 2003) it is imposed that the matrix $Q := \left[(\tilde{A}_\zeta - L_\zeta C_\zeta)^\top P_\zeta + P_\zeta(\tilde{A}_\zeta - L_\zeta C_\zeta)\right] + \dot{P}_\zeta$ is PE along the output trajectories, here denoted by $\tilde{y}(t,\lambda)$. The conditions imposed above guarantee that Q is actually positive definite — cf. the proof of Proposition 1. It shall be clear from previous discussions on parameterized systems, that the conditions imposed (as well as e.g. in (Khalil 1996*b*, Janković 1996, Ortega and Fradkov 1993, Loría and de León Morales 2003)) are required to hold along trajectories. However, notice that as it is shown in (Loría and Panteley 2002), at least for the exponential stability of the origin of (12), the imposed PE properties are also necessary. An interesting open question is whether one can relax the λ-PE assumptions to a form of PE independent of the trajectories (e.g. in the spirit of (Loría *et al.* 2002*b*)), to conclude UGAS.

Proof of Proposition 1. We provide here the main steps of the proof, which follows a standard Lyapunov analysis. Detailed computations and intermediary steps are provided in the appendix. Considering that the system is forward complete, let

$$V_\zeta(t,\bar{\zeta},\lambda) := \bar{\zeta}^\top P_\zeta(t,\lambda)\bar{\zeta}\,, \tag{16}$$

where

$$P_\zeta := \begin{pmatrix} P_x & -P_x\Lambda \\ -\Lambda^\top P_x & P_\theta + \Lambda^\top P_x\Lambda \end{pmatrix}. \tag{17}$$

Claim 1 Define $T := \max\{T_\theta, T_x\}$. Then, there exist positive numbers α_1 and α_2 such that for any $\bar{\zeta} \in \mathbb{R}^{n+m}$, $t \geq T$ and $\lambda \in \mathcal{D}$, the function $V_\zeta(t,\bar{\zeta},\lambda)$ defined in (16) satisfies

$$\alpha_1 \left|\bar{\zeta}\right|^2 \leq V_\zeta(t,\bar{\zeta},\lambda) \leq \alpha_2 \left|\bar{\zeta}\right|^2\,. \tag{18}$$

Proof. See Appendix[2] A. △

Long but straight-forward computations show that the total derivative of $\dot{V}_\zeta(t,\bar{\zeta},\lambda)$ along the trajectories of (12) satisfies $\dot{V} = \bar{\zeta}^\top Q\bar{\zeta}$ where

$$Q := \begin{pmatrix} -\rho_x P_x - C^\top C & \rho_x P_x\Lambda \\ \rho_x\Lambda^\top P_x & -\rho_\theta P_\theta - \rho_x\Lambda^\top P_x\Lambda \end{pmatrix}.$$

So defining $\rho := \min\{\rho_x\,;\,\rho_\theta\}$ it follows, from (16) and (17), that for all $t \geq T$, $\bar{\zeta} \in \mathbb{R}^{n+m}$ and all $\lambda \in \mathcal{D}$,

$$\dot{V}_\zeta(t,\bar{\zeta},\lambda) \leq -\rho V_\zeta(t,\bar{\zeta},\lambda)\,. \tag{19}$$

This inequality together with (18) imply that

$$\left|\bar{\zeta}(t)\right| \leq \frac{\alpha_2}{\alpha_1}\left|\bar{\zeta}_\circ\right| e^{-\rho(t-t_\circ)} \tag{20}$$

for all $t_\circ \geq T$ and all $t \geq t_\circ$. For any $t_\circ < T$ and all $t \in [t_\circ,\,T]$ we have, from forward completeness (uniformly in λ), that there exist c_1, $c_2 > 0$ independent of λ, such that[3] $\left|\bar{\zeta}(t)\right| \leq c_1\left|\bar{\zeta}_\circ\right| e^{c_2(t-T)}$. From this and (20) we conclude that the origin of (12) is λ-UGES. ■

2.3 Output feedback control

We follow a cascades approach to output feedback control, that is, we design the controller so that the overall closed loop system has a cascaded structure formed, on one hand, by the plant in closed loop with a state feedback controller and, on the other hand, by the estimation error dynamics.

To that end, consider again system (6) in closed loop with $u = k(t,\hat{x},\hat{\theta})$, $\hat{x} = \Pi^{-1}(t,\hat{\xi})$ and the observer (10) that is,

$$\begin{array}{rcl} \dot{x} & = & f(t,x,\theta) + g(t,x,\theta)k(t,x,\theta) \\ & & +\, g(t,x,\theta)[k(t,\hat{x},\hat{\theta}) - k(t,x,\theta)]\,. \end{array} \tag{21}$$

Define $\bar{x} := x - \hat{x}$, and

$$\begin{array}{rcl} F(t,x,\theta) & := & f(t,x,\theta) + g(t,x,\theta)k(t,x,\theta) \\ \bar{\alpha}(t,x,\theta,\bar{x},\bar{\theta}) & := & k(t,\bar{x}+x,\bar{\theta}+\theta) - k(t,x,\theta) \end{array}.$$

Then, equation (21) becomes

$$\dot{x} = F(t,x,\theta) + g(t,x,\theta)\bar{\alpha}(t,x,\theta,\bar{x},\bar{\theta}) \tag{22}$$

where the states $\bar{x}$ and $\bar{\theta}$ are generated by the estimation error dynamics (11) and $\Pi^{-1}(t,\bar{\xi})$.

Next, to exhibit the cascaded structure of the overall closed loop system let us introduce $\chi :=$

[2] An explicit expression of a convenient α_1 is also given in Appendix A.

[3] This may be established integrating $\dot{V}_\zeta \leq V_\zeta$, using the comparison theorem and (18).

$\text{col}[\chi_1, \chi_2]$ with $\chi_1 := x$ and $\chi_2 := \overline{\zeta}$, $F_\chi(t, \chi_1) := F(t, x, \theta)$, $G_\chi(t, \chi) := g(t, x, \theta)\bar{\alpha}(t, x, \theta, \bar{x}, \bar{\theta})$ and $H_\chi(t, \chi_2, \lambda) := A_\zeta(t, \lambda) - \tilde{L}_\zeta(t, \lambda)C(t)\overline{\zeta}$. Then, the closed loop system can be written as

$$\begin{aligned} \dot{\chi}_1 &= F_\chi(t, \chi_1) + G(t, \chi) && \text{(23a)} \\ \dot{\chi}_2 &= H_\chi(t, \chi_2, \lambda) \,. && \text{(23b)} \end{aligned}$$

Under the conditions of Proposition 1, $\bar{x}(t)$ and $\bar{\theta}(t)$ (hence $\chi_2(t)$) tend to zero exponentially fast. More precisely, the origin, $\chi_2 = 0$, of (23b) is λ-UGES. By assumption, the origin of $\dot{x} = F(t, x, \theta)$ is UGAS hence, so is the origin of $\dot{\chi}_1 = F_\chi(t, \chi_1)$. We also have, under the standing regularity assumptions on the system's dynamics and the control function $k(\cdot, \cdot, \cdot)$ and the interconnection term $g(\cdot, \cdot, \cdot)$, that for each $r > 0$ there exists a continuous non decreasing function α_r such that, for all $t \geq t_\circ \geq 0$ and all $|\chi_2| \leq r$,

$$|G(t, \chi(t))| \leq \alpha_r(|\chi_1(t)|) \,.$$

Finally, we remark that by construction, $\bar{\alpha}(t, x, \theta, 0, 0) \equiv 0$ hence, $G(t, \chi) \equiv 0$ if $\chi_2 = 0$. Following these observations, the following result holds.

Theorem 1 Consider the system (6) in closed loop with $k(t, \hat{x}, \hat{\theta})$ and the estimator from Proposition 1 under the following assumptions.

Assumption 2 For the system (7) assume that there exist: a Lyapunov function $V_x(t, x, \theta)$, a positive semidefinite function $W(\cdot)$, class $\mathcal{K}_\infty$ functions α_4, α_5 and a class $\mathcal{K}$ function α_6 such that

$$\begin{gathered} \alpha_4(|x|) \leq V_x(t, x, \theta) \leq \alpha_5(|x|) \\ \dot{V}_{x(7)}(t, x, \theta) \leq -W(x) \\ \left| \frac{\partial V_x}{\partial x}(t, x, \theta) \right| \leq \alpha_6(|x|) \,. \end{gathered}$$

Moreover, for any r the functions α_5 and α_r are such that

$$\int_0^\infty \frac{ds}{\alpha_5 \circ \alpha_4^{-1}(s) \alpha_r \circ \alpha_4^{-1}(s)} = \infty \,. \qquad (24)$$

Assumption 3 (growth of F_χ and G_χ) There exist numbers $q \geq 0$, $c_1, c_2 > 0$ such that, for all $\theta \in \mathbb{R}^m$,

$$\frac{|k(t, x, \theta)|}{|z|^q} \geq c_1, \qquad \forall \; |x| \geq c_2 \,, \; t \geq 0 \,,$$

Under these conditions, the origin of the closed loop system is UGAS. □

Sketch of proof . Assumption 2 implies forward completeness: let $t_{\max} < \infty$ be such that $[t_\circ, t_{\max})$ is the maximal interval of definition for the solutions $\overline{\zeta}(t)$ and $x(t)$. On this interval we have that $\tilde{A}_\zeta(\cdot, \lambda)$, $\tilde{L}(\cdot, \lambda)$ and $C(\cdot)$ are uniformly bounded for all $\lambda \in \mathcal{D}$. Also, $V_\zeta(t, \overline{\zeta}, \lambda)$ is well defined and satisfies (18) for all $t \in [t_\circ, t_{\max})$, $\lambda \in \mathcal{D}$ and $\overline{\zeta} \in \mathbb{R}^{n+m}$. Further, define $v(t, \lambda) := V_\zeta(t, \overline{\zeta}(t), \lambda)$. It follows from (19) that $\dot{v}(t, \lambda) \leq v(t, \lambda)$ for all $t \in [t_\circ, t_{\max})$ and $\lambda \in \mathcal{D}$. Integrating the latter and using (18) once more, we obtain that there exists $c_\zeta \in \mathcal{K}_\infty$ such that $|\overline{\zeta}(t)| \leq c_\zeta(|\overline{\zeta}_\circ|)$. Clearly, such reasoning holds for any finite $t_{\max}$. Consider next the function $V_x(t, x, \theta)$ from Assumption 2. Let $r := c_\zeta(|\overline{\zeta}_\circ|)$. Define $v_x(t, \lambda, \theta) := V_x(t, x(t), \theta)$; its time derivative along the trajectories of (23a) satisfies, for all $t \in [t_\circ, t_{\max})$, $\lambda \in \mathcal{D}$ and $\theta \in \mathbb{R}^m$, $\dot{v}_x(t, \lambda, \theta) \leq \alpha_6 \circ \alpha_4^{-1}(v_x(t, \lambda, \theta))\alpha_r \circ \alpha_4^{-1}(v_x(t, \lambda, \theta))$. Integrating on both sides from $t_\circ$ to $t_{\max}$ and using the fact that $\lim_{t \to t_{\max}} v_x(t, \lambda, \theta) = \infty$ (uniformly for all λ and θ) we obtain that

$$\int_{v_x(t_\circ, \lambda, \theta)}^\infty \frac{dv_x}{\alpha_5 \circ \alpha_4^{-1}(v_x) \alpha_r \circ \alpha_4^{-1}(v_x)} = t - t_{\max} < \infty \,,$$

which contradicts (24), hence $t_{\max} = \infty$.
The rest of the proof follows along the lines of the proof of (Loría and de León Morales 2003, Theorem 3), invoking (Panteley and Loría 2001, Theorem 2) and observing that Assumption 3 implies that the growth rate of $F_\chi(t, \cdot)$ is similar to that of $G(t, \cdot)$, as function of χ_1 (i.e., for each fixed t and χ_2). ■

3 Conclusion

We have addressed the problem of adaptive observer design and output feedback control for nonlinear time-varying systems. We have established that, under certain persistency of excitation conditions, uniform global asymptotic stability may be obtained. Such conditions are imposed along the output trajectories of the system. Undergoing further research is focussed on relaxing such condition for a property of persistency of excitation, independent of the trajectories.

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A Proof of Claim 1

It is sufficient to establish that all the functions involved in the definition of P_ζ are uniformly bounded and that P_ζ is positive definite for all $t \geq T$ and $\lambda \in \mathcal{D}$. The former follows from the definitions of P_x, P_θ and the assumption that $A_\xi(t,u,y)$ and $B(t,u,y)$ are uniformly bounded hence, the existence of α_2. The latter can be shown computing the Schur complement of P_ζ, that is, P_ζ is positive definite if, for all $t \geq 0$ and $\lambda \in \mathcal{D}$, $P_x(t,\lambda)$ is positive definite and $P_\theta(t,\lambda) + \Lambda^\top P_x(t,\lambda)\Lambda > \Lambda^\top P_x(t,\lambda) P_x(t,\lambda)^{-1} P_x(t,\lambda)\Lambda$. Notice that both of these conditions are fulfilled if $P_\theta(t,\lambda)$ and $P_\theta(t,\lambda)$ are positive definite for all $t \geq 0$ and $\lambda \in \mathcal{D}$.

Expression of α_1. Recall that $T := \max\{T_x; T_\theta\}$. It can be shown, by integrating (14) and (15), that

$$P_\theta(t,\lambda) > \mu_\theta e^{-\rho_\theta T_\theta} I \qquad \forall t \geq T_\theta,\ \lambda \in \mathcal{D}\,. \quad (25)$$

and

$$P_x(t,\lambda) \geq \mu_x e^{-\rho_x T_x} I\,, \quad (26)$$

for all $t \geq T_x$ and all $\lambda \in \mathcal{D}$. From (17), (25) and (26) we have that, for all $t \geq T$ and all $\lambda \in \mathcal{D}$,

$$\begin{aligned} P_\zeta(t,\lambda) \;\; &\geq \;\; \begin{pmatrix} \frac{\mu_x}{2} e^{-\rho_x T_x} I & 0 \\ 0 & \frac{\mu_\theta}{2} e^{-\rho_\theta T_\theta} I \end{pmatrix} \\ &+ \;\; \begin{pmatrix} \frac{1}{2} P_x & -P_x \Lambda \\ -\Lambda^\top P_x & \frac{1}{2} P_\theta + \Lambda^\top P_x \Lambda \end{pmatrix}. \end{aligned}$$

Computing the Schur complement, for the second matrix, we obtain that it is positive definite if

$$\frac{1}{2} P_x > 0 \quad \text{and} \quad \frac{1}{2} P_\theta + \Lambda^\top P_x \Lambda > \Lambda^\top P_x P_x^{-1} P_x \Lambda\,,$$

which is equivalent to the positiveness of P_x and P_θ. Hence, for any $t \geq T$ and $\lambda \in \mathcal{D}$,

$$P_\zeta(t,\lambda) \geq \begin{pmatrix} \frac{\mu_x}{2} e^{-\rho_x T_x} I & 0 \\ 0 & \frac{\mu_\theta}{2} e^{-\rho_\theta T_\theta} I \end{pmatrix} \geq \alpha_1 I\,, \quad (27)$$

where

$$\alpha_1 := \frac{1}{2} \min\{\mu_x e^{-\rho_x T_x}; \mu_\theta e^{-\rho_\theta T_\theta}\}\,.$$